The Borcherds lift

Lecture 7 • Charlotte Chan • March 14, 2016

These are rough notes of a lecture in which we explain how to define the Borcherds lift from the regularized theta lift discussed in Lecture 6. We give very few numerics in this exposition and instead give specific references to [Br] and [B2] for formulas.

Contents

1. The regularized theta lift

Recall from the previous lecture that we constructed a real-analytic function $\Phi^L_{\beta,m}$ on $\mathrm{Gr}(L) \smallsetminus H(\beta, m)$ by defining it to be the constant term in the Laurent expansion of the regularized theta lift $\Phi^L_{\beta,m}(v,s)$ at $s = 1 - \frac{k}{2}$. From this real-analytic function $\Phi^L_{\beta,m}$, we'd like to construct an automorphic form for $\Gamma(L) := \mathrm{O}^+(L \otimes \mathbb{R}) \cap \mathrm{O}_d(L)$, where $\mathrm{O}^+(L \otimes \mathbb{R}) \subset \mathrm{O}(L \otimes \mathbb{R})$ is the connected component of the identity and $\mathrm{O}_d(L)$ is the discriminant kernel of $\mathrm{O}(L) := \{g \in \mathrm{O}(L \otimes \mathbb{R}) : gL = L\}$, i.e., $\mathrm{O}_d(L)$ is the subgroup of $\mathrm{O}(L)$ consisting of all elements that act trivially on L'/L.

However, to construct such an automorphic form, we want to extract from $\Phi^L_{\beta,m}$ a function that has holomorphicity properties. There are multiple issues to handle:

(a) The lift $\Phi^L_{\beta,m}$ is a function on $\mathrm{Gr}(L) \smallsetminus H(\beta, m)$, so we need a complex structure on the space $\mathrm{Gr}(L) \smallsetminus H(\beta, m)$.

(b) Even if we did have a complex structure on $\mathrm{Gr}(L) \smallsetminus H(\beta, m)$, the function $\Phi^L_{\beta,m}$ is real-valued, so it has no hope of being holomorphic. A natural guess would be to show that $\Phi^L_{\beta,m}$ is related to the real part of some automorphic form of $\Gamma(L)$.

We will deal with (a) in Section 2 and (b) in Section 3.

2. Tube domains and Heegner divisors

Assume that L has an anisotropic vector. When L has rank at least 5, this is automatic. Fix a primitive anisotropic vector $z \in L$ and pick $z' \in L'$ with $(z, z') = 1$. Note that z' exists since L is non-degenerate. Consider the sublattice

$$K := L \cap z^\perp \cap z'^\perp$$

of signature $(1, b^- - 1)$. We

Consider the following spaces:

$$\mathrm{Gr}(L) := \{\text{2-dimensional positive-definite subspaces of } L \otimes \mathbb{R}\}$$
$$\mathcal{K}^+ \subset \mathcal{K} := \{[Z_L] \in \mathbb{P}(L \otimes \mathbb{C}) : Z_L^2 = 0 \text{ and } (Z_L, \overline{Z}_L) > 0\}$$
$$\mathbb{H}_{b^-} \subset \{X + iY \in K \otimes \mathbb{C} : Y^2 > 0\}$$

where we choose a connected component of $\mathcal{K}$ and call it $\mathcal{K}^+$, and we choose $\mathbb{H}_{b^-}$ to be the connected component corresponding to $\mathcal{K}^+$ under the maps below:

$$\begin{array}{ccc}
\mathrm{Gr}(L) & v & \\
\downarrow & \downarrow & \\
\mathcal{K}^+ & [X_L + iY_L] & [Z_L = Z + z' - (q(Z) + q(z'))z] \\
\uparrow & & \uparrow \\
\mathbb{H}_{b^-} & & Z
\end{array}$$

Here, X_L, Y_L is an orthogonal basis of v with $X_L^2 = Y_L^2$; we choose the order X_L, Y_L that identifies $\mathrm{Gr}(L)$ with our chosen connected component $\mathcal{K}^+$.

Lemma 1. *For $\lambda \in L'$, write $\lambda = \lambda_K + az' + bz$ for some $\lambda_K \in K'$. Under the above bijections,*
$$\lambda^\perp = \{Z \in \mathbb{H}_{b^-} : a \cdot q(Z) - (Z, \lambda_K) - a \cdot q(z') - b = 0\}.$$

Proof. This is an easy computation. By definition $Z_L = Z + z' + (-q(Z) - q(z'))z$, so
$$\begin{aligned}
(Z_L, \lambda) &= (Z + z' + (-q(Z) - q(z'))z, \lambda_K + az' + bz) \\
&= (Z, \lambda_K) + a(z', z') + b - a \cdot q(Z) - a \cdot q(z'),
\end{aligned}$$
since Z is perpendicular to z, z' and z is isotropic. □

Definition 2. Following [B2] and [Br], we define the *Heegner divisor of discriminant* (β, m) to be
$$\sum_{\substack{\lambda \in \beta + L \\ q(\lambda) = m}} \lambda^\perp.$$
It is a $\Gamma(L)$-invariant divisor on $\mathbb{H}_{b^-}$ with support $H(\beta, m)$.

3. Borcherds products

We fix a choice of z, z' as in the previous section. Via the bijections in the previous section, $\Phi^L_{\beta,m}$ is a real-valued real analytic function on $\mathbb{H}_{b^-} \smallsetminus H(\beta, m)$. Looking at Theorem 3.8 of [Br], we can write
$$\Phi^L_{\beta,m}(Z) = \xi^L_{\beta,m}(Z) + \psi^L_{\beta,m}(Z), \qquad \text{for } Z \in \mathbb{H}_{b^-} \smallsetminus H(\beta, m),$$
where $\xi^L_{\beta,m}$ is real analytic on $\mathbb{H}_{b^-}$ and $\psi^L_{\beta,m}$ is the sum of terms of the form $\log|\dots|$.

Theorem 3 (Thm 3.15 of [Br])**.** *There exists a holomorphic function $\Psi^L_{\beta,m}$ on $\mathbb{H}_{b^-}$ such that*
$$\log|\Psi^L_{\beta,m}(Z)| = -\frac{1}{4}\left(\psi^L_{\beta,m}(Z) - C_{\beta,m}\right),$$
where $C_{\beta,m}$ is a constant independent of Z. Moreover, the zeroes of $\Psi^L_{\beta,m}$ can be calculated: for any open $U \subset \mathbb{H}_{b^-}$ with compact closure $\overline{U} \subset \mathbb{H}_{b^-}$, let
$$\mathcal{S}(\beta, m, U) := \{\lambda \in L' : \lambda + L = \beta,\ q(\lambda) = m,\ \lambda^\perp U \neq \varnothing\}.$$

Then

$$\Psi^L_{\beta,m}(Z) \cdot \prod_{\lambda \in \mathcal{S}(\beta,m,U)} (\lambda, Z_L)^{-1}$$

is holomorphic and nonvanishing on U.

It seems to me that once one can calculate the Fourier expansion of the regularized theta lift, the statements in the above theorem hold essentially by appealing to the theorem that a real-valued function is pluriharmonic if and only if it is the real part of a holomorphic function.

Definition 4. We call $\psi^L_{\beta,m}$ the *Borcherds lift* of the weight-k vector-valued nearly holomorphic form $F^L_{\beta,m}(1-\frac{k}{2})$. The divisor of $\psi^L_{\beta,m}$ on $\mathbb{H}_{b^-}$ is the Heegner divisor of discriminant (β, m).

Let $F\colon \mathbb{H} \to \mathbb{C}[L'/L]$ be a nearly holomorphic modular form of weight k with principal part

$$\sum_{\beta \in L'/L} \sum_{\substack{n-q(\beta)\in\mathbb{Z} \\ n<0}} c(\beta, n) \cdot e_\beta(n\tau), \qquad c(\beta, n) \in \mathbb{Z}.$$

Then

$$F(\tau) = \frac{1}{2} \sum_{\beta \in L'/L} \sum_{\substack{m-q(\beta)\in\mathbb{Z} \\ m<0}} c(\beta, m) \cdot F_{\beta,m}(\tau, 1-\tfrac{k}{2})$$

and we definite the Borcherds lift of F to be

$$\Psi_F := \prod_{\beta \in L'/L} \prod_{\substack{m-q(\beta)\in\mathbb{Z} \\ m<0}} \left(\Psi^L_{\beta,m}\right)^{c(\beta,m)/2}.$$

References

[B2] Borcherds, Richard. *Automorphic forms with singularities on Grassmannians.* Inventiones, 1998.

[Br] Brunier, Jan Hendrik. *Borcherds products on $O(2,l)$ and Chern classes of Heegner divisors.* 2000.

[Ku] Kudla, Stephen S. *Integrals of Borcherds forms.* Compositio, 2003.