The regularized theta lift

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These are rough notes of a lecture in which we describe the regularized theta lift used to define Borcherds products. We try to explain the parallelism between the theta lift in this context and the classical theta lift on $(\mathrm{Sp}, \mathrm{O})$ pairs (following [Ku]).

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1. Theta lifts for $(\mathrm{Sp}, \mathrm{O})$ pairs

Recall the following general set-up, noting that some sentences in this part may need further adjectives. Let A be an abelian group and let V be an abelian group with an A-valued symplectic form. Recall that for a Heisenberg group defined by the short exact sequence

$$0 \to A \to H \to V \to 0$$

together with the 2-cocycle $V \times V \to A$ given by the form on V, we can define a center-fixing action of $\mathrm{Sp}(V)$ on H. The irreducible representations of H with nontrivial central character can be realized on the space $\mathcal{S}(X)$ of Schwartz functions $f : X \to \mathbb{C}$, where we choose X, Y Lagrangian (i.e., maximal isotropic) subspaces of V with $V = X + Y$. We have an induced action of $\mathrm{Sp}(V)$ on $\mathcal{S}(X)$ that is well-defined up to scaling. We therefore obtain a projective representation of $\mathrm{Sp}(V)$ on the $\mathbb{C}$-vector space $\mathcal{S}(X)$. As discussed in Lecture 3, in some contexts, this lifts to an honest representation of $\mathrm{Sp}(V)$; in other contexts, we only get a lifting to a representation of the double cover $\mathrm{Mp}(V)$, the metaplectic group. The genuine representation is called the Weil representation and we denote it by ω_ψ.

Recall the following two examples we discussed in Lecture 3.

Example 1. Let F be a field and let $A = F$, $V = F^{\oplus 2}$ with symplectic form

$$\langle (a_1, b_1), (a_2, b_2) \rangle = a_1 b_2 - a_2 b_1.$$

Then the Weil representation ω_ψ is given by an action of the metaplectic group $\mathrm{Mp}_2(F)$ on the space $\mathcal{S}(F)$ of $\mathbb{C}$-valued functions on F. ◊

Example 2. Let L be an even lattice with non-degenerate bilinear form $(\cdot, \cdot)$ and let L' denote its dual. We have an induced $\mathbb{Q}/\mathbb{Z}$-valued bilinear form on L'/L. We can take $A = \mathbb{R}/\mathbb{Z}$ and $V = \mathbb{Z}^{\oplus 2} \otimes L'/L$ with symplectic form

$$\langle (a_1, b_1), (a_2, b_2) \rangle = a_1 b_2 - a_2 b_1,$$

where the multiplication $a_i b_j$ is shorthand for the pairing coming from L'/L. We saw earlier that the Weil representation ω_ψ of $\mathrm{Mp}_2 \mathbb{Z}$ on $\mathbb{C}[L'/L]$ is the one coming from $\mathrm{Mp}_2 \mathbb{Z}$ acting on $\mathcal{S}(L'/L) = \mathbb{C}[L'/L]$. ◊

Example 2 is somewhat misleading. The Weil representation coming from

$$0 \to \mathbb{R}/\mathbb{Z} \to H \to \mathbb{Z}^{\oplus 2} \otimes L'/L \to 0$$

is a representation of $\mathrm{Mp}(\mathbb{Z}^{\oplus 2} \otimes L'/L)$, and the ω_ψ in Example 2 is the pullback along the natural map

$$\mathrm{Mp}_2 \mathbb{Z} \times \widetilde{\mathrm{O}}(L'/L) \to \mathrm{Mp}(\mathbb{Z}^{\oplus 2} \otimes L'/L).$$

This construction also holds in the context of Example 1. For instance, we can take V to be a symplectic space over a field F and take W to be an orthogonal space over F. We can choose a polarization $V = X + Y$ (i.e., X, Y isotropic) and consider the tensor product $\mathbb{W} = V \otimes_F W$. Endowing $\mathbb{W}$ with the product of the symplectic form on V and the bilinear form on W, we see that $\mathbb{W}$ is a symplectic space and we have a natural map

$$\mathrm{Sp}(V) \times \mathrm{O}(W) \to \mathrm{Sp}(\mathbb{V}).$$

In this setting, we in fact have a map

$$\mathrm{Sp}(V) \times \mathrm{O}(W) \to \mathrm{Mp}(\mathbb{W}),$$

and hence we may consider ω_ψ as a representation of $\mathrm{Sp}(V) \times \mathrm{O}(W)$. A natural representation-theoretic question is: given an irreducible representation π of $\mathrm{Sp}(V)$, what can we say about the $\mathrm{O}(V)$-representation $\omega_\psi[\pi]$? It turns out that $\omega_\psi[\pi]$ has a unique irreducible quotient ρ_π. The correspondence $\pi \mapsto \rho_\pi$ is an instance of the Howe correspondence.

Remark 1. We can play this game for any subgroups $G, H \subset \mathrm{Mp}$ that centralize each other. There is a lot of work in this direction, for example, Waldspurger, Howe, Gan, ... ♢

However, we'd like more than just a correspondence on the level of representations. Suppose f is an automorphic form of $\mathrm{Sp}(V_\mathbb{A})$. Then f determines an automorphic representation π of $\mathrm{Sp}(V_\mathbb{A})$ and the Howe correspondence lifts π to ρ_π. What we'd like to do is to define a way to lift f to a form $\Phi(f) \in \rho_\pi$. This is called the theta correspondence and we usually denote $\Phi(f)$ by something like $\theta(f)$, but in these notes, we write $\Phi(f)$ to match with the notation in [Br] and [B2] for the regularized theta lift and to stress the parallelism between the constructions.

For ease of notation, let V be a 2-dimensional symplectic space over $\mathbb{Q}$ and let W be a $\mathbb{Q}$-vector space equipped with a non-degenerate bilinear form. Every line in V is isotropic, so by picking a line Y in V, we may identify $Y \otimes W$ with W and view the representation space of ω_ψ as $\mathcal{S}(W_\mathbb{A})$. We have

$$\mathrm{Sp}(V_\mathbb{A}) \times \mathrm{O}(W_\mathbb{A}) \to \mathrm{Mp}(\mathbb{W}_\mathbb{A}^{\oplus 2}).$$

We define the Siegel theta function attached to $\varphi \in \mathcal{S}(W_\mathbb{A})$ to be

$$\Theta(g, h; \varphi) := \sum_{x \in W} (\omega_\psi(g, h) \cdot \varphi)(x), \qquad \text{where } (g, h) \in \mathrm{SL}_2(\mathbb{A}) \times \mathrm{O}(W_\mathbb{A}).$$

Then for any automorphic form $f\colon \mathrm{SL}_2(\mathbb{Q}) \backslash \mathrm{SL}_2(\mathbb{A}) \to \mathbb{C}$, the function

$$(\Phi_\varphi(f))(h) := \int_{\mathrm{SL}_2(\mathbb{Q}) \backslash \mathrm{SL}_2(\mathbb{A})} f(g) \cdot \Theta(g, h; \varphi) \cdot dg, \qquad \text{where } h \in \mathrm{O}(W_\mathbb{A})$$

defines an automorphic form of $\mathrm{O}(W_\mathbb{A})$.

2. Theta lifts for vector-valued modular forms of $\mathrm{SL}_2(\mathbb{Z})$

We try to explain here that the regularized theta lift for $(\mathrm{SL}_2(\mathbb{Z}), \mathrm{O}(L'/L))$ can be recovered from a set-up like the construction in Section 1. (See Section 1 of [Ku] for more details and fewer mistakes than this exposition on the relationship between lifts of automorphic forms and lifts of forms on $\mathrm{SL}_2(\mathbb{Z})$.) We will define a lift of a vector-valued modular form of $\mathrm{SL}_2(\mathbb{Z})$, following [Br].

Recall from Example 2 that the Weil representation ω_ψ pulls back to an action of $\mathrm{Mp}_2(\mathbb{Z}) \times \widetilde{\mathrm{O}}(L'/L)$ on $\mathcal{S}(L'/L) = \mathbb{C}[L'/L]$. Recall that if L has signature (b^+, b^-), then $\mathrm{Gr}(L)$ is the set of b^+-dimensional positive-definite subspaces of $L \otimes \mathbb{R}$. For $\gamma \in L'/L$ and $(\tau, v) \in \mathbb{H} \times \mathrm{Gr}(L)$, define

$$\theta_{L,\gamma}(\tau, v) := \sum_{\lambda \in \gamma + L} e(\tau \cdot q(\lambda_v) + \bar{\tau} \cdot q(\lambda_{v^\perp})),$$

where λ_v and $\lambda_{v^\perp}$ denote the orthogonal projections of λ onto v and $v^\perp$. We define the Siegel theta function in this setting to be the $\mathbb{C}[L'/L]$-valued function $\mathbb{H} \times \mathrm{Gr}(L)$ given by summing $\theta_{L,\gamma}$ over $\gamma \in L'/L$:

$$\Theta_L(\tau, v) := \sum_{\gamma \in L'/L} e_\gamma \cdot \theta_\gamma(\tau, v).$$

Here, $e_\gamma = e^{2\pi i \tau} \cdot \gamma \in \mathbb{C}[L'/L]$. It is well known that Θ_L is a real analytic function in $(\tau, v) \in \mathbb{H} \times \mathrm{Gr}(L)$.

Remark 2. Note that $\Theta(g, h; \varphi)$ in Section 1 had an obvious compatibility with the Weil representation. For $\Theta_L(\tau, v)$, however, such a compatibility does not follow formally from the definition. ◊

Theorem 3 (Thm 4.1 of [B2])**.** *Let $(M, \phi) \in \mathrm{Mp}_2(\mathbb{Z})$ and $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Recall that ϕ is a holomorphic function on $\mathbb{H}$ with $\phi(\tau)^2 = c\tau + d$. Then $\Theta_L(\tau, v)$ satisfies*

$$\Theta_L(M\tau, v) = \phi(\tau)^{b^+} \overline{\phi(\tau)}^{b^-} (\omega_\psi(M, \phi) \cdot \Theta_L)(\tau, v).$$

In particular, $\overline{\Theta_L}$ is a vector valued modular form of weight $(\frac{b^-}{2}, \frac{b^+}{2})$.

Let F be a vector-valued nearly holomorphic modular form of weight $k := 1 - \frac{b^-}{2}$. To define a lift $\Phi(F)$ analogous to the lift $\Phi_\varphi(f)$ of the previous section, we'd like to consider an integral like

$$\int_{\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}} \langle F(\tau), \Theta_L(\tau, v) \rangle \, \frac{dx\, dy}{y^2}.$$

However, an integral like this may diverge as $F(\tau)$ may blow up as $y \to \infty$.

Recall from Lectures 4 and 5 that the space of vector-valued nearly holomorphic forms of weight k are spanned by the non-holomorphic Poincaré series $F^L_{\beta,m}(\tau, s)$ at $s = 1 - \frac{k}{2}$. It is therefore enough to define lifts of $F^L_{\beta,m}(\tau, s)$.

3. Regularized theta lifts of non-holomorphic Poincaré series

We now set $b^+ = 2$. The constructions in this section can be generalized to b^+ arbitrary, and this is done in [B2].

For the Poincaré series $F^L_{\beta,m}(\tau,s)$, where $\tau\in\mathbb{H}$ and $s\in\mathbb{C}$, we define the regularized theta lift:

$$\Phi^L_{\beta,m}(v,s):=\lim_{u\to\infty}\int_{\mathcal{F}_u}\left\langle F^L_{\beta,m}(\tau,s),\Theta_L(\tau,v)\right\rangle\cdot y^{\frac{b^+}{2}}\cdot\frac{dx\,dy}{y^2},$$

where $\mathcal{F}_u:=\{\tau=x+iy\in\mathcal{F}:y\le u\}\subset\mathcal{F}:=\{\tau=x+iy:|x|\le\frac{1}{2},\,|\tau|\ge 1\}$.

It is difficult to compute integrals over $\mathcal{F}_u$, so we apply a standard trick to change the domain of integration to a rectangle. Suppose we had a weight k modular form $\Theta(\tau)$ and a weight $-k$ modular form $F(\tau)$. Suppose furthermore that we can write

$$\Theta(\tau)=\sum_{(c,d)=1}(c\tau+d)^k\cdot g\left(\tfrac{a\tau+b}{c\tau+d}\right)\tag{1}$$

for some function g. Then it follows formally that

$$\int_{\mathcal{F}}\Theta(\tau)\cdot F(\tau)\cdot\frac{dx\,dy}{y^2}=\int_{y>0}\int_{x\in\mathbb{R}/\mathbb{Z}}g(\tau)\cdot F(\tau)\cdot\frac{dx\,dy}{y^2}.$$

We can check that the weights of the components of the integrand sum to 0:

$$\Big\langle\underbrace{F^L_{\beta,m}(\tau,s)}_{(1-\frac{b^-}{2},0)},\underbrace{\Theta_L(\tau,v)}_{(\frac{b^-}{2},\frac{b^+}{2})}\Big\rangle\cdot\underbrace{y^{\frac{b^+}{2}}}_{(-\frac{b^+}{2},-\frac{b^+}{2})}.$$

We would like to make use of the change-of-domain trick to get explicit expressions for the Fourier coefficients of $\Phi^L_{\beta,m}$. These computations therefore reduce to two main steps:

1. Find an expression like (1) for $\Theta_L(\tau,v)$. (See Section 5 of [B2].)
2. Compute the resulting integral over $\int_{y>0}\int_{x\in\mathbb{R}/\mathbb{Z}}$. (See Section 7 of [B2].)

Remark 3. It seems to me that the most important part of the construction of Borcherds products are the explicit formulas for the regularized theta lift $\Phi^L_{\beta,m}$. ◊

It's natural to study the holomorphic/real analytic behavior of $\Phi^L_{\beta,m}(v,s)$. It turns out that $\Phi^L_{\beta,m}(v,s)$ is real analytic in $v\in\mathrm{Gr}(L)$ outside the union of some sub-Grassmannians and has a meromorphic continuation to $\Re(s)>1$ with a pole of order 1.

Define

$$H(\beta,m):=\bigcup_{\substack{\lambda\in L'\\ \lambda+L=\beta\\ q(\lambda)=m}}\lambda^\perp\subset\mathrm{Gr}(L),$$

where $\lambda^\perp\subset\mathrm{Gr}(L)$ consists of all b^+-dimensional positive-definite subspaces orthogonal to λ.

Proposition 4 (Prop 2.7 of [Br])**.** *Let $v\in\mathrm{Gr}(L)\setminus H(\beta,m)$ and assume $k:=1-\frac{b^-}{2}<0$.*

(a) The regularized theta integral converges for $\Re(s)>1-\frac{k}{2}$ and defines a holomorphic function in s in this region.

(b) $\Phi^L_{\beta,m}(v,s)$ has a holomorphic continuation to $\{s\in\mathbb{C}:\Re(s)>1\}\setminus\{1-\frac{k}{2}\}$ that has a pole of order 1 at $s=1-\frac{k}{2}$. The residue at $s=1-\frac{k}{2}$ is (roughly speaking) constant term of the Fourier expansion of $F^L_{\beta,m}$ at $s=1-\frac{k}{2}$.

Definition 5. For $v\in\mathrm{Gr}(L)\setminus H(\beta,m)$, define

$$\Phi^L_{\beta,m}(v):=\text{constant term of the Laurent expansion of }\Phi^L_{\beta,m}(v,s)\text{ at }s=1-\frac{k}{2}.$$

Important Remark 6. By Theorem 2.14 of [Br], the function $\Phi^L_{\beta,m}(v)$ is real-valued. ♢

References

[B2] Borcherds, Richard. *Automorphic forms with singularities on Grassmannians*. Inventiones, 1998.

[Br] Brunier, Jan Hendrik. *Borcherds products on $O(2,l)$ and Chern classes of Heegner divisors*. 2000.

[Ku] Kudla, Stephen S. *Integrals of Borcherds forms*. Compositio, 2003.