Theta lifts and currents on Shimura varieties

Lecture 9 • Speaker: Luis Garcia • Notes by: Charlotte Chan • March 25, 2016

This is a write-up of a very nice talk given by Luis Garcia. The abstract of the talk is:

We will review work of Borcherds and Bruinier using regularised theta lifts for the pair $(\mathrm{SL}_2, \mathrm{O}(V))$ to construct Green currents for special divisors on some Shimura varieties. Then we will explain how to construct other interesting currents on X using the dual pair $(\mathrm{Sp}_4, \mathrm{O}(V))$. We will show that one obtains currents in the image of the regulator map of a certain motivic complex of X. Finally, we will describe how an argument using the Siegel-Weil formula allows to relate the values of these currents to the product of a special value of an L-function and a period on a certain subgroup of Sp_4.

Any errors in the notes are introduced by me in the preparation of these notes.

Contents

1. Motivation

Let X be a smooth projective surface over $\mathbb{Q}$ and consider the higher Chow group

$$\mathrm{CH}^2(X,1) := Z^2(X,1)/B^2(X,1),$$

where

$$\begin{aligned} Z^2(X,1) &:= \big\{ \textstyle\sum n_i(C_i, f_i) : n_i \in \mathbb{Q},\, C_i \subset X,\, \dim C_i = 1, \\ &\qquad\qquad f_i \in \mathbb{Q}(C_i)^\times,\, \textstyle\sum \mathrm{div}(f_i) = 0 \in Z^2(X) \big\}, \\ B^2(X,1) &:= \{\text{tame symbols}\}. \end{aligned}$$

Example 1. We give an example of an element of $Z^2(X,1)$. Let C be a curve and consider $X = C \times C$. Pick $P, Q \in C$ satisfying $n(P-Q) = \mathrm{div}(f)$ for some $n \in \mathbb{Q}$ and $f \in \mathbb{Q}(C)^\times$. We can construct a cycle

$$\Delta_{P,Q} = (\Delta, f) - (C \times \{P\}, f) - (\{Q\} \times C, f).$$

From this, it is easy to see that $\sum \mathrm{div}(f_i) = 0$, and thus the above cycle is an element in $Z^2(X,1)$. ◊

We can define a subgroup

$$\mathrm{CH}^2(X,1)_{\mathbb{Z}} := \mathrm{Im}(\mathrm{CH}^2(\mathcal{X},1) \to \mathrm{CH}^2(X,1)) \subset \mathrm{CH}^2(X,1),$$

where $\mathcal{X}$ is a "nice" model of X over $\operatorname{Spec} \mathbb{Z}$.

There is a very interesting map called the *regulator map*:

$$\begin{aligned} \mathrm{reg}\colon \mathrm{CH}^2(X,1) &\to (H^{1,1}(X_{\mathbb{C}})^+)^\vee \\ \sum n_i(C_i,f_i) &\mapsto \left(\alpha \mapsto \tfrac{1}{2\pi i}\sum n_i \int_{C_i} \alpha \cdot \log|f_i|\right). \end{aligned}$$

Note that this regulator map is well-defined because of the imposed $\sum \mathrm{div}(f_i) = 0$ condition. Here, $H^{1,1}(X_{\mathbb{C}})$ denotes the subgroup of $H^2(X_{\mathbb{C}}, \mathbb{C})$ consisting of harmonic forms of type $(1,1)$.

Recall also the usual cycle class map:

$$\begin{aligned} d\colon Z^1(X)/\sim_{\mathrm{hom}} &\to (H^{1,1}(X_{\mathbb{C}})^+)^\vee, \\ \sum n_i C_i &\mapsto \left(\alpha \mapsto \tfrac{1}{2\pi i}\sum n_i \int_{C_i} \alpha\right) \end{aligned}$$

Now define

$$r := \mathrm{reg}|_{\mathrm{CH}^2(X,1)_{\mathbb{Z}}} \oplus d\colon \mathrm{CH}^2(X,1)_{\mathbb{Z}} \oplus Z^1(X)/\sim_{\mathrm{hom}} \to (H^{1,1}(X_{\mathbb{C}})^+)^\vee.$$

Conjecture 2 (Beilinson–Tate)**.** *(1)* $\mathrm{Im}(r)$ *is a* $\mathbb{Q}$*-lattice in* $(H^{1,1}(X_{\mathbb{C}})^+)^\vee$

(2) $\det(\mathrm{Im}(r)) = L^*(H^2(X)(1),0)\cdot B$*, where* B *is a* $\mathbb{Q}$*-structure coming from differential forms "defined over* $\mathbb{Q}$*"*

(3) $-\mathrm{ord}\, L(H^2(X)(1),1) = \dim Z^1(X)/\sim_{hom}$

The notation L^* denotes the first nonzero term in the Taylor series.

Very little is known about this conjecture. A case when this is known is the following. Let $C = X_0(N)$ and take P, Q to be cusps of $X_0(N)$.

Theorem 3 (Manin–Drinfeld)**.** $n(P-Q) = \mathrm{div}(f_{P,Q})$ *for some* $n \in \mathbb{N}$ *and* $f_{P,Q} \in \mathbb{Q}(X_0(N))^\times$.

Now we can consider the surface $X = C\times C$ and the divisor $\Delta_{P,Q}$ as in Example 1. To see if this is nontrivial, we can try to compute its image under the regulator map.

The interesting part of $H^{1,1}(X_{\mathbb{C}})^+$ is the subgroup coming from cusp forms:

$$\bigoplus_{f_1,f_2\in\mathcal{S}_2(\Gamma_0(N))} H^{1,1}(X_{\mathbb{C}})^+(f_1,f_2) \subset H^{1,1}(X_{\mathbb{C}})^+,$$

where each summand is generated by the $(1,1)$-form $\omega_{1,1} := f_1(z_1)dz_1 \wedge \overline{f_2(z_2)}\, d\bar{z}_2$.

Say f_1, f_2 are normalized newforms. Then there are two cases, and these two cases behave in very different ways:

(1) $f_1 = f_2$. Then for $\Delta \in Z^1(X)/\sim_{\mathrm{hom}}$,

$$(\Delta, \omega_{1,2}) := \int_\Delta \omega_{1,2} = * \cdot (f_1, f_1) > 0,$$

where the righthand form is the Petersson inner product. Also, $L(f_1\times f_2, s)$ has a pole at $s=1$. (Can think of this L-function as the Rankin–Selberg L-function.)

(2) $f_1 \neq f_2$. Then the trick in (1) doesn't work since $(f_1,f_2) = 0$, so pairings like $(\Delta,\omega_{1,2})$ won't tell us when Δ is nontrivial.

We instead appeal to the Beilinson–Tate conjecture. The L function $L(f_1\times f_2, s)$ has a pole at $s=1$, so the conjecture predicts that there exists some cycle $Z \in \mathrm{CH}^2(X,1)_{\mathbb{Z}}$ such that $\mathrm{reg}(Z)(\omega_{1,2}) := (Z, \omega_{1,2}) \neq 0$. When we take $Z = \Delta_{P,Q}$ and

$$\mathrm{reg}(\Delta_{P,Q})(\omega_{1,2}) = \int_{X_0(N)} f_1(z)\overline{f_2(z)}\log|f_{P,Q}(z)|.$$

It turns out that we can compute the righthand side!! The reason is that in the case $X = C \times C$, we have the Kronecker limit formula, which says:

$$\log|f_{P,Q}| = E_{P,Q}(z,s)|_{s=0},$$

where $E_{P,Q}(z,s)$ is an Eisenstein series. Then by a Rankin–Selberg-type argument, we have

$$\mathrm{reg}(\Delta_{P,Q}(\omega_{1,2})) = \int_{X_0(N)} f_1(z)\overline{f_2(z)}E_{P,Q}(z,s)|_{s=0} = * \cdot L^*(f_1 \times f_2, 0),$$

where the $*$ factor is an easy constant. This is one of the examples Beilinson gives in the original paper to support the Conjecture.

The problem with this approach is that this essentially only works for $X = X_0(N) \times X_0(N)$ and a handful of other examples as there is no known analogue of the Kronecker limit formula in general. This leads to the following question:

> Can one use Borcherds lifts (wherein by construction one gets an expression of the form $\log|-|$) to compute the regulator map in other settings?

2. Theta correspondences for $(\mathrm{Sp}, \mathrm{O})$ pairs

Let F be a totally real field and denote by F_v its v-adic completion. For simplicity, we set $F = \mathbb{Q}$.

We first recall the construction of theta correspondences for dual reductive pairs of the form $(\mathrm{Sp}, \mathrm{O})$. This was discussed in Lectures 2, 3, and 6. The following is a slightly different exposition.

Let (V, Q) be a quadratic vector space over Q of even dimension. For $n \in \mathbb{N}$, denote by $\mathcal{S}(V(\mathbb{A})^n) = \bigotimes' \mathcal{S}(V(\mathbb{Q}_v)^n)$ the corresponding Schwartz space. Let O_V be the orthogonal group of (V, Q). We have an action of $O_V(\mathbb{A})$ on $\mathcal{S}(V(\mathbb{A})^n)$ via

$$\omega(h)\varphi(v) := \varphi(h^{-1}v)$$

that extends to an action of $\mathrm{Sp}_{2n}(\mathbb{A}) \times O_V(\mathbb{A})$ on $\mathcal{S}(V(\mathbb{A})^n)$. These are the actions induced by pulling back the Weil representation of $\mathrm{Mp}(V(\mathbb{A})^{2n})$ on $\mathcal{S}(V(\mathbb{A})^n)$ to the subgroup $\mathrm{Sp}_{2n} \times O_V$.

We have a theta functional that intertwines the $\mathrm{Sp}_{2n}(\mathbb{A}) \times O_V(\mathbb{A})$ actions:

$$\begin{aligned} \theta\colon \mathcal{S}(V(\mathbb{A})^n) &\to \mathcal{C}^\infty(\mathrm{Sp}_{2n}\mathbb{Q}\backslash \mathrm{Sp}_{2n}(\mathbb{A}) \times O_V(\mathbb{Q})\backslash O_V(\mathbb{A})) \\ \varphi &\mapsto \Big((g,h) \mapsto \theta(g,h;\varphi) := \textstyle\sum_{v\in V(\mathbb{Q})^n} \omega(g,h)\cdot\varphi(v)\Big). \end{aligned}$$

By integrating against θ, we obtain a map

$$\mathcal{A}_0(\mathrm{Sp}_{2n}) \otimes \mathcal{S}(V(\mathbb{A})^n) \to \mathcal{A}(O_V), \qquad f \otimes \varphi \mapsto \theta(f;\varphi),$$

where $\mathcal{A}_0(\mathrm{Sp}_{2n})$ denotes the space of cusp forms on $\mathrm{Sp}_{2n}(\mathbb{A})$ and $\mathcal{A}(O_V)$ denotes the space of automorphic forms on $O_V(\mathbb{A})$. (Note that $\theta(f;\varphi)$ might not be a cusp form! We will come back to this comment later.)

Now given an automorphic representation $\pi \subset \mathcal{A}_0(\mathrm{Sp}_{2n})$, we can define its theta lift

$$\theta(\pi) := \langle \theta(f,\varphi) : f \in \pi,\ \varphi \in \mathcal{S}(V(\mathbb{A})^n)\rangle.$$

By appealing to the Rallis inner product formula, one can prove the following basic global result: If $\theta(\pi)$ is cuspidal, then for $f = \otimes_v f_v$, $\varphi = \otimes_v \varphi_v$, we have

$$||\theta(f;\varphi)||^2 = * \cdot ||f||^2 \cdot L^S(\pi, \mathrm{std}, s_0) \cdot \prod_v Z_v(f_v, \varphi_v).$$

Now we specialize to the situation where we take (V, Q) to be the 4-dimensional quadratic space defined by $V = B$ an indefinite quaternion algebra over $\mathbb{Q}$ with quadratic form given by the reduced norm on B. (For F totally real, we take B to be indefinite at exactly one infinite place so that X_B is a curve.) Then we have

$$\mathrm{GSO}(V) \cong (B^\times \times B^\times)/\mathbb{Q}^\times.$$

This isomorphism can be realized by $B^\times$ acting on $V = B$ by left and right multiplication. It follows then that an automorphic representation π of $\mathrm{GSO}(V)$ is of the form $\pi = \pi_1 \boxtimes \pi_2$, where π_i is an automorphic representation of $B(\mathbb{A})^\times$ with compatible central characters (i.e., $\omega(\pi_1) \cdot \omega(\pi_2) = 1$).

Let X_B be a full level Shimura curve and consider the surface $X = X_B \times X_B$. Pick $f_i \in \pi_i^K$, where K is a maximal compact subgroup of $B(\mathbb{A})^\times$. Then $\pi = \pi_1 \boxtimes \pi_2$ has two different behaviors:

(1) We have $\pi_2 \cong \pi_1^\vee$ if and only if $L(\pi_1 \otimes \pi_2, s)$ has a pole at $s = 1$. This happens if and only if $\pi = \theta(\Pi)$ for some $\Pi \subset \mathcal{A}_0(\mathrm{GL}_2)$. In other words, in this situation, π can be realized the theta lift of a cuspidal representation of GSp_2.

(2) Conversely, we have $\pi_2 \not\cong \pi_1^\vee$ if and only if $L(\pi_1 \otimes \pi_2, s)$ has no pole at $s = 1$. This happens if and only if $\pi = \theta(\Pi)$, where $\Pi \subset \mathcal{A}_0(\mathrm{GSp}_4)$. In other words, in this situation, π can be realized as the theta lift of a cuspidal representation of GSp_4.

Remark 1. It is natural to ask what happens if we consider the theta correspondence for $(\mathrm{Sp}_{2n}, \mathrm{O}(4))$, where $n > 2$. On the level of representations (i.e. by considering unique irreducible subquotient of the π-isotypic component of the Weil representation, where π is an irreducible representation of one of the groups), we obtain a correspondence

$$\{\text{reps coming from Eis series of } \mathrm{Sp}_{2n}\} \longleftrightarrow \{\text{cuspidal reps of } \mathrm{O}(4)\}.$$

We can only define the theta lift for cusp forms (otherwise the theta integral may not converge), and so while we can define the $\mathrm{O}(4) \rightsquigarrow \mathrm{Sp}_{2n}$ on the level of forms to obtain the above correspondence, we cannot define $\mathrm{Sp}_{2n} \rightsquigarrow \mathrm{O}(4)$ on the level of forms. Along the same lines, the lift of a cusp form on Sp_{2n} to $\mathrm{O}(4)$ vanishes. $\Diamond$

With (2) in mind, our goal is to construct currents related to Beilinson's regulator currents by lifting forms of GSp_4. The hope is that this will allow us to test when cycles are nonzero in $\mathrm{CH}^2(X, 1)$.

3. Cycles and currents

Let (V, Q) be a quadratic space of signature $(N, 2)$ and let $G = O(V, Q)$. We have an associated Hermitian symmetric space $\mathbb{D} = O(N, 2)/(O(N) \times O(2))$ and we can realize this space as

$$\mathbb{D} = \{Z \subset V_\mathbb{R} : \dim Z = 2,\ Q|_Z \text{ is negative definite, } Z \text{ oriented}\}.$$

(In Lecture 7, we called this $\mathrm{Gr}(L)$, where L is an integral lattice in $V_\mathbb{R}$.) For $v \in V(\mathbb{Q})$ with $Q(v) > 0$, the subspace

$$\mathbb{D}_v := \{z \in \mathbb{D} : v \perp z\} \subset \mathbb{D}$$

is an analytic divisor of $\mathbb{D}$.

Now let L be a lattice in $V(\mathbb{Q})$ and let $\Gamma := \Gamma_L := \mathrm{stab}_G(L)$. This is a discrete subgroup of $O(V_\mathbb{R})$. We then have a diagram

$$\begin{array}{ccc} \mathbb{D}_v & \hookrightarrow & \mathbb{D} \\ \downarrow & & \downarrow \\ X(v)_L := \Gamma_v\backslash\mathbb{D}_v & \hookrightarrow & \Gamma\backslash\mathbb{D} =: X_L \end{array}$$

Example 4. Here are some possible outcomes for X_L. If $N = 1$, we can get X_L a Shimura curve and $X(v)_L$ a CM point. If $N = 2$, we can get $X_L = X_B \times X_B$ and $X(v)_L$ is a special divisor (coming from, for example, Hecke correspondences). $\Diamond$

Define certain cycles:

$$Z(v)_L := [X(v)_L] \in Z^1(X_L),$$
$$Z(n)_L := \textstyle\sum_{\{v\in L,\, Q(v)=n\}/\Gamma} X(v)_r \in Z^1(X_L).$$

Now consider a pair of vectors $v, w \in V(\mathbb{Q})$ spanning a positive definite plane. Then we have a diagram

$$\begin{array}{ccccc} \mathbb{D}_{v,w} := \mathbb{D}_v \cap \mathbb{D}_w & \hookrightarrow & \mathbb{D}_v & \hookrightarrow & \mathbb{D} \\ \downarrow & & \downarrow & & \downarrow \\ X(v,w)_L := \Gamma_{v,w}\backslash\mathbb{D}_{v,w} & \hookrightarrow & X(v)_L := \Gamma_v\backslash\mathbb{D}_v & \hookrightarrow & \Gamma\backslash\mathbb{D} =: X_L \end{array}$$

Each containment in the bottom row is of codimension 1. We can define

$$Z(v,w)_L := [X(v,w)_L] \in \mathrm{Div}(X(v)_L).$$

In 2009, Bruinier constructed Green functions on $G(v,w)_L$ for $Z(v,w)_L$ in $X(v)_L$ with log singularities at $Z(v,w)_L$. We would like to understand the current

$$[\Phi(v,w)_L] := G(v,w)_L \cdot \delta_{X(v)_L},$$

which is defined to be the linear functional on differential forms $\alpha \in A_c^{N-1,N-1}(X_L)$ given by

$$[\Phi(v,w)_L](\alpha) = \int_{X(v)_L} G(v,w)_L \cdot \alpha.$$

The upshot of Bruinier's results is that we obtain many currents of the form

$$\sum \log|\psi_i| \cdot \delta_{X(v_i)_L}$$

as $\mathbb{Q}$-linear combinations of $[\Phi(v,w)_L]$.

If for $T \in \mathrm{Sym}_2(\mathbb{Z})_{>0}$ we define

$$[\Phi(T)_L] = \sum_{\{Q(v,w)=T,\, v,w\in L\}/\Gamma} [\Phi(v,w)_L],$$

we can obtain $[\Phi(T)_L] = G_{\mathrm{CM}(\mathbb{Z}[-n])} \cdot \delta_\Delta$ for certain choices of T, L.

Theorem 5 (Garcia)**.** *There exists a $(1,1)$-form $\Phi(T,s)_L$ defined over a Zariski open $U \subset X_B \times X_B$ such that*

(1) $\Phi(T,s)_L$ is locally integrable (and so $[\Phi(T,s)_L]$ is defineable.)
(2) $[\Phi(T,s)_L]$ admits a meromorphic continuation to $s \in \mathbb{C}$

(3) *The constant term at* $s = s_0$ *of* $[\Phi(T,s)_L]$ *is* $[\Phi(T)_L]$ *modulo* $\partial + \bar{\partial}$.

(4) $\Phi(T,s)$ *arises as a kind of regularized theta lift* $(\mathcal{M}_T(s), f)^{\mathrm{reg}}$, *where replace the theta kernel with a function* $\mathcal{M}_T(s)\colon A(\mathbb{R})^\circ \times N(\mathbb{Z})\backslash N(\mathbb{R}) \to \mathbb{C}$, *where*

$$A := \left\{ \begin{pmatrix} a & & & \\ & b & & \\ & & a^{-1} & \\ & & & b^{-1} \end{pmatrix} : a, b \in \mathbb{R}_{>0} \right\}, \qquad N := \left\{ \begin{pmatrix} 1_2 & X \\ & 1_2 \end{pmatrix} : X \in \mathrm{Sym}_2 \mathbb{R} \right\}.$$

So what is the upshot? The point is that we can use this to evaluate currents: If we specialize to the case $X_L = X_B \times X_B$ (where X_B is the full level Shimura variety), we have

$$[\Phi(T)](\omega_{1,2}) = * \cdot L^*(f_1 \times f_2, 0) \cdot \mathrm{CT}\,|_{s=s_0} (\mathcal{M}_T(s), f)^{\mathrm{reg}},$$

where CT denotes the constant term.