Borcherds products learning seminar
Working syllabus • Winter 2016

Speakers are encouraged to write up their talks. These will be put on the webpage.

1. Broad goals

(1) Cover ideas of [Br], skipping gory calculations. (1+8 lectures)
(2) Discuss Kudla–Millson theory and its relation to Borcherds products [BF]. (2+3 lectures)

2. Suggested lectures

2.1. **February 5: The big picture (Brandon).**

2.2. **February 12: Lattice model of the Weil representation for $\mathrm{Mp}_2(\mathbb{R})$ (Cameron).**

(a) Explain the Weil representation for $\mathrm{Mp}_2(\mathbb{R})$ described in §1.1 of [Br]. In the literature, this is sometimes called the lattice model of the Weil representation. Mention that there is a Weil representation for $\mathrm{Mp}_{2n}(\mathbb{R})$ and that this is one of the key ingredients to defining the theta lift.
(b) The Weil representation for $\mathrm{Mp}_2(\mathbb{R})$ attached to $(L/L', q)$ induces an action of $\mathrm{Mp}_2(\mathbb{R})$ on the space of functions $f\colon \mathbb{H} \to \mathbb{C}[L/L']$. This leads to the definition of a vector-valued modular form for $\mathrm{Mp}_2(\mathbb{R})$.
(c) Discuss vector-valued modular forms $\longleftrightarrow$ modular forms?
(d) Reference: §1.1 of [Br]

2.3. **February 15: Holomorphic and non-holomorphic Poincaré series (Cameron).**

This is a special time: 3-4pm, Monday, February 15. Location TBA.

(a) Construct Poincaré series and Eisenstein series in the space of vector-valued modular forms.
(b) Construct Maass-Poincaré series (non-holomorphic analogue of the classical Poincaré series).
(c) Reference: §1.2, 1.3 of [Br].

2.4. **February 19: The regularized theta lift (Charlotte).**

(a) Discuss classical theta lift, focusing on the dual reductive pair $(\mathrm{SL}_2, O(2,l))$. Reference: Kudla, D. Prasad expository papers, etc.
(b) Discuss the Harvey–Moore regularized theta lift obtained by integrating a Poincaré series against a Siegel theta function to get Φ_m (real analytic outside $H(m)$). Reference: §2 of [Br].
(c) Discuss relation between Harvey–Moore and original definition of the Borcherds lift. Reference: original constructions in [B1], but maybe a comparison is in [B2].

2.5. **February 26: Applications and connections to algebraic geometry (Igor).**

2.6. March 18: Borcherds products (Jeff).

(a) Construct the splitting $\Phi_m(Z) = \psi_m(Z) + \eta_m(Z)$, where $\eta_m(Z)$ is real analytic, and $-\frac{1}{4}\psi_m(Z)$ is the logarithm of the absolute value of a holomorphic function $\Psi_m(Z)$ whose divisor is $H(m)$.
(b) Discuss relation to Heegner divisors.
(c) Borcherds product expansion of $\Psi(Z)$.
(d) Reference: §3 of [Br]

2.7. March 25: Examples of Borcherds products.

(a) Reference: §3 of [Br]

2.8. April 1: Heegner divisors and automorphic forms.

(a) Φ_m is an eigenfunction of the $O(2, l)$-invariant Laplace operator Ω
(b) "Any automorphic form whose divisor is a linear combination of Heegner divisors is given by the regularized theta lift of a (possibly non-holomorphic) Maass form with singularity at ∞" (Theorem 4.22)
(c) Reference: §4 of [Br]

2.9. April 8: Chern classes of Heegner divisors (Kartik).

(a) Construct the map ϑ. This is a generalization of the Doi–Nagunuma map?
(b) Mention Kudla–Millson map. (Adjoint to ϑ? In correct formulation, yes!)
(c) Reference: §5.1 of [Br]

2.10. April 15: Every Heegner divisor can be realized in a Borcherds product.

(a) Discuss Theorem 5.11 (compare to Theorem 4.22) from the geometric viewpoint.
(b) Reference: §5.2 of [Br] + some earlier sections

References

[B1] Borcherds, Richard. *Automorphic forms on $O_{s+2,2}(\mathbb{R})$ and infinite products.* Inventiones, 1995.
[B2] Borcherds, Richard. *Automorphic forms with singularities on Grassmannians.* Inventiones, 1998.
[Br] Brunier, Jan Hendrik. *Borcherds products on $O(2, l)$ and Chern classes of Heegner divisors.* 2000.
[BF] Brunier and Funke. *Two geometric theta lifts.* Duke, 2004.