BorchERDS products learning seminar
The Weil representation: Heisenberg groups, lattices, and vector-valued modular forms

Lecture 3 • Charlotte Chan • February 15, 2016

1. Introduction

I found it very hard to find the content of Sections 2 and 3 in the literature. Bizarrely, the only place I know that discusses Section 2 in detail is my undergraduate thesis [C]. I could not find any references that discuss 3, so in the preparation of this talk, I made some guesses about the construction. Several dodgy guesses were clarified by Andrew during the talk, and the conclusions are written here.

Our goals are the following:

(A) Use Heisenberg groups to motivate the formulas for the Weil representation of $\mathrm{SL}_2(\mathbb{Z})$

(B) Give the definition of a vector-valued modular form and relate this picture to classical modular forms

All representations are over $\mathbb{C}$.

2. Heisenberg groups and the Weil representation of $\mathrm{SL}_2(F)$

In this section we describe the construction of the Weil representation of $\mathrm{SL}_2(F)$ for F a field. In this set-up, the Heisenberg group $H(F)$ is

$$H(F) := \left\{ \begin{pmatrix} 1 & a & c \\ & 1 & b \\ & & 1 \end{pmatrix} : a, b, c \in F \right\}.$$

Note that $H(F)$ sits in a short exact sequence

$$1 \to \left\{ \begin{pmatrix} 1 & & c \\ & 1 & \\ & & 1 \end{pmatrix} \right\} \to \left\{ \begin{pmatrix} 1 & a & c \\ & 1 & b \\ & & 1 \end{pmatrix} \right\} \to \left\{ \begin{pmatrix} 1 & a & * \\ & 1 & b \\ & & 1 \end{pmatrix} \right\} \to 1.$$

Equivalently, we can describe $H(F)$ by the short exact sequence

$$0 \to F \to H(F) \to F^{\oplus 2} \to 0$$

together with a choice of a 2-cocycle on $Q := F^{\oplus 2}$. In this situation, we can pick the cocycle

$$f \colon Q \times Q \to F, \qquad ((a_1, b_1), (a_2, b_2)) \mapsto \tfrac{1}{2} \det \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} = \tfrac{1}{2}(a_1 b_2 - a_2 b_1).$$

From this perspective, since the natural action of $\mathrm{SL}_2(F)$ on Q preserves f, it is clear that this action extends to a center-fixing action of $\mathrm{SL}_2(F)$ on $H(F)$.

Theorem 1 (Stone–von Neumann)**.** *For each $\psi \colon F \to \mathbb{C}^1$ nontrivial, there exists a unique irreducible representation (π_ψ, V_ψ) of $H(F)$ with central character ψ.*

Proof for $F = \mathbb{F}_p$. It is easy to see that $H(F)$ has $p + (p^2 - 1)$ conjugacy classes. Thus $H(F)$ has $p + (p^2 - 1)$ irreducible representations. Since $H(F)$ has a quotient isomorphic to $F^{\oplus 2}$, an abelian group of order p^2, it follows that $H(F)$ has p^2 distinct one-dimensional representations obtained by pulling back along $H(F) \to F^{\oplus 2}$. Note that each of these representation has trivial central character. We have $p - 1$ remaining representations to account for. Let n_i, $1 \le i \le p-1$ denote the dimension of these representations. We know $p^2 + \sum n_i^2 = p^3$ and $n_i \mid \#H(F) = p^3$. It follows that $n_i = p$ for all i.

To finish the proof, it is enough to show that there are $p-1$ irreducible representations of $H(F)$ with distinct central characters. This is easy: Given any nontrivial character ψ of the center $Z(H(F)) \cong F$ (note that there are $p-1$ of these), the induced representation $\mathrm{Ind}_{Z(H(F))}^{H(F)}(\psi)$ has central character ψ and is isomorphic to p copies of an irreducible representation π_ψ, necessarily with central character ψ. $\square$

Now fix $\psi\colon F \to \mathbb{C}^1$ nontrivial. For each $g \in \mathrm{SL}_2(F)$, we may consider representation π_ψ^g of $H(F)$ that arises from precomposing by the action of g:

$$\pi_\psi^g\colon H(F) \xrightarrow{g} H(F) \to \mathrm{GL}(V_\psi).$$

By construction, π_ψ^g is an irreducible representation of $H(F)$ with central character ψ, so by the Stone–von Neumann theorem,

$$\pi_\psi^g \cong \pi_g, \qquad \text{for all } g \in \mathrm{SL}_2(F).$$

Explicitly, this means there exists $\Phi_g \in \mathrm{GL}(V_\psi)$ such that

$$\Phi_g \cdot \pi_\psi^g \cong \pi_\psi \cdot \Phi_g.$$

By Schur's lemma, Φ_g is unique up to scaling and we therefore have a group homomorphism

$$[\rho_\psi]\colon \mathrm{SL}_2(F) \to \mathrm{PGL}(V_\psi), \qquad g \mapsto [\Phi_g].$$

This defines the *projective Weil representation* for $\mathrm{SL}_2(F)$.

Remark 1. This is a remark that is meant to convince you that the formulas for representation of $\mathrm{SL}_2(\mathbb{Z})$ ([Br]) come from a Weil representation construction. As such, we take some liberties with precision in this remark. Let's take $F = \mathbb{F}_p$. We can realize π_ψ on $\mathbb{C}[F]$ by defining an action of $H(F)$ via the following generators:

$$\begin{pmatrix} 1 & a & \\ & 1 & \\ & & 1 \end{pmatrix} \cdot f(x) = f(x-a),$$
$$\begin{pmatrix} 1 & & \\ & 1 & b \\ & & 1 \end{pmatrix} \cdot f(x) = \psi(-bx) f(x),$$
$$\begin{pmatrix} 1 & & c \\ & 1 & \\ & & 1 \end{pmatrix} \cdot f(x) = \psi(c) f(x).$$

Recall that $\mathrm{SL}_2(F)$ is generated by $T = \left(\begin{smallmatrix} 1 & 1 \\ & 1 \end{smallmatrix}\right)$ and $S = \left(\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right)$. Moreover: $(a,b)T = (a, a+b)$ and $(a,b)S = (b,-a)$. In particular, we see that via ρ_ψ, the element $T \in \mathrm{SL}_2(F)$ should act on $\mathbb{C}[F]$ by something like $\gamma \mapsto \psi(-)\gamma$ for $\gamma \in F$. The element $S \in \mathrm{SL}_2(F)$ swaps the $\psi(ax)$ and $f(x-a)$ and so should be a kind of Fourier transform. These vague descriptions agree with the formulas given in Section 1.1 of [Br]. $\Diamond$

The question now is: Can we lift this to an honest representation? That is,

$$\begin{array}{ccc} & & \mathrm{GL}(V_\psi) \\ & \overset{\exists ?}{\nearrow} & \downarrow \\ \mathrm{SL}_2(F) & \xrightarrow[{[\rho_\psi]}]{} & \mathrm{PGL}(V_\psi) \end{array}$$

It turns out we can always lift $[\rho_\psi]$ to a double cover $\mathrm{Mp}_2(F)$ (metaplectic group) of $\mathrm{SL}_2(F) = \mathrm{Sp}_2(F)$, so maybe the better question is:

$$\begin{array}{ccccc} & & \mathrm{Mp}_2(F) & \longrightarrow & \mathrm{GL}(V_\psi) \\ & \overset{\exists ?}{\nearrow} & \downarrow & & \downarrow \\ G & \hookrightarrow & \mathrm{SL}_2(F) & \xrightarrow[{[\rho_\psi]}]{} & \mathrm{PGL}(V_\psi) \end{array}$$

Sometimes we can realize the Weil representation on $\mathrm{SL}_2(F)$ itself. For example, when $F = \mathbb{F}_q$ or $\mathbb{C}$, there is a splitting $\mathrm{SL}_2(F) \to \mathrm{Mp}_2(F)$ and so $[\rho_\psi]$ lifts to an honest representation of $\mathrm{SL}_2(F)$. (Reason: $H_2(\mathrm{SL}_2(\mathbb{F}_q), \mathbb{Z}) = 0$ and every central extension of a complex semisimple Lie algebra splits.)

Summarizing, the recipe for constructing a Weil representation is:

(A) Define a Heisenberg group H together with a center-fixing action of SL_2.
(B) Establish a Stone–von Neumann theorem about H.
(C) Fix a central character ψ and construct the projective Weil representation $[\rho_\psi]$.
(D) Lift $[\rho_\psi]$ to a representation ρ_ψ on Mp_2.

This recipe generalizes to many contexts. For instance, we could take the Heisenberg group to be a central F-extension of $F^{\oplus 2n}$ and consider a center-fixing action of $\mathrm{Sp}_{2n}(F)$. From this, we can construct Weil representations of $\mathrm{Mp}_{2n}(F)$. In the next section, we describe how to construct Weil representations attached to lattices.

3. Lattices and the Weil representation of $\mathrm{SL}_2(\mathbb{Z})$

Let L be an even lattice of finite rank and let L' be its dual. Let $\langle \cdot, \cdot \rangle$ denote the symmetric bilinear form on L and let $q(x) := \frac{1}{2}\langle x, x\rangle$ be the associated quadratic form. Let N be the smallest integer such that $Nq(\gamma) \in \mathbb{Z}$ for all $\gamma \in L'$.

Recall that the Weil representation of $\mathrm{SL}_2(\mathbb{Z})$ defined in [Br] is a representation on the group algebra $\mathbb{C}[L'/L]$ of the discriminant group L'/L (a finite abelian group). We will now describe a set up wherein we can run (A) through (D) to obtain a representation of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{C}[L'/L]$.

Let the Heisenberg group H associated to L be the group determined by the short exact sequence

$$0 \to \mathbb{R}/\mathbb{Z} \to H \to (L'/L) \otimes \mathbb{Z}^{\oplus 2} \to 0$$

together with the 2-cocycle

$$f \colon Q \times Q \to \mathbb{R}/\mathbb{Z}, \qquad ((a_1, b_1), (a_2, b_2)) \mapsto \det \left(\begin{smallmatrix} a_1 & a_2 \\ b_1 & b_2 \end{smallmatrix}\right) = \langle a_1, b_2 \rangle - \langle a_2, b_1 \rangle,$$

where $Q = (L'/L) \otimes \mathbb{Z}^{\oplus 2}$. We have a natural action of $\mathrm{SL}_2(\mathbb{Z})$ on $(L'/L) \otimes \mathbb{Z}^{\oplus 2}$. This action factors through an action of $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ and furthermore, since the action preserves f, lifts to a center-preserving action on H.

The Stone–von Neumann theorem holds for H: for every nontrivial character $\psi\colon \mathbb{R}/\mathbb{Z} \to \mathbb{C}^1$, there exists a unique irreducible representation π_ψ of H with central character ψ.

Now fix a nontrivial ψ. By replacing $\mathbb{F}_p$ with L'/L, the construction given in Remark 1 gives us a realization π_ψ on $\mathbb{C}[L'/L]$. We therefore obtain a projective Weil representation of $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ on $\mathbb{C}[L'/L]$. In general, this lifts to a representation on a double cover $\mathrm{Mp}_2(\mathbb{Z}/N\mathbb{Z})$ of $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ which then pulls back to a representation ρ_ψ on $\mathrm{Mp}_2(\mathbb{Z})$.

Remark 2. If L has even rank, then in fact the Weil representation can be realized on $\mathrm{SL}_2(\mathbb{Z})$. If L has odd rank, there is no splitting $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{Mp}_2(\mathbb{Z})$. ◊

With Remark 1 in mind, the representation ρ_L in [Br] is ρ_ψ with $\psi\colon \mathbb{R}/\mathbb{Z} \to \mathbb{C}^1$ chosen to be $\psi(x) = e^{2\pi i x}$.

4. Vector valued modular forms for $\mathrm{SL}_2(\mathbb{Z})$

In Igor's Lecture 2 notes, there is a description of vector-valued modular forms in terms of sections of vector bundles on $\mathbb{H}$. Here we give a brief review of the approach taken in [Br]. More or less, a vector-valued modular form with respect to an even lattice L is a function $\mathbb{H} \to \mathbb{C}[L'/L]$ that is stable, up to an automorphic factor, under the Weil representation ρ_L.

We have an explicit description of $\mathrm{Mp}_2(\mathbb{Z})$:

$$\mathrm{Mp}_2(\mathbb{Z}) := \{(M, \phi) : M \in \mathrm{SL}_2(\mathbb{Z}),\ \phi\colon \mathbb{H} \to \mathbb{C} \text{ holomorphic s.t. } \phi(\tau)^2 = c\tau + d\}.$$

For each $(M, \phi) \in \mathrm{Mp}_2(\mathbb{Z})$ and $k \in \frac{1}{2}\mathbb{Z}$, we define an operator on the space of functions $F\colon \mathbb{H} \to \mathbb{C}[L'/L]$ via:

$$\left(F|_k^*(M, \phi)\right)(\tau) := \phi(\tau)^{-2k} \rho_L^*(M, \phi)^{-1} F(M\tau).$$

Here, ρ_L^* is the dual representation. (Question: Why not have $\rho_L(M, \phi)$ in place of $\rho_L^*(M, \phi)^{-1}$?)

Definition 2. A holomorphic *modular form of weight k* (with respect to ρ_L^*) is a function $F\colon \mathbb{H} \to \mathbb{C}[L'/L]$ satisfying

(i) $F|_k^*(M, \phi) = F$ for all $(M, \phi) \in \mathrm{Mp}_2(\mathbb{Z})$
(ii) F is holomorphic on $\mathbb{H}$
(iii) F is holomorphic at infinity

Note that (iii) gives us a Fourier expansion

$$F(\tau) = \sum_{\gamma \in L'/L} \sum_{n + q(\gamma) \in \mathbb{Z}} a(\gamma, n) e^{2\pi i n \tau} \cdot \gamma.$$

Remark 3. We will sometimes relax (ii) to "meromorphic." ◊

Remark 4. Note that if L is unimodular, then it is self-dual so its discriminant is trivial. In this setting, the Heisenberg group degenerates to its center and we take $\rho_L = 1$. Then the above definition of a $\mathbb{C}[L'/L]$-valued modular form becomes the standard definition of a classical $\mathbb{C}$-valued modular form. ♢

5. Classical modular forms for $\Gamma_0(p)$

We briefly discuss the relation between vector valued modular forms for $\mathrm{Mp}_2(\mathbb{Z})$ and classical modular forms on the congruence subgroup $\Gamma_0(p) \subset \mathrm{SL}_2(\mathbb{Z})$ consisting of upper-triangular matrices modulo p. Reference: [BB].

Let $F = \mathbb{Q}(\sqrt{p})$ with $p \equiv 1 \pmod 4$ and consider the lattice $L = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathcal{O}_F$ together with the quadratic form $q(a, b, \gamma) = ab - \mathrm{Nm}_{F/\mathbb{Q}}(\gamma)$. We have $L'/L \cong \mathbb{Z}/p\mathbb{Z}$ and the induced quadratic form is $q(\gamma) = -\frac{1}{p}\gamma^2 \in \mathbb{Q}/\mathbb{Z}$. Recall that this was considered at length in Lecture 1 (Brandon's talk). Consider the vector-valued modular form F with Fourier expansion

$$F(\tau) = \sum_{\gamma \in L'/L} \sum_{n + q(\gamma) \in \mathbb{Z}} a(\gamma, n) q^n \cdot \gamma = \sum_{\gamma \in L'/L} F_\gamma(\tau) \cdot \gamma.$$

As usual, we take $q = e^{2\pi i \tau}$. Define a function $f(F)\colon \mathbb{H} \to \mathbb{C}$ via

$$f(F)(\tau) := \frac{1}{2} \sum_{\gamma \in L'/L} F_\gamma(p\tau).$$

One can check that $f(F)$ has the following properties:

(a) It is a modular form for $\Gamma_0(p)$ of nebentypus χ_p, the quadratic character associated to $F/\mathbb{Q}$.

(b) The coefficient of q^n is $\frac{1}{2} \sum_{\substack{\gamma \in L'/L \\ p \cdot q(\gamma) \equiv n \pmod p}} a(\gamma, n)$. In particular, if n is not a square modulo p, then the coefficient of q^n in $f(F)$ is 0; i.e. if $\chi_p(n) = -1$, then the coefficient of q^n is 0.

This means that $f(F)$ satisfies the "plus condition" of Brandon's talk (Lecture 1).

More precisely: Let $M(p, \chi_p)$ be the space of modular forms for $\Gamma_0(p)$ of nebentypus χ_p. We have a direct sum decomposition

$$M(p, \chi_p) = M^+(p, \chi_p) \oplus M_k^-(p, \chi_p)$$

where

$$M^\pm(p, \chi_p) := \{f = \sum a(n) q^n \in M(p, \chi_p) : a(n) = 0 \text{ whenever } \chi_p(n) = \mp 1.\}.$$

Theorem 3 (Bruinier–Bundschuh [BB])**.** *The assignment $F \mapsto f(F)$ gives a bijection*

$$\{\textit{vector valued modular form wrt } \rho_L\} \to M^+(p, \chi_p).$$

Remark 5. From this point onwards, we will mainly work with vector-valued modular forms. The reason for this choice is that we want to always think of forms transforming appropriately under all of $\mathrm{SL}_2(\mathbb{Z})$, rather than keep track of congruence subgroups and make modifications to the construction of lifts depending on this. Thank you to Kartik for pointing this out. ♢

References

[Br] Brunier, Jan Hendrik. *Borcherds products on $O(2, l)$ and Chern classes of Heegner divisors*. 2000.

[BB] Bruinier and Bundschuh. *On Borcherds products associated with lattices of prime discriminant.* arXiv:0309178.

[C] Chan, Charlotte. *The Weil representation*. http://www-personal.umich.edu/~charchan/TheWeilRepresentation.pdf