LECTURES 4 & 5: POINCARÉ SERIES

ANDREW SNOWDEN

These are notes from my two lectures on Poincaré series from the 2016 Learning Seminar on Borcherds products. I begin by reviewing classical Poincaré series, then redo the theory in the vector-valued setting, and finally discuss a non-holomorphic variant.

References:

- Bruinier, Borcherds products on O(2,ℓ) and Chern classes of Heegner divisors (Lecture Notes in Math 1780). I basically cover §1.2 and §1.3 here.
- Leis, The Poincaré series (notes).

Warning: There are many formulas and computations in these notes. I have not verified them myself, but simply copied them from the sources I used. They're probably mostly right, but there could be some mistakes.

1. Classical Poincaré series

Suppose we want to construct modular forms for some group $\Gamma \subset \mathbf{SL}_2(\mathbf{R})$. One way to try to construct a function with the right invariance is to start with any function and then average it over Γ. Unfortunately, in this particular situation, such averages will usually not converge. However, a modification of this idea works: we instead start with a function that's already invariant under Γ_∞ (assuming ∞ is a cusp), namely an exponential function, and then average over the cosets of Γ_∞ in Γ. These averages are the Poincaré series, and they span the space of modular (or cusp) forms. While this theory works for quite general Γ, we only present the case of $\Gamma = \mathbf{SL}_2(\mathbf{Z})$ here.

1.1. **Notation.** Before getting into things, we set some notation. We let $\Gamma = \mathbf{SL}_2(\mathbf{Z})$ and let $\Gamma_\infty \subset \Gamma$ be the subgroup of upper-triangular matrices. These groups act on the upper half-plane $\mathfrak{h}$ by linear fractional transformations:

$$\gamma z = \frac{az+b}{cz+d}, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Note that Γ_∞ is the full stabilizer in Γ of the cusp ∞.

We let $j_\gamma(z) = cz + d$ (in the above notation) be the usual automorphy factor. For a function f on $\mathfrak{h}$, we define $f|_k\gamma$ to be the function on $\mathfrak{h}$ given by

$$(f|_k\gamma)(z) = j_\gamma(z)^{-k} f(z).$$

This defines an action of Γ on functions, and the transformation property of modular forms of weight k is exactly invariance under this action.

Finally, we put $e(z) = e^{2\pi i z}$.

1.2. **Definition.** Let $k > 2$ and $m \geq 0$ be integers, with k even. The **Poincaré series** is defined by

$$P_{m,k}(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} e(m\gamma z)|_k \gamma \tag{1.1}$$

Note that the function $e(mz)$ is invariant under the $|_k$ aciton of Γ_∞, and so the sum is well-defined. We have a bijection

$$\begin{aligned} \Gamma_\infty \backslash \Gamma &\to \{(c, d) \in \mathbf{Z}^2 \mid c \geq 0, (c, d) = 1\} \\ \Gamma_\infty \gamma &\mapsto \pm(0, 1)\gamma, \end{aligned}$$

where the sign is taken to make the first coordinate positive (and $(0, 1)$ denotes a row vector). Using this, we find

$$P_{m,k}(z) = \sum_{\substack{c,d \in \mathbf{Z}^2 \\ c \geq 0, (c,d)=1}} \frac{e(m\gamma_{c,d} z)}{(cz + d)^k}, \tag{1.2}$$

where $\gamma_{c,d}$ denotes any element of Γ with bottom row (c, d). From this we see that $P_{0,k}(z)$ is the usual Eisenstein series $E_k(z)$ of weight k. Moreover, it is clear that the the terms of $P_{m,k}(z)$ are bounded (in absolute value) by those of $E_k(z)$, and thus (appealing to the standard convergence result for E_k) we see that the series (1.2) converges absolutely. Thus $P_{m,k}(z)$ is a well-defined holomorphic function on the upper half-plane invariant under the $|_k$ action of Γ. Furthermore, the bound $|P_{m,k}(z)| \leq |E_k(z)|$ implies that $P_{m,k}(z)$ is holomorphic at the cusps, since the same is true for $E_k(z)$, and so $P_{m,k}(z)$ is a modular form of weight k.

1.3. **Fourier expansion.** Fix m and k. Being a modular form, the Poincaré series admits a Fourier expansion:

$$P_{m,k}(z) = \sum_{n \geq 0} a_n e(nz).$$

We now give a formula for the coefficients a_n. We need to introduce some auxiliary quantities. The **Kloosterman sum** $k(m, n, c)$ is defined by

$$k(m, n, c) = \sum_{d \in (\mathbf{Z}/c\mathbf{Z})^\times} e\left(\frac{nd + md^{-1}}{c}\right).$$

This is a finite sum of roots of unity. The **Bessel function** is defined by

$$J_{k-1}(x) = \frac{1}{2\pi i} \int_{|z|=1} z^{-k} e^{(z - z^{-1})x/2} dz.$$

We let

$$\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$$

be the usual function, and let ζ be the Riemann zeta function. We can now give the formula:

Theorem 1.3. *For $m > 0$ we have*

$$a_n = \left(\frac{n}{m}\right)^{(k-1)/2} \sum_{c=1}^{\infty} \left(\frac{2\pi i}{c}\right)^k k(m, n, c) J_{k-1}\left(\frac{4\pi\sqrt{nm}}{c}\right),$$

while for $m = 0$ we have

$$a_n = \frac{(2\pi i)^k \sigma_{k-1}(n)}{(k-1)!\zeta(k)}.$$

Proof. We just provide a sketch of the proof. Let $y > 0$ be a real number. We have

$$\begin{aligned} a_n &= \int_{0+iy}^{1+iy} P_{m,k}(z)e(-nz)dz \\ &= \sum_{\substack{c,d\in\mathbf{Z}^2 \\ c\geq 0,(c,d)=1}} \int_{0+iy}^{1+iy} \frac{e(m\gamma_{c,d}z)}{(cz+d)^k} e(-nz)dz \end{aligned}$$

The $c = 0$ terms are

$$\sum_{d=\pm 1} \int_{0+iy}^{1+iy} e(mz)e(-nz)dz = 2\delta_{n,m}.$$

Write $a_n = 2\delta_{n,m} + a'_n$, so that a'_n is the sum over $c > 0$. We break this sum up over cosets mod c. Precisely, write $d = \overline{d} + c\ell$ where $0 \leq \overline{d} < c$ is prime to c and $\ell \in \mathbf{Z}$. We have

$$cz + d = c(z+\ell) + \overline{d}, \qquad \gamma_{c,d}z = \gamma_{c,\overline{d}}(z+\ell).$$

Thus

$$a'_n = \sum_{c=1}^{\infty} \sum_{\substack{0\leq\overline{d}<c, \\ (c,\overline{d})=1}} \sum_{\ell\in\mathbf{Z}} \int_{0+iy}^{1+iy} \frac{e(m\gamma_{c,\overline{d}}(z+\ell))}{(c(z+\ell)+\overline{d})^k} e(-nz)dz$$

Changing z to $z - \ell$ (and dropping the overline on d), we find

$$a'_n = \sum_{c,d} \sum_{\ell\in\mathbf{Z}} \int_{\ell+iy}^{\ell+1+iy} \frac{e(m\gamma_{c,d}z)}{(cz+d)^k} e(-nz)dz$$

$$a'_n = \sum_{c,d} \int_{-\infty+iy}^{\infty+iy} \frac{e(m\gamma_{c,d}z)}{(cz+d)^k} e(-nz)dz$$

Next, we have

$$(cz+d)^k = c^k(z+\tfrac{d}{c})^k, \qquad \gamma_{c,d}z = \frac{a}{c} - \frac{1}{c^2(z+\frac{d}{c})}.$$

We change z to $z - \frac{d}{c}$ in the integral, to obtain

$$\begin{aligned} a'_n &= \sum_{c=1}^{\infty} \sum_{d\in(\mathbf{Z}/c\mathbf{Z})^\times} \int_{-\infty+iy}^{\infty+iy} \frac{e(\frac{ma}{c} - \frac{m}{c^2 z})}{c^k z^k} e(-nz + \tfrac{nd}{c})dz \\ &= \sum_{c=1}^{\infty} c^{-k} \left[\sum_{d\in(\mathbf{Z}/c\mathbf{Z})^\times} e(\tfrac{ma}{c})e(\tfrac{nd}{c}) \right] \int_{-\infty+iy}^{\infty+iy} z^{-k} e(-\tfrac{m}{c^2 z} - nz)dz \end{aligned}$$

Note that in the first sum, a is the inverse of d modulo c. Thus this sum is simply $k(m,n,c)$. If $m > 0$ then, after some manipulation, the integral turns into the Bessel function and other factors. [What happened to the $\delta_{n,m}$ in the formula for a_n?] If $m = 0$ then the integral is easy to evaluate using the residue theorem, and one reaches the final formula by applying the identity

$$\sum_{c=1}^{\infty} \frac{k(0,n,c)}{c^k} = \left(-\frac{2\pi i}{n}\right)^{k-1} \frac{\sigma_{k-1}(n)}{\zeta(k)}.$$

[This doesn't seem quite right...] □

Corollary 1.4. *For $m > 0$ the series $P_{m,k}$ is a cusp form.*

Proof. It is clear from the formulas that $a_n = 0$ if $m > 0$. □

Corollary 1.5. *We have $P_{m,k} = 0$ for all $m > 0$ if $k \in \{4, 6, 8, 10\}$.*

Proof. There are no non-zero cusp forms for Γ for these weights. □

Remark 1.6. This vanishing is not at all clear from the definition of $P_{m,k}$ or the formula for its Fourier coefficients. Apparently, it is a difficult problem in general to determine when Poincaré series vanish identically. □

1.4. **Petersson inner product.** Let f and g be weight k modular forms, at least one of which is cuspidal. Recall that their **Petersson inner product** is defined by

$$(f, g) = \int_{\Gamma\backslash\mathfrak{h}} f(z)\overline{g(z)} y^{k-2} dx dy.$$

The factor y^{k-2} ensures that this is well-defined, i.e., independent of the choice of fundamental domain. The Petersson inner product defines a non-degenerate hermitian inner product on the space $S_k(\Gamma)$ of cusp forms of weight k. The following result describes how it interacts with Poincaré series

Theorem 1.7. *Let f be a cusp form of weight k. Then*

$$(f, P_{m,k}) = \frac{\Gamma(k-1)}{(4\pi m)^{k-1}} a_m(f),$$

where Γ is the usual Γ-function and $a_m(f)$ denotes the mth Fourier coefficient of f.

Proof. We have

$$(f, P_{m,k}) = \int_{\Gamma\backslash\mathfrak{h}} y^k f(z) \sum_{\gamma \in \Gamma_\infty\backslash\Gamma} \left(\overline{j_\gamma(z)}\right)^{-k} \overline{e(m\gamma z)} \frac{dx dy}{y^2}$$

We now make use of the identity

$$\mathrm{Im}(\gamma z) = \frac{\mathrm{Im}(z)}{|j_\gamma(z)|^2}$$

to obtain

$$(f, P_{m,k}) = \int_{\Gamma\backslash\mathfrak{h}} \sum_{\gamma \in \Gamma_\infty\backslash\Gamma} \mathrm{Im}(\gamma z)^k f(\gamma z)\overline{e(m\gamma z)} \frac{dx dy}{y^2}.$$

Note that the measure $\frac{dxdy}{y^2}$ is invariant under the action of Γ. We can therefore "unfold" the integral to obtain

$$(f, P_{m,k}) = \int_{\Gamma_\infty\backslash\mathfrak{h}} y^{k-2} f(z)\overline{e(mz)} dx dy.$$

Now, the action of Γ_∞ on $\mathfrak{h}$ is generated by the translation $z \mapsto z+1$, and so has for a fundamental domain the region defined by $0 \le x \le 1$. We thus find

$$(f, P_{m,k}) = \int_0^\infty \int_0^1 y^{k-2} f(x+iy) e^{-2\pi imx - 2\pi my} dx dy.$$

Write $f(z) = \sum_{n \ge 0} a_n e(nz)$ with $a_n = a_n(f)$. Then we obtain

$$(f, P_{m,k}) = \sum_{n \ge 0} a_n \int_0^\infty \int_0^1 y^{k-2} e^{2\pi i(n-m)x} e^{-2\pi(n+m)y} dx dy.$$

The x-integral is $\delta_{n,m}$ and the y-integral (for $n = m$) is $\Gamma(k-1)(4\pi m)^{-(k-1)}$. This gives the stated formula. □

Corollary 1.8. *The series $P_{m,k}$ with $m > 0$ span the space $S_k(\Gamma)$ of cusp forms.*

Proof. Suppose $f \in S_k(\Gamma)$ is orthogonal to all of the $P_{m,k}$. Then $a_m(f) = 0$ for all $m > 0$ by the Theorem, and so $f = 0$. The non-degeneracy of the inner product thus implies that the $P_{m,k}$ span. □

Example 1.9 ($k = 12$)**.** Let $\Delta(z)$ be the modular discriminant, the unique normalized cusp form of weight 12. We have

$$\Delta(z) = q \prod_{n \ge 1} (1 - e(nz))^{24} = \sum_{n \ge 1} \tau(n) e(nz),$$

where $\tau(n)$ is the customary notation for the nth Fourier coefficient. (The function τ is called the Ramanujan function.) From the Theorem, we find

$$(\Delta, P_{m,12}) = \frac{10! \tau(m)}{(4\pi m)^{11}}$$

Since $S_{12}(\Gamma)$ is 1-dimensional, $P_{m,12}$ is a scalar multiple of Δ. We find

$$P_{m,12}(z) = \frac{10! \cdot \tau(m)}{(2\pi m)^{11} \cdot (\Delta, \Delta)} \cdot \Delta(z).$$

In particular, we find $P_{m,12} = 0$ if and only if $\tau(m) = 0$. Lehmer conjectured $\tau(m) \ne 0$ for all $m \ge 1$. This has been verified for all m up to about 10^{24}, but has not been proved. □

2. Vector-valued Poincaré series

We now give variants of the constructions and results from the previous section in the vector-valued setting. The proofs are exactly the same, so we omit them.

2.1. **Notation.** We let Γ and Γ_∞ be as before, though we now write M for a typical element of Γ. We let $\widetilde{\Gamma}$ be the metaplectic cover of Γ. Its elements are pairs (M, ϕ) where $M \in \Gamma$ and ϕ is a continuous square-root of the automorphy factor j_γ on $\mathfrak{h}$. We let $\widetilde{\Gamma}_\infty \subset \widetilde{\Gamma}$ be the subgroup consisting of elements of the form $(M, 1)$ with $M \in \Gamma_\infty$. We put

$$T = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \right), \qquad Z = \left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, i \right).$$

The element Z generates the center of $\widetilde{\Gamma}$ and $\{1, Z^2\} = \ker(\widetilde{\Gamma} \to \Gamma)$.

Fix an even lattice L of signature (b_+, b_-). We assume (b_+, b_-) is of the form $(2, \ell)$, $(1, \ell-1)$, or $(0, \ell-2)$. We let $q(x) = \frac{1}{2}\langle x, x\rangle$ be the quadratic form. We put $\kappa = 1 + \frac{\ell}{2}$. This will be the weight of the modular forms we produce.

Let L' be the dual lattice and let ρ be the Weil representation of $\widetilde{\Gamma}$ on the group algebra $\mathbf{C}[L'/L]$. For $\gamma \in L'/L$ we let e_γ be the corresponding basis vector in the group algebra, and put $e_\gamma(\tau) = e(\tau) \cdot e_\gamma$.

For a function $f \colon \mathfrak{h} \to \mathbf{C}[L'/L]$ and $(M, \phi) \in \widetilde{\Gamma}$, we define $f|^*_\kappa(M, \phi)$ to be the function defined by

$$(f|^*_\kappa(M,\phi))(\tau) = \phi(\tau)^{-2k}\rho^*(M,\phi)^{-1} f(M\tau).$$

Here $\rho^*(M, \phi)$ is just the complex conjugate of the matrix $\rho(M, \phi)$, since the Weil representation is unitary and the standard basis is orthonormal. The transformation law for a weight κ vector-valued modular form is exactly invariance under the the $|^*_\kappa$ action of $\widetilde{\Gamma}$.

Remark 2.1. (1) The assumption on the signature of L does not seem to be used. (2) The weight κ of the form is derived from the signature of the lattice, but this seems unnecessary. That is, it seems that one can fix the lattice L and then work with arbitrary choices of κ. $\square$

2.2. **Definition (m positive).** Fix $\beta \in L'/L$ and $m \in \mathbf{Q}$ positive such that $m + q(\beta) \in \mathbf{Z}$. Then $e_\beta(m\tau)$ is invariant under $|^*_\kappa T$. Indeed, we have $\rho^*(T)^{-1}e_\beta = e(q(\beta))e_\beta$ [is this right, or should it be $e(-q(\beta))$?], and so

$$e_\beta(m\tau)|^*_\kappa T = \rho^*(T)^{-1}e_\beta e(m(\tau+1)) = e(q(\beta)+m)e_\beta e(m\tau) = e_\beta(\tau),$$

as $e(q(\beta) + m) = 1$ since $q(\beta) + m$ is an integer. We define the **Poincaré series** by

$$P^L_{\beta,m}(\tau) = \tfrac{1}{2} \sum_{(M,\phi)\in\widetilde{\Gamma}_\infty\backslash\widetilde{\Gamma}} e_\beta(m\tau)|^*_\kappa(M,\phi).$$

The sum is well-defined since $e_\beta(m\tau)$ is invariant under $|^*_\kappa T$. One easily sees that this defines a weight κ vector-valued modular form. We often drop the L from the notation.

2.3. **Fourier expansion.** We have a Fourier expansion

$$P_{\beta,m}(\tau) = \sum_{\gamma\in L'/L} \sum_{\substack{n>0,\\ n+q(\gamma)\in\mathbf{Z}}} p_{\beta,m}(\gamma,n)e_\gamma(n\tau).$$

Define the **generalized Kloosterman sum** by

$$H^*_c(\beta,m,\gamma,n) = \frac{e^{-i\pi\,\mathrm{sgn}(c)\kappa/2}}{|c|} \sum_{M\in\Gamma_\infty\backslash\Gamma/\Gamma_\infty} \rho_{\beta,\gamma}(\widetilde{\gamma})e\left(\frac{ma+nd}{c}\right), \qquad M = \begin{pmatrix} a & b \\ c & d\end{pmatrix}.$$

Here $\widetilde{\gamma}$ denotes a lift of γ to $\widetilde{\Gamma}$ and $\rho_{\beta,\gamma}(\widetilde{M})$ denotes the entry of the matrix $\rho(\widetilde{M})$ in row β and column γ. As previously mentioned, $\Gamma_\infty\backslash\Gamma$ is in bijection with relatively prime pairs (c, d) with $c \ge 0$. Since c is fixed, summing over $M \in \Gamma_\infty\backslash\Gamma/\Gamma_\infty$ is the same as summing over $d \in (\mathbf{Z}/c\mathbf{Z})^\times$, and then a is just d^{-1}.

Theorem 2.2. *We have*

$$p_{\beta,m}(\gamma,n) = \delta_{m,n}(\delta_{\beta,\gamma} + \delta_{-\beta,\gamma}) + 2\pi\left(\frac{n}{m}\right)^{(\kappa-1)/2} \sum_{c\in\mathbf{Z}\backslash\{0\}} H^*_c(\beta,m,\gamma,n)J_{k-1}\left(\tfrac{4\pi\sqrt{mn}}{|c|}\right),$$

where J_{k-1} is the Bessel function used previously.

2.4. **Petersson inner product.** Let $\langle , \rangle$ be the Hermitian inner product on $\mathbf{C}[L'/L]$ for which the basis vectors e_γ are orthonormal. We then define the **Petersson inner product** of weight κ vector-valued modular forms f and g (at least one of which is cuspidal) by

$$(f, g) = \int_{\Gamma\backslash\mathfrak{h}} \langle f(\tau), g(\tau)\rangle y^{\kappa-2} dxdy.$$

This defines a non-degenerate hermitian form on the space of cusp forms $S_{\kappa,L}$. We have:

Theorem 2.3. *Let $f \in S_{\kappa,L}$ be a cusp form and let $c(\beta, m)$ denote its Fourier coefficients. Then*

$$(f, P_{m,\beta}) = 2\frac{\Gamma(\kappa-1)}{(4\pi m)^{\kappa-1}} c(\beta, m).$$

Corollary 2.4. *The series $P_{m,\beta}$ with $m > 0$ and $\beta \in L'/L$ span $S_{\kappa,L}$.*

2.5. **Eisenstein series ($m = 0$).** As we have seen, the Poincaré series with $m > 0$ span the space of cusp forms. We'd like to be able ot span all modular forms by including the Poincaré series at $m = 0$, as in the classical case. However, there is an issue here: the function $e_\beta(m\tau)$ is only invariant under $|^*_\kappa T$ if $q(\beta) + m \in \mathbf{Z}$. Thus we can only hope to define the Poincaré series at $m = 0$ under the assumption $q(\beta) \in \mathbf{Z}$. However, this turns out to be enough.

We now give some details. Fix $\beta \in L'/L$ with $q(\beta) \in \mathbf{Z}$. We define the **Eisenstein series** by

$$E^L_\beta(\tau) = \tfrac{1}{2} \sum_{(M,\phi)\in\widetilde{\Gamma}_\infty\backslash\widetilde{\Gamma}} e_\beta|^*_\kappa(M, \phi).$$

Here e_β denotes the constant function $\mathfrak{h} \to \mathbf{C}[L'/L]$ with value e_β. This is easily seen to be a modular form. It is a theorem that the E_β and $P_{m,\beta}$ span the space $M_{\kappa,L}$ of all modular forms.

Consider the Fourier expansion of E_β:

$$E_\beta(\tau) = \sum_{\gamma\in L'/L} \sum_{\substack{n\geq 0\\ q(\gamma)+n\in\mathbf{Z}}} q_\beta(\gamma, n) e_\gamma(ntau).$$

We have

$$q_\beta(\gamma, 0) = \delta_{\beta,\gamma} + \delta_{-\beta,\gamma}$$

and, for $n > 0$,

$$q_\beta(\gamma, n) = \frac{(2\pi)^\kappa n^{\kappa-1}}{\Gamma(\kappa)} \sum_{c\in\mathbf{Z}\backslash\{0\}} |c|^{1-\kappa} H^*_c(\beta, 0, \gamma, n).$$

It is a theorem that this quantity is in fact a rational number.

Remark 2.5. It is not difficult to see that $E_\beta = E_{-\beta}$. It is a theorem that there are no other linear dependencies among the Eisenstein series. Thus the dimension of the space of Eisenstein series (i.e., the dimension of the complement of $S_{\kappa,L}$ in $M_{\kappa,L}$) is equal to

$$\#\{\beta \in L'/L \mid 2\beta = 0, q(\beta) \in \mathbf{Z}\} + \tfrac{1}{2}\#\{\beta \in L'/L \mid 2\beta \neq 0, q(\beta) \in \mathbf{Z}\}. \qquad \square$$

3. Non-holomorphic vector-valued Poincaré series

We now give one more variant of the Poincaré series: non-holomorphic series of negative weight. We only do the vector-valued theory, though there is a simpler scalar theory (and perhaps it would have been good to do this first for pedagogical reasons).

3.1. **Notation.** We fix an even lattice L as before, with the same assumptions on the signature. We now let $k = 1 - \frac{\ell}{2}$, which will be the weight of our forms. For a function $f : \mathfrak{h} \to \mathbf{C}[L'/L]$, we put

$$(f|_k(M,\phi))(\tau) = \phi(\tau)^{-2k}\rho(M,\phi)^{-1}f(M\tau)$$

We let

$$\Delta_k = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

be the weight k Laplacian. It satisfies

$$\Delta_k(f|_k(M,\phi)) = (\Delta_k f)|_k(M,\phi).$$

We let $M_{\nu,\mu}(z)$ and $W_{\nu,\mu}(z)$ be the Whittaker functions. They span the space of solutions of the Whittaker differential equation

$$\frac{d^2w}{dz^2} + \left(-\frac{1}{4} + \frac{\nu}{z} - \frac{\mu^2 - 1/4}{z^2}\right)w = 0$$

and are distinguished from each other by different asymptotic behavior at 0 and ∞. For $s \in \mathbf{C}$ and $y > 0$ we put

$$\mathcal{M}_s(y) = y^{-k/2}M_{-k/2,s-1/2}(y),$$

and for $s \in \mathbf{C}$ and $y \in \mathbf{R} \setminus \{0\}$ we put

$$\mathcal{W}_s(y) = |y|^{-k/2}W_{k\,\mathrm{sgn}(y)/2,s-1/2}(|y|).$$

3.2. **Definition.** Fix $\beta \in L'/L$ and $m \in \mathbf{Q}$ *negative* such that $m + q(\beta) \in \mathbf{Z}$. The function

$$\mathcal{M}_s(-4\pi my)e_\beta(mx)$$

is invariant under $|_kT$ (same calculation as previous section, the $\mathcal{M}_s$ factor is irrelevant) and an eigenfunction of Δ_k of eigenvalue $s(1-s) + (k^2 - 2k)/4$ (this is where the $\mathcal{M}_s$ factor is important). We define the **Poincaré series** by

$$F_{\beta,m}(\tau,s) = \frac{1}{2\Gamma(2s)}\sum_{(M,\phi)\in\widetilde{\Gamma}_\infty\backslash\widetilde{\Gamma}}(\mathcal{M}_s(-4\pi my)e_\beta(mx))|_k(M,\phi),$$

for $\mathrm{Re}(s) > 1$. This is invariant under the $|_k$ action of $\widetilde{\Gamma}$ and an eigenfunction for Δ_k of eigenvalue $s(1-s) + (k^2 - 2k)/4$. Since it is an eigenfunction of Δ_k, it is real-analytic in τ. It is holomorphic in s.

3.3. **Fourier expansion.** Since $F_{\beta,m}(\tau,s)$ is invariant under $x \mapsto x + 1$, it admits a Fourier expansion

$$F_{\beta,m}(\tau,s) = \sum_{\gamma\in L'/L}\sum_{n+q(\gamma)\in\mathbf{Z}}c(\gamma,n,y)e_\gamma(nx).$$

Note that the coefficients depend on y, and we have $e_\gamma(nx)$ instead of $e_\gamma(n\tau)$.

Bruinier gives a formula for the Fourier coefficients. It is quite involved, taking an entire page to state fully. It again uses generalized Kloosterman sums and (different) Bessel functions, and now also involves the Whittaker function $\mathcal{W}_s(y)$. When $s = 1 - k/2$ the formula simplifies somewhat, but is still more complicated than I want to try to reproduce here! The important point seems to be that it has the following form:

$$F_{\beta,m}(\tau,s) = e_\beta(m\tau) + e_{-\beta}(m\tau) + (\text{bounded as } \tau \to \infty).$$

(Recall that $m < 0$, so $e(m\tau)$ is not bounded as $\tau \to \infty$.) Using this, Bruinier obtains the following result:

Theorem 3.1. *Let f be a vector-valued modular form of weight k that is holomorphic on $\mathfrak{h}$ and meromorphic at ∞ (or "nearly holomorphic" in Bruinier's terminology). Then f is a linear combination of the series $F_{\beta,m}(\tau, 1-k/2)$ with $m < 0$.*

Proof. The idea of the proof seems to be the following. By the previous formula for the first terms in the Fourier expansion of $F_{\beta,m}$, one can conconct a linear combination g of the $F_{\beta,m}$'s that has the same principal part of f. ("Principal part" meaning "negative terms in Laurent expansion.") The difference $f - g$ is then killed by Δ_k (since $s = 1-k/2$), invariant under the $|_k$ action of $\widetilde{\Gamma}$, and bounded as $\tau \to \infty$, and therefore vanishes by some sort of maximum modulus principal. $\square$