

Borcherds products learning seminar
Applications to Algebraic Geometry
 Lecture 8 • Igor Dolgachev • March 28, 2016

1. THE LEECH LATTICE

A *Leech lattice* is a definite (we assume it is negative definite) even unimodular lattice of rank 24 with no vectors of norm -2 . There are 24 isomorphism classes of even unimodular negative definite lattices of rank 24, the remaining 23 (called *Niemeyer lattices*) have vectors of norm -2 . Let Λ be a Leech lattice, an even unimodular lattice of signature $(1, 25)$ (resp. $(2, 26)$) is unique up to isomorphism and it is isomorphic to the orthogonal sum $II_{2,26} := \Lambda \oplus U$ (resp. $U \oplus U$), where U is a hyperbolic plane. Applying Milnor's Theorem, we see that it is also isomorphic to $E_8^{\oplus 3} \oplus U$ (resp. $E_8^{\oplus 3} \oplus U^{\oplus 2}$). We denote such a lattice by $II_{1,25}$ (resp. $II_{2,26}$).¹

One can construct the Leech lattice as follows. Let A be the incidence matrix of the graph with vertices (edges) equal to the set of vertices (edges) of the icosahedron. We view its entries as elements of the field \mathbb{F}_2 of two elements. Let J be the 12×12 -matrix with all ones as its entries and I_{12} be the identity matrix. Consider the linear subspace \mathcal{G} of \mathbb{F}_2^{24} spanned by the rows of the 12×24 matrix $[I_n J - A]$. It defines the *extended Golay linear binary code*. Let $\mathbb{Z}^{24} \rightarrow \mathbb{F}_2^{24}$ be the natural surjection. Let Γ be the pre-image of \mathcal{G} in \mathbb{Z}^{24} , considered as a quadratic lattice with the inner product defined by the dot-product multiplied by $1/2$. One can check that the sum of the coordinates of each vector in Γ is divisible by 2, taking the half-sum, we obtain a homomorphism $\alpha : \Gamma \rightarrow \mathbb{Z}$. The Leech lattice can be defined now as a sublattice of vectors in the union of the subsets $A = \alpha^{-1}(0)$ and $N = \alpha^{-1} + (\frac{1}{2}, \dots, \frac{1}{2})$.

We can see the Leech lattice inside of the lattice $II_{1,25}$ as follows. First we realize any unimodular even lattice $II_{1,n}$ of signature $(1, n)$ (we must have $n = 8k + 1$) as the set of vectors $(x_0, \dots, x_{n-1}|x_n)$ in \mathbb{Z}^{1+n} with coordinates x_i in $\frac{1}{2}\mathbb{Z}$ satisfying $\sum_{i=0}^{n-1} x_i - x_n \in 2\mathbb{Z}$. We take for the inner product in \mathbb{Z}^{1+n} the Lorentzian dot-product $x \cdot y = x_n y_n - \sum_{i=0}^{n-1} x_i y_i$. For example, when $n = 8$, the the lattice E_8 is isomorphic to the sublattice of $II_{1,9}$ equal to the orthogonal complement of the isotropic vector $k = (-1, \dots, -1|3)$ modulo $\mathbb{Z}k$. In the case $n = 25$, we take $k = (0, 1, 2, \dots, 23, 24|70)$ to obtain that $k^\perp/\mathbb{Z}k$ is isomorphic to the Leech lattice.

Let U be a sublattice of $II_{1,25}$ such that its orthogonal complement is isomorphic to Λ (other possibility is that it is isomorphic to $E_8^{\oplus 3}$). Let (f, g) be two isotropic vectors in U with $(f, g) = 1$. Then the *norm* $v^2 = (v, v)$ of the vector $r = \lambda + mf + ng \in II_{1,25}$ is equal to $\lambda^2 + 2mn$, so taking $m = 1, n = -1 - \frac{1}{2}\lambda^2$, we obtain $v^2 = -2$. A vector of this form is called a *Leech root* (of course, not every vector of norm -2 is a Leech root). The vector g is denoted by ρ and is called the *Weyl vector*. It has the property that $(w, r) = 1$ for all Leech roots r . Let W be the *reflection subgroup* of the orthogonal group $O(II_{1,25})$

¹ II indicates that it is an even lattice.

generated by reflections $s_r : v \mapsto v + (v, r)r$, where r is a Leech root. We have

$$\mathrm{O}(II_{1,25})' \cong W \rtimes A(P),$$

where $A(P)$ is the symmetry group of a fundamental polyhedron of W . A remarkable fact due to J. Conway is that this group is isomorphic to the group $\Lambda \rtimes \cdot \mathrm{O}(\Lambda)$, where $\cdot \mathrm{O}(\Lambda)$ is the *first Conway sporadic simple group* isomorphic to the index 2 subgroup of the group of automorphisms of the Leech lattice.

Let T be an even quadratic lattice of signature $(2, b_-)$ and l_T be the minimal number of generators of the discriminant group T'/T . Applying some general techniques from the theory of quadratic lattices (due to V. Nikulin), one obtains that L can be primitively embedded into $II_{2,26}$ provided $l_T \leq 28 - \mathrm{rk} T$. In particular, if T is a sublattice of $L_{K3} := E_8^{\oplus 2} \oplus U^{\oplus 3}$ isomorphic to the lattice of transcendental 2-cycles on a K3 surface, we get $l_T = l_{T^\perp} \leq 11$, so we can always primitively embed T in $II_{2,26}$. The orthogonal complement of T in $II_{2,26}$ is a negative definite lattice R of rank $28 - \mathrm{rk} T$.

2. DENOMINATOR OF THE FAKE MONSTER LIE ALGEBRA

Let

$$\Delta = \eta(q)^{24} = q \prod_{m=1}^{\infty} (1 - q^m)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n = (2\pi)^{-12} g_2^3 - 27g_4$$

be the discriminant modular form of weight 24 with respect to the modular group $\Gamma = \mathrm{PSL}(2, \mathbb{Z})$. Here g_2 and g_4 are modular forms of weights 4 and 6 defining a Weierstrass equation

$$y^2 - 4x^3 + g_2(\tau)x + g_3(\tau) = 0$$

of an elliptic curve $\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$. We have $g_2 = 60E_4, g_3 = 140E_6$, where E_k are the Eisenstein modular forms. The function Δ does not vanish on the upper-half plane \mathbb{H} and has simple zero at the unique Γ -orbit of cusps. If \mathcal{M} is a moduli space of some algebra-geometrical objects and $\overline{\mathcal{M}}$ is its some partial compactification with normal crossing divisorial boundary, a section of some line bundle on \mathcal{M} that can be extended to the compactification with simple zeros at the boundary is called a *discriminant*. A K3 surface is a 2-dimensional analog of an elliptic curve. In Lecture 2 we discussed the moduli space $\mathcal{M}_{K3, M}$ parameterizing isomorphism classes of K3 surfaces X together with a primitive embedding of a hyperbolic lattice $j : M \hookrightarrow H^2(S, \mathbb{Z})$ such that $j(M)$ contains an ample line bundle. It is isomorphic to the orbit space

$$(\Omega_T \setminus \bigcup_{\beta \in T_{-2}} \beta^\perp) / \Gamma_T,$$

where $T = M_{L_{K3}}^\perp$, L_m denotes the set of vectors in a lattice L with norm m , and $\Omega_T = \mathrm{Gr}(2, T_{\mathbb{R}})^+$ is the Grassmannian of oriented positive definite planes on $T_{\mathbb{R}}$ (equipped with a complex structure of an open subset of a quadric in $\mathbb{P}(T_{\mathbb{C}})$). In the notation from the previous lectures,

$$\bigcup_{\beta \in T_{-2}} \beta^\perp = H(0, -2)$$

is the *Heegner divisor of discriminant* $(0, -2)$. So, a natural generalization of the modular form $\Delta(\tau)$ is a modular form on Ω_T with respect to the group $\Gamma_T = \text{Ker}(\text{O}(T)' \rightarrow \text{O}(T'/T))$ which vanishes of order 1 on the Heegner divisor $H(0, -2)$.

Unfortunately, it turns out that it is impossible to construct a discriminant modular form for a general lattice (even in the case when $\text{rk } T = 19, \text{rk } M = 1$ corresponding to a general K3 surface with Picard group generated by an ample divisor class). A result of Nikulin says that it is possible to do it only for a finite number of isomorphism classes of lattices T (or M).

However, using his theory, Borcherds constructs such a discriminant modular form in the case when $M = \langle 2 \rangle$. In this case the moduli space is the moduli space of K3 surfaces that admit a degree 2 cover of the projective plane branched along a nonsingular plane curve of degree 6. In other words, it is given by the equation

$$w^2 + f_6(x, y, z) = 0,$$

where $f_6(x, y, z)$ is a homogeneous polynomial of degree 6 which defines a nonsingular plane curve of degree 6 in the projective plane \mathbb{P}^2 . This equation has to be considered as a homogeneous equation of degree 6 in the weighted projective space $\mathbb{P}(3, 1, 1, 1)$ (the same as an elliptic curve $z^2 + f_4(x, y) = 0$ has to be considered as a curve of degree 4 in the weighted projective plane $\mathbb{P}(2, 1, 1)$).

First Borcherds constructs the analog of the discriminant form for the lattice $T = II_{2,26}$ (although it is not known whether there are any algebraic varieties whose periods are represented by an open subspace of $\Omega_{II_{2,26}}$). Then we restrict it to the subdomain $\Omega_T \subset \Omega_{II_{2,26}}$.

Let Φ be the modular form on $\Omega_{II_{2,26}}$ of weight $-12 = 1 - \frac{26}{2}$ associated to the modular form

$$\frac{1}{\Delta} = q^{-1} + 24 + 324q + \dots$$

We know from the previous lectures that its divisor is equal to $H(0, -2)$. In the tube domain realization \mathbb{H}_{26} corresponding to the decomposition $II_{2,26} = U \oplus II_{1,25} = U \oplus (U \oplus \Lambda)$, it is given by the Fourier expansion

$$\begin{aligned} \Phi(z) &= e^{2\pi i(r,z)} \prod_{r>0} (1 - e^{-2\pi i(r,z)})^{p_{24}(1-\frac{r^2}{2})} \\ &= \sum_{w \in W} \sum_{n>0} \det(w) \tau(n) e^{-2\pi i n(w(\rho), z)}, \end{aligned}$$

where r are positive integer linear combinations of Leech roots (*positive roots*) and ρ is the Weyl vector. Also here $p_{24}(n)$ is the number of partitions of a number n in 24 parts and $\tau(n)$ is the Ramanujan τ -function given by the Fourier coefficients of the discriminant function Δ .

For any $\beta \in (II_{2,26})_{-2}$ we denote by β_T a primitive vector in T which is equal to the projection to T of some integer multiple of β under the orthogonal decomposition $(II_{2,26})_{\mathbb{Q}} = T_{\mathbb{Q}} \oplus (T^{\perp})_{\mathbb{Q}}$. So, we have $k\beta = \beta_T + \beta_T^*$, where $\beta_T^* \in T^{\perp}$. Since T^{\perp} is negative definite lattice, we have $(\beta_T^*, \beta_T^*) < 0$. This shows that $\beta_T^2 \leq -2$. Thus Φ may vanish

on hyperplanes H_α with $\alpha \leq -2$ and not only on the discriminant hyperplanes H_α with $\alpha^2 = -2$. However, assume that

$$T = \langle 2 \rangle_{L_{K3}}^\perp \cong E_8^{\oplus 2} \oplus U^{\oplus 2} \oplus \langle -2 \rangle.$$

Then we can embed T into $II_{2,26} \cong E_8^{\oplus 3} \oplus U^{\oplus 2}$ (such embeddings are unique up to isometry of $II_{2,26}$) as follows. We embed a generator of $\langle -2 \rangle$ to a copy of E_8 with orthogonal complement isomorphic to the root lattice of type E_7 with discriminant 2. We embed the other summands of T in the natural way. Then $T_{II_{2,26}}^\perp \cong E_7$. For any $\beta \in (II_{2,26})_{-2}$ we have $k\beta = \beta_T + \beta_T^*$, where $k = 1$ or 2 . Taking the norms, we get $-2k^2 = (\beta_T, \beta)^2 + (\beta_T^*, \beta_T^*)$. If $k = 1$, we get $\beta_T = \beta$, hence $\beta_T^2 = -2$. If $k = 2$, then $\beta_T^2 = \beta_T^{*2} = -4$, but $\beta_T + \beta_T^*$ is not divisible by 2. So, we see that the projection of β has norm -2 . This shows that the restriction of Φ vanishes only on $H(0, -2)$.

An immediate corollary of this is that the moduli space $\mathcal{M}_{K3, \langle 2 \rangle}$ is a *quasi-affine* algebraic variety, i.e. it is a complement to a closed subset of an affine variety. In fact, the Borcherds modular form Φ is a section of an ample line bundle on a partial compactification Ω_T/Γ_T of the moduli space. One can show that some positive tensor power of the line bundle extends to a compactification, and hence the complement of its zero becomes a hyperplane section of a projective embedding of the compactification, hence its complement is an affine subset. It contains the part of the boundary and the open subset isomorphic to $\mathcal{M}_{K3, \langle 2 \rangle}$.

Note that, for a general T it could happen that the lattice T is contained in the orthogonal complement in $II_{2,26}$ of some vector $\beta \in (II_{2,26})_{-2}$. Then the Borcherds discriminant form Φ vanishes on Ω_T . The construction can be modified by dividing the Fourier expansion by some product of linear factors.

3. APPLICATIONS

Our first application is related to the following problem. Since the coarse moduli space of elliptic curves is isomorphic to the affine line, any smooth family of elliptic curves over a complete base must be isotrivial. A natural question is: does this result extend to K3 surfaces (or abelian varieties). Let $f : \mathcal{X} \rightarrow B$ be a smooth family of K3 surfaces over a projective base B . Then, we can choose a relatively ample line bundle \mathcal{L} on \mathcal{X} such that its restriction \mathcal{L}_b to the fiber \mathcal{X}_b is an ample line bundle with $(\mathcal{L}_b, \mathcal{L}_b) = 2d$. After a base change $C \rightarrow B$ we get a morphism from C to $\mathcal{M}_{K3, \langle 2d \rangle}$ that assigns to a point $c \in C$ the isomorphism class of the fiber \mathcal{X}_c with embedding $\langle 2d \rangle$ into $\text{Pic}(\mathcal{X}_c)$. Since the boundary of the period domain is of dimension ≤ 1 , we may construct a family such that the image of C misses the boundary. This will produce a smooth non-isotrivial family of K3 surfaces.

The following result was proven by R. Borcherds, L. Katzarkov, T. Pantev, and N. Shepherd-Barron (JAG, 7 (1998)):

Theorem 1. *Assume the rank ρ of the Picard group of fibers is constant. Then a smooth family must be isotrivial.*

Here is a sketch of the proof. Assume the family is not isotrivial. We have a local coefficient system L_B of rank ρ over the base whose fiber is the Picard lattice. Over

some complex analytic base change $U \rightarrow B$ defined by the monodromy group of the local coefficient system, it trivializes, i.e. becomes isomorphic to M_U for some quadratic lattice M . However, we can choose a basis in M such that each member is represented by a divisor class whose irreducible components are finite covers of B . This shows that the family is trivialized over a finite cover $B' \rightarrow B$. Replacing B with B' , we may assume that the family is a family of M -lattice polarized K3 surface, and after a further base change $C \rightarrow B'$, we get a map $C \rightarrow \mathcal{M}_{K3,M}$. Now we apply Borchard's construction of an automorphic form with zeros on the Heegner divisor $H(0, -2)$. Since it defines a section of an ample line bundle over $\mathcal{M}_{K3,M}$, we obtain that the image of C hits it. Thus, we see a vector δ in the orthogonal complement of the period map, it should represent a divisor class D in the orthogonal complement of $M = \text{Pic}(X_c)$ for the point c whose period point lies in the hyperplane. This shows that the rank of the Picard lattice jumps, contradicting the assumption.

Note that the assumption that the rank of the Picard lattice does not change in the family is important. The loc.cit. paper contains several explicit examples of smooth non-isotrivial families of K3 surfaces.