Chern classes of Heegner divisors

Lecture 11 • Speaker: Kartik Prasanna • Notes by: Charlotte Chan • April 1, 2016

This is a write-up of a very nice talk given by Kartik Prasanna. The purpose of this talk is to explain Chapter 5 of [Br] and explain the motivation behind [Br].

We are interested in studying cycles on Shimura varieties attached to orthogonal groups $\mathrm{O}(V)$, where V is a quadratic space over $\mathbb{Q}$ with form Q of signature $(2, n)$. (More generally, we can consider, for example, a quadratic space over a totally real field F that has signature $(2, n)$ at ∞_1 and is definite at all other infinite places.) Then, via the embeddings $\mathrm{O}(2, l) \hookrightarrow \mathrm{O}(2, l)$, the Heegner divisors will give rise to cycles of every codimension in the Shimura variety for $\mathrm{O}(2, n)$.

Pick a lattice $L \subseteq V$. For simplicity, we assume L is unimodular. Let $\Gamma(L) \subset \mathrm{O}(V)$ be the stabilizer of L in $\mathrm{O}(V)$; this is a discrete subgroup. Recall that we have an n-dimensional symmetric space $\mathbb{H}$ associated to L and that the quotient $X_L = \mathbb{H}/\Gamma(L)$ is a connected component of the orthogonal Shimura variety. Recall also that for each negative integer m such that $m \in Q(V)$, we have an associated Heegner divisor $H(m) \in \mathrm{CH}^1(X_L)$. Once and for all, set $\kappa := 1 + n/2 > 0$ and $k := 2 - \kappa = 1 - n/2 < 2$. Then the Borcherds lift in [B2] is a given by a map

$$\begin{array}{ccc} M_k^!(\mathrm{SL}_2 \mathbb{Z}) & \longrightarrow & \{\text{mero aut form for } \Gamma(L)\} \\ f & \longmapsto & \Psi_f, \end{array}$$

where $M_k^!(\mathrm{SL}_2 \mathbb{Z})$ denotes the space of weakly holomorphic modular forms of weight k (i.e. f may have poles at the cusps but is otherwise holomorphic). Writing

$$f = \sum_{n \gg -\infty} c(n) q^n,$$

we get a meromorphic automorphic form Ψ_f of weight $c(0)/2$ with divisor

$$\operatorname{div} \Psi_f = \sum_{m<0} c(m) H(m).$$

This is the Borcherds lift. Note that to define the Borcherds lift, we need to assume that $c(m) \in \mathbb{Z}$ for $m < 0$. We also remark that the divisors $H(m)$ were defined as divisors on $\mathbb{H}$ and were later shown to be $\Gamma(L)$-invariant. Hence the Heegner divisors $H(m)$ can be viewed as divisors on the quotient X_L.

Remark 1. What do we mean by an automorphic form? We can think of an automorphic form as a section of an automorphic vector bundle on X_L. The Shimura variety X_L carries a family a Hodge structures of weight 2 with Hodge numbers $h^{2,0} = 1$, $h^{1,1} = n$, $h^{0,2} = 1$. There is an automorphic line bundle ω over X_L that corresponds to taking the first step in the Hodge filtration, and the Borcherds lift Ψ_f is a section of $\omega^{\otimes c(0)/2}$. ◊

The following result of Borcherds gives a criterion for what can appear as the principal part of a weakly holomorphic modular form. (Note that the stated result below is a simplification of the general statement, which does not require L to be unimodular.)

Theorem 1. *There exists an $f \in M_k^!(\mathrm{SL}_2 \mathbb{Z})$ with principal part $\sum_{-\infty \ll n < 0} c(n) q^n$ if and only if $\sum_{n<0} c(n) a(-n) = 0$ for all cusp forms $\sum_{n>0} a(n) q^n \in \mathcal{S}_\kappa(\mathrm{SL}_2 \mathbb{Z})$.*

Example 2. Consider the special case when $L = H \oplus H$ is the sum of two copies of the hyperbolic plane. Then $V = L \otimes \mathbb{R}$ has signature $(2, 2)$ and we have $\kappa = 2$ and $k = 0$. The group $\mathrm{SO}(V)$ is (roughly) $\mathrm{SL}_2 \times \mathrm{SL}_2$ and so the associated symmetric space is (roughly) a product of two copies of the upper-half plane $\mathfrak{h}$. Consider the modular j-function

$$j(\tau) = q^{-1} + 744 + 196884q + \cdots$$

and consider the cusp form

$$f := j - 744 \in M_0^!(\mathrm{SL}_2 \mathbb{Z}).$$

Then the Borcherds lift Ψ_f is a form of weight 0 on $\mathfrak{h} \times \mathfrak{h}$ and

$$\Psi_f(z_1, z_2) = j(z_1) - j(z_2).$$

So, the divisor is just the diagonal $\mathfrak{h} \overset{\Delta}{\hookrightarrow} \mathfrak{h} \times \mathfrak{h}$. We can say something more. Since $\mathcal{S}_2(\mathrm{SL}_2 \mathbb{Z}) = 0$, then by Theorem 1, the principal part of a weakly holomorphic function of weight k can be anything. In particular, for any linear combination of Heegner divisors on $\mathfrak{h} \times \mathfrak{h}$, there exists a meromorphic automorphic form for $\Gamma(L)$ with this specified divisor. $\Diamond$

The main question that Bruinier answers in [Br] is:

> Given a Heegner divisor, can one find an explicit section of the corresponding line bundle with this specified divisor?

To do this, we need to enlarge $M_k^!(\mathrm{SL}_2 \mathbb{Z})$ to the space of weakly harmonic Maass forms. This space is spanned by the Poincaré series F_m (discussed in Lectures 4 & 5), which are *not* weakly holomorphic. It will turn out that we can define lifts of F_m analogous to the Borcherds lifts of weakly holomorphic modular forms, and that these lifts will have divisor exactly equal to the Heegner divisor $H(m)$. This is the construction given in [Br] (see also Lectures 6 & 7 of this seminar) and we review the construction of this lift now.

We can define a regularized theta lift (using a different regularization to the one in [B2])

$$F_m \mapsto \Phi_m,$$

where Φ_m is a $\Gamma(L)$-invariant function on $\mathbb{H}$ that is real-valued and real-analytic outside $H(m)$. As in Lecture 7, there is a decomposition

$$\Phi_m = \psi_m + \xi_m,$$

where ξ_m is real-analytic on $\mathbb{H}$ and ψ_m has the form

$$\log|\Psi_m(z)| = -\frac{1}{4}(\psi_m(z) - C_m)$$

for some holomorphic function Ψ_m on $\mathbb{H}$ and some specified constant C_m. Note however that ψ_m and ξ_m are not $\Gamma(L)$-invariant. We have

$$\mathrm{div}(\Psi_m) = \pi^* H(m),$$

where $\pi\colon \mathbb{H} \to \mathbb{H}/\Gamma(L)$.

Lemma 3. *Suppose f is a meromorphic function on $\mathbb{H}$ whose divisor is $\pi^* H(m)$. Let*

$$J(\gamma, z) = \frac{f(\gamma z)}{f(z)}.$$

Then to give a hermitian metric on the line bundle $\mathcal{L}(H(m))$ associated to the Heegner divisor $H(m)$ is the same as giving a C^∞-function $h\colon \mathbb{H} \to \mathbb{R}^+$ satisfying

$$h(\gamma z) = |J(\gamma, z)| \cdot h(z).$$

Furthermore, a representative for $c_1(\mathcal{L}(H(m)))$ *is given by* $\partial\overline{\partial}\log|h(z)|$.

In our situation, $f = \Psi_m$ so that $J(\gamma, z) = \Psi_m(\gamma z)/\Psi_m(z)$. Hence we want to find h such that $h(z)\cdot|\Psi_m(z)|$ is $\Gamma(L)$-invariant. But we already know that

$$|\Psi_m(z)| \cdot e^{-\frac{1}{4}\xi_m(z)}$$

is $\Gamma(L)$-invariant, and so we can take

$$h(z) = e^{-\frac{1}{4}\xi_m(z)}.$$

Now the Chern class $c_1(\mathcal{L}(H(m)))$ is given by

$$\partial\overline{\partial}\log e^{-\frac{1}{4}\xi_m(z)} = -\frac{1}{4}\partial\overline{\partial}\xi_m(z) = -\frac{1}{4}\partial\overline{\partial}\Phi_m,$$

and Φ_m is $\Gamma(L)$-invariant. So to summarize what we've done, we've explicitly constructed a section Ψ_m of the line bundle, and moreover, we know the Chern class of this line bundle is given by $-\frac{1}{4}\partial\overline{\partial}\Phi_m$. So Φ_m is a Green's function for the divisor $H(m)$.

The point of all this is that in arithmetic intersection theory, one wants to write down not only the divisor but also a Green's function. Bruinier's Borcherds lift gives rise to such a function (and we moreover know a lot of explicit information about this function: it's Fourier expansion, etc.).

Remark 2. For higher codimension, one expects that the lift should come from $\mathrm{Sp}_{2n}(\mathbb{Z})$ for $n > 1$. There is very little known about how to construct the Green's functions. $\Diamond$

There is another thread of this story that relates Borcherds lifts to Kudla–Millson theory. We may pick up this thread in the summer.

References

[B2] Borcherds, Richard. *Automorphic forms with singularities on Grassmannians.* Inventiones, 1998.

[Br] Brunier, Jan Hendrik. *Borcherds products on* $O(2,l)$ *and Chern classes of Heegner divisors.* 2000.

[BB] Bruinier and Bundschuh. *On Borcherds products associated with lattices of prime discriminant.* arXiv:0309178.