The Weil Representation

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Abstract

This is my senior honors thesis done in my final year as an undergraduate at Stanford University, under the direction of Professor Akshay Venkatesh. We will construct the Weil representation of $\mathrm{SL}_2(\mathbb{R})$ through a natural action of the Heisenberg group $\mathrm{Heis}(\mathbb{R})$ on the space of square-integrable complex-valued functions $\mathscr{L}^2(\mathbb{R})$, together with the celebrated Stone-von Neumann theorem of functional analysis. Our approach will be to explicitly construct the analogous representation of $\mathrm{SL}_2(\mathbb{F}_p)$ on $\mathscr{L}^2(\mathbb{F}_q)$, for p an odd prime, using the finite-field equivalents of the aforementioned ingredients. This will allow us to separate the functional analytic complications of unitary representation theory of Lie groups from the representation theoretic and purely algebraic ideas. As the final arc to our triptych of stories, we will construct the Lie algebra analogue of the Weil representation through the Heisenberg algebra. Throughout this paper, we will be as explicit as possible while simultaneously giving motivation to our computations, allowing us to speak of concepts concretely without sacrificing the overarching philosophy. In spite of these efforts to make this paper self-contained, some proofs are omitted to avoid straying from the main arc of the story of the Weil representation. We will draw upon ideas from homological algebra, group theory, algebraic topology, functional analysis, measure theory, Lie theory, and, of course, representation theory. What is remarkable is that this topic touches as many subject areas as this paper will use, including number theory, harmonic analysis, topology, and physics. Because of its place in the intersection of a multitude of different fields, the Weil representation is also known, in the literature, as the Segal-Shale Weil representation, the metaplectic representation, the harmonic representation, and the oscillator representation.

Contents

Notation and Conventions

K, a field

G, a group

$\mathfrak{g}$, a Lie algebra

$\mathrm{Heis}(K) := \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in K \right\}$, the Heisenberg group of K

$\mathrm{Z}(G)$, the center of G

ψ, an irreducible character of $\mathrm{Z}(\mathrm{Heis}(K))$

$\mathscr{L}^2(K)$, square-integrable $\mathbb{C}$-valued functions on K (see further comments on page 13)

$\mathrm{SL}_2(K) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in K, ad - bc = 1 \right\}$

$\mathrm{GL}(V)$, the space of invertible K-linear transformations $V \to V$ for a vector space V over K

$\mathrm{End}(V)$, the space of K-linear transformations $V \to V$ for a vector space V over K

$\mathscr{S}(\mathbb{R})$, the Schwartz space, the space of rapidly decreasing functions on $\mathbb{R}$

$\mathfrak{sl}_2(K) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in K, a + d = 0 \right\}$, the Lie algebra of $\mathrm{SL}_2(K)$

$\mathfrak{sl}(2) := \mathfrak{sl}_2(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$, the complexified Lie algebra of $\mathrm{SL}_2(\mathbb{R})$

$\mathfrak{heis}(K) := \left\{ \begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} : a, b, c \in K \right\}$, the Heisenberg algebra of K, the Lie algebra of the Heisenberg group $\mathrm{Heis}(K)$

$\mathfrak{heis} := \mathfrak{heis}(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$, the complexified Lie algebra of $\mathrm{Heis}(K)$

$\mathrm{Isom}(\mathscr{H}) := \{\phi : \mathscr{H} \to \mathscr{H} : \langle \phi(v), \phi(w) \rangle = \langle v, w \rangle\}$, where $\langle \cdot, \cdot \rangle$ is the inner product with which $\mathscr{H}$ is endowed.

$\mathbb{R}^+$, the real numbers viewed as an additive group

V_λ, a lowest weight module of weight λ

Unless otherwise stated, all representations will be complex representations.

1 Introduction

In this paper, we give three parallel constructions of Weil representation through the representation theory of the Heisenberg group (or, in the Lie algebra case, the Heisenberg algebra). Namely, we construct the Weil representation for $\mathrm{SL}_2(\mathbb{F}_p)$, $\mathrm{SL}_2(\mathbb{R})$, and for $\mathfrak{sl}(2) := \mathfrak{sl}_2(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$. In each of these cases, we begin with a Heisenberg object ($\mathrm{Heis}(\mathbb{F}_p)$, $\mathrm{Heis}(\mathbb{R})$, and $\mathfrak{heis} := \mathfrak{heis}(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$, respectively) and study its representation theory. From this, we will construct a projective representation (of $\mathrm{SL}_2(\mathbb{F}_p)$, $\mathrm{SL}_2(\mathbb{R})$, and $\mathfrak{sl}(2)$, respectively), and then lift this projective representation to a linear representation. This resulting representation is exactly the Weil representation and has a surprisingly significant role in many fields of mathematics, including number theory, topology, harmonic analysis, and physics. Unless otherwise stated, all representations in our discussion are complex representations.

We will begin by discussing the Heisenberg group

$$\mathrm{Heis}(K) = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in K \right\}$$

for a field K with characteristic away from 2. Section 2 is completely focused on this discussion. We will explicitly analyze its structure as a central extension of $K^{\oplus 2}$ by K, where $K^{\oplus 2}$ and K are viewed as additive groups. Hence we have a short exact sequence

$$0 \to K \to \mathrm{Heis}(K) \to K^{\oplus 2} \to 0. \tag{1.1}$$

Writing $Z = \mathrm{Z}(\mathrm{Heis}(K))$ and $Q = \mathrm{Heis}(K)/Z$, we obtain isomorphisms $Z \cong K$ and $Q \cong K^{\oplus 2}$. Letting $\mathrm{SL}_2(K)$ act by matrix multiplication on $Q \cong K^{\oplus 2}$ and trivially on $Z \cong K$, we obtain an action of $\mathrm{SL}_2(K)$ on $\mathrm{Heis}(K)$. This is described explicitly by unravelling the second cohomology class in $\mathrm{H}^2(Q, Z)$ associated to the isomorphism class of central extensions described by (1.1). This action of $\mathrm{SL}_2(K)$ on $\mathrm{Heis}(K)$ is the first main ingredient in our construction of the Weil representation.

The second main ingredient is the following Heisenberg representation associated to a central character. For an irreducible character $\psi : \mathrm{Z}(\mathrm{Heis}(K)) \to \mathbb{C}$, we will define a representation π_ψ on the $\mathbb{C}$-vector space $\mathscr{L}^2(K)$ by the following action on $f(x) \in \mathscr{L}^2(K)$:

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \cdot f(x) := \psi(-bx + c) f(x - a).$$

In this paper, we will only discuss the specializations to $K = \mathbb{F}_p$ and $K = \mathbb{R}$. In the former case, $\mathscr{L}^2(K)$ has the obvious meaning as a p-dimensional $\mathbb{C}$-vector space of $\mathbb{C}$-valued functions on K, and in the latter case, $\mathscr{L}^2(K)$ is defined to be the $\mathbb{C}$-vector space of square-integrable functions on K with respect to the Lebesgue measure. This Heisenberg representation is discussed in Section 2.3.

Given these two ingredients, the natural subsequent question is, "What happens when we put them together?" That is, what is the relationship between the representation π_ψ and the representation $\pi_\psi \circ g$ obtained by precomposing with the action of $g \in \mathrm{SL}_2(K)$? This is where the story ramifies into the finite-field case and the real case.

The (complex) representation theory of $\mathrm{Heis}(\mathbb{F}_p)$ is simple. From the representation theory of finite groups, we know that the dimension of an irreducible representation of some finite group G must divide the order of G. Furthermore, the decomposition of the regular representation gives us that

$$\sum \dim(V)^2 = |G|,$$

where V ranges over a transversal of the isomorphism classes of irreducible representations of G. Since $|\mathrm{Heis}(\mathbb{F}_p)| = p^3$, we obtain arithmetically that $\mathrm{Heis}(\mathbb{F}_p)$ has p^2 irreducible representations of dimension 1 and $p-1$ irreducible representations of dimension p. We in fact can say more about the representations of $\mathrm{Heis}(\mathbb{F}_p)$:

Theorem. *For a non-trivial irreducible character $\psi\colon \mathrm{Z}(\mathrm{Heis}(\mathbb{F}_p)) \to \mathbb{C}$, there exists a unique, up to isomorphism, representation of $\mathrm{Heis}(\mathbb{F}_p)$ such that $\mathrm{Z}(\mathrm{Heis}(\mathbb{F}_p))$ acts by ψ.*

This is the finite-field analogue of the celebrated Stone-von Neumann theorem from functional analysis.

It follows from the above theorem that the representation $(\pi_\psi, \mathscr{L}^2(\mathbb{F}_p))$ is irreducible and that this representation is isomorphic to $(\pi_\psi \circ g, \mathscr{L}^2(\mathbb{F}_p))$ for any $g \in \mathrm{SL}_2(\mathbb{F}_p)$. This means that there exists an intertwining operator $\Phi_g : \mathrm{Heis}(\mathbb{F}_p) \to \mathrm{GL}(\mathscr{L}^2(\mathbb{F}_p)) = \mathrm{GL}_p(\mathbb{C})$ such that

$$\Phi_g(h)\pi_\psi(h) = (\pi_\psi \circ g)(h)\Phi_g(h), \quad \text{for all } h \in H.$$

By Schur's lemma, an intertwining operator between isomorphic irreducible representations is uniquely determined up to a scalar. Hence, from the Heisenberg representation together with the action of $\mathrm{SL}_2(\mathbb{F}_p)$ on $\mathrm{Heis}(\mathbb{F}_p)$, we get the projective representation

$$\mathrm{SL}_2(\mathbb{F}_p) \to \mathrm{PGL}_p(\mathbb{C}), \quad g \mapsto [\Phi_g].$$

(The notation $[\Phi_g]$ means the image of Φ_g with respect to the surjection $\mathrm{GL}_p(\mathbb{C}) \to \mathrm{PGL}_p(\mathbb{C})$.) In Section 3.2, we explicitly compute this projective representation.

We would like to lift this projective representation to a linear representation. In Section 3.3, we discuss the problem of lifting projective representations of a group G in general. This is centered around the Schur multiplier, which is defined to be the second integral homology group $\mathrm{H}_2(G, \mathbb{Z})$, and its relation to group extensions of G by $\mathbb{C}^\times$ and $\mathrm{H}^2(G, \mathbb{C}^\times)$. In particular,

$$\mathrm{H}^2(G, \mathbb{C}^\times) \cong \mathrm{Hom}(\mathrm{H}_2(G, \mathbb{Z}), \mathbb{C}^\times).$$

This result hinges upon the Universal Coefficient Theorem. To finish our discussion of the finite-field model of the Weil representation, we compute the Schur multiplier of $\mathrm{SL}_2(\mathbb{F}_p)$ in Section 3.4. We show that $\mathrm{H}_2(\mathrm{SL}_2(\mathbb{F}_p),\mathbb{Z}) = 0$ and hence $\mathrm{H}^2(\mathrm{SL}_2(\mathbb{F}_p),\mathbb{C}^\times) = 0$, which therefore means that every extension of $\mathrm{SL}_2(\mathbb{F}_p)$ by $\mathbb{C}^\times$ splits. Hence the projective Weil representation $\mathrm{SL}_2(\mathbb{F}_p) \to \mathrm{PGL}_p(\mathbb{C})$ lifts to a linear Weil representation $\mathrm{SL}_2(\mathbb{F}_p) \to \mathrm{GL}_p(\mathbb{C})$.

When we pass to the real case, almost everything from the finite-field model carries over. The exception is that we must now be more careful about the nature of the representation spaces since all of the representations we are considering are infinite-dimensional. But because we have a solid understanding of the finite-field case, we may separate the representation theoretic aspects of this construction of the Weil representation from the functional analytic aspects of working with infinite-dimensional unitary representations.

Our discussion of the representation theory of the Heisenberg group now relies on the real Stone-von Neumann theorem, whose proof relies on functional analytic methods, rather than dimension-counting techniques that we could use in the finite-field analogue.

Theorem. *For a non-trivial irreducible character $\psi : \mathrm{Z}(\mathrm{Heis}(\mathbb{R})) \to \mathbb{C}$, there exists a unique, up to unitary equivalence, infinite-dimensional irreducible unitary representation of* $\mathrm{Heis}(\mathbb{R})$ *such that* $\mathrm{Z}(\mathrm{Heis}(\mathbb{R}))$ *acts by ψ.*

We prove that the Heisenberg representation π_ψ defined in Section 2.3 is an irreducible unitary representation, and from the Stone-von Neumann, together with the formulation of Schur's lemma for compact groups, we may construct a projective representation of $\mathrm{SL}_2(\mathbb{R})$. As in the finite-field case, this is done by mapping each $g \in \mathrm{SL}_2(\mathbb{R})$ to the intertwining operator between π_ψ and $\pi_\psi \circ g$, where the latter denotes precomposition by the action of $g \in \mathrm{SL}_2(\mathbb{R})$ on $\mathrm{Heis}(\mathbb{R})$. In Section 4.2, we will explicitly compute these intertwining operators for a generating set of $\mathrm{SL}_2(\mathbb{R})$.

Unfortunately, unlike the case of $\mathrm{SL}_2(\mathbb{F}_p)$, the Schur multiplier for $\mathrm{SL}_2(\mathbb{R})$ is nontrivial. Hence not all projective representations lift to linear representations. In particular, the projective representation obtained from the intertwiner operators of the Heisenberg representation with the $\mathrm{SL}_2(\mathbb{R})$ action on $\mathrm{Heis}(\mathbb{R})$ does *not* lift to a linear representation of $\mathrm{SL}_2(\mathbb{R})$. In order to realize the Weil representation as a linear representation of a group, we must pass to a double cover of $\mathrm{SL}_2(\mathbb{R})$. This is discussed in Section 4.3.

In the third and final arc of this exposition, we formulate the Weil representation construction for the (complexified) Lie algebra of $\mathrm{SL}_2(\mathbb{R})$. We will denote this by $\mathfrak{sl}(2)$. We would like to apply the Lie derivative to the projective representation of $\mathrm{SL}_2(\mathbb{R})$ on $\mathscr{L}^2(\mathbb{R})$, but without the finite-dimensionality to control the behavior of $\mathscr{L}^2(\mathbb{R})$, there is no reason that this representation should be differentiable. Hence we must pass to the Schwartz space $\mathscr{S}(\mathbb{R})$, the space of rapidly decreasing functions on $\mathbb{R}$. On this (dense) subspace of $\mathscr{L}^2(\mathbb{R})$, we construct the Weil representation for $\mathfrak{sl}(2)$

through studying the Heisenberg algebra representation, which is exactly the Lie derivation of the Heisenberg group representation described in Section 2.3. To obtain a projective representation from this set-up, we take the formula for the intertwining operators and apply the Lie derivative. We then obtain the Lie algebra analogue of the construction of the projective Weil representation.

Note that the description above is exactly the Lie algebra analogue of the construction we have explicated for $SL_2(\mathbb{F}_p)$ and $SL_2(\mathbb{R})$; indeed, in Section 5 (in particular, Section 5.4), we will essentially apply the Lie derivative to each of our steps in Section 4.

We spend the remainder of Section 5 unpacking the Weil representation by explicitly computing the action of two sets of standard bases for $\mathfrak{sl}(2)$ and describing the module structure of the representation (it splits up into two lowest weight modules). In our analysis, we use the Weil representation to compute the eigenvectors and eigenvalues of the Hermite operator $x^2 - \frac{d^2}{dx^2}$, which alludes to the well-known Schrödinger's equation from quantum mechanics (see Griffiths' *An Introduction to Quantum Mechanics* [Gri04]).

For further reading, learning, and discovery on topics on or related to the Weil representation, we refer the reader to the Bibliography, wherein we give articles and books that may interest the eager reader.

2 The Heisenberg Group

Throughout this section, we will take K to be any field with $\mathrm{char}(K) \neq 2$. The Heisenberg group $\mathrm{Heis}(K)$ is defined to be

$$\mathrm{Heis}(K) := \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in K \right\}.$$

The terminology of this group comes from the mathematical formulation of the Heisenberg uncertainty principle in the context of the relationship between the behavior of an $\mathscr{L}^2$ function f and the behavior of its Fourier transform $\hat{f}$. This relationship is closely tied to the representations of $\mathrm{Heis}(K)$. This will be illuminated in Sections 3 and 4.

When the field K is clear from the context, we will write:

$$\begin{aligned} H &= \mathrm{Heis}(K), \\ Z &= \mathrm{Z}(H), \\ Q &= H/Z. \end{aligned}$$

It is easy to see that the following maps are isomorphisms:

$$Z = \left\{ \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : c \in K \right\} \xrightarrow{\sim} K, \qquad \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mapsto c, \tag{2.1}$$

$$Q = \left\{ \begin{pmatrix} 1 & a & * \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b \in K \right\} \xrightarrow{\sim} K \oplus K = K^{\oplus 2}, \qquad \begin{pmatrix} 1 & a & * \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mapsto (a, b). \tag{2.2}$$

The asterisk $*$ in $\left(\begin{smallmatrix} 1 & a & * \\ 0 & 1 & b \\ 0 & 0 & 1 \end{smallmatrix}\right)$ means that we ignore the upper right-hand entry.

2.1 The Structure of the Heisenberg Group as a Central Extension

From group cohomology, we know that for any group G acting trivially on an additive abelian group A, there is a bijective correspondence between the cocycle classes of the second cohomology group $\mathrm{H}^2(G, A)$ and the isomorphism classes of central extensions of G by A. This correspondence is given by the following. Recall that a central extension E of G by A is a short exact sequence of groups

$$0 \to A \to E \to G \to 0,$$

with the image of A being central in E. For a section σ of the surjection $E \to G$, we may define a function

$$f : G \times G \to A, \quad (q, q') \mapsto \sigma(q) + \sigma(q) - \sigma(qq'). \tag{2.3}$$

Recall that a 2-cocycle is a function $\omega : G \times G \to A$ satisfying

$$\omega(g_1, g_2) + \omega(g_1 g_2, g_3) = \omega(g_2, g_3) + \omega(g_1, g_2 g_3) \tag{2.4}$$

for all g_1, g_2, and g_3 in G. One can prove, by straighforward computation, that f is a 2-cocycle, and that a different choice of section changes the induced 2-cocycle by a 2-coboundary. We will leave the former of these two claims to the reader and verify the latter. Let $\sigma' : G \to E$ be another section and let $f' : G \times G \to E$ be the 2-cocycle induced by σ'. Necessarily, for any $g \in G$, $\sigma'(g)$ and $\sigma(g)$ differ only by an element of A, and hence we have a function

$$\phi : G \to A, \quad g \mapsto \sigma'(g) - \sigma(g),$$

which gives rise to the 2-coboundary

$$\omega : G \times G \to Q, \quad (g, h) \mapsto \phi(g) + \phi(h) - \phi(gh).$$

By construction, we have

$$f'(g, h) = f(g, h) + \omega(g, h),$$

and this completes our check. We have therefore shown that the 2-cocycle f determines a unique cohomology class in $\mathrm{H}^2(G, A)$.

Specializing to our case, this means that the Heisenberg group H is determined, up to isomorphism, by the short exact sequence

$$0 \to Z \to H \to Q \to 0,$$

together with a 2-cocycle $f : Q \times Q \to Z$.

2.2 The Action of $\mathrm{SL}_2(K)$

We will first define the action of $\mathrm{SL}_2(K)$ on $\mathrm{Heis}(K)$. After this definition, we will spend the rest of this section motivating the formula.

For

$$g = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \in \mathrm{SL}_2(K), \quad M = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{Heis}(K),$$

define the action of g on M as

$$g\cdot A := \begin{pmatrix} 1 & ax_{11}+bx_{12} & \frac{1}{2}(ax_{11}+bx_{12})(ax_{21}+bx_{22})+c-\frac{1}{2}ab \\ 0 & 1 & ax_{21}+bx_{22} \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.5}$$

It is a straightforward (though painfully tedious) computation to check that (2.5) indeed defines an action.

Alone, the formula in (2.5) is terribly unsatisfying. We will spend the remainder of this section making sense of this formula and ultimately reformulate (2.5) in a more enlightening way. This will essentially come to explicit computations in unravelling group cohomology.

From Section 2.1, the Heisenberg group H is determined, up to isomorphism, by the short exact sequence

$$0 \to Z \to H \to Q \to 0,$$

together with a (normalized) 2-cocycle $f : Q\times Q \to Z$. Recall from equations (2.1) and (2.2) that

$$Z \cong K, \quad \text{and} \quad Q \cong K^{\oplus 2}.$$

It is natural to let $\mathrm{SL}_2(K)$ act naturally (by matrix multiplication) on Q and trivially on Z. The question now becomes: how can we extend this to an action on the Heisenberg group H?

To do this, we will need to unravel the relationship between the the second cohomology group $\mathrm{H}^2(Q,Z)$ and the Heisenberg as a central extension of Q by Z.

Let us begin by picking the most naïve section of the surjection $H \to Q$. Namely, consider

$$\sigma_1 : Q \to H, \quad (a,b) \mapsto \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}.$$

Then the induced 2-cocycle

$$f_1 : Q\times Q \to Z, \quad (q,q') \mapsto \sigma_1(q)\sigma_1(q')\sigma_1(qq')^{-1} = \begin{pmatrix} 1 & 0 & ab' \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

allows us to write H as the set $\sigma_1(Q)\times Z$, together with the multiplication rule

$$(\sigma_1(q),z)(\sigma_1(q'),z') = (\sigma_1(qq'), zz'f_1(q,q')).$$

The problem with this 2-cocycle is that it does not commute with the action of $\mathrm{SL}_2(K)$ we described at the beginning of this subsection. That is to say,

$$f_1(g\cdot q, g\cdot q') \neq f_1(q,q').$$

Hence this presentation of H does not help us to describe the action of $\mathrm{SL}_2(K)$ on H, as we wish to do.

We would like to find a choice of section so that the induced 2-cocycle is $\mathrm{SL}_2(K)$-invariant. We may view $\mathrm{SL}_2(K)$ as

$$\mathrm{SL}_2(K) = \{A \in \mathrm{GL}_2(K) : \det(A) = 1\}. \tag{2.6}$$

But what is more enlightening is the equivalent description of $\mathrm{SL}_2(K)$ as *the set of all linear transformations from $K^{\oplus 2}$ to itself that preserve a volume form*. Volume forms are alternating bilinear forms on $K^{\oplus 2}$ and are unique up to scalar multiples. Since the determinant is a volume form, then we know that the form of the 2-cocycle we would like to end up with is

$$f_2 : Q \times Q \to Z, \quad (q, q') \mapsto C \det(q, q') = C \det \begin{pmatrix} a & a' \\ b & b' \end{pmatrix} = C(ab' - a'b),$$

for $q = (a, b), q' = (a', b')$, $C \in K$. We can show that this is $\mathrm{SL}_2(K)$-invariant using the description of $\mathrm{SL}_2(K)$ in (2.6). Indeed, for any $g = \left(\begin{smallmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{smallmatrix}\right)$,

$$\begin{aligned} f_2(g \cdot q, g \cdot q') &= f_2\big((ax_{11} + bx_{12}, ax_{21} + bx_{22}), (a'x_{11} + b'x_{12}, a'x_{21} + b'x_{22})\big) \\ &= C\left((ax_{11} + bx_{12})(a'x_{21} + b'x_{22}) - (ax_{21} + bx_{22})(a'x_{11} + b'x_{12})\right) \\ &= C\left(ab'(x_{11}x_{22} - x_{12}x_{21}) - a'b(x_{11}x_{22} - x_{12}x_{21})\right) \\ &= f_2(q, q'). \end{aligned}$$

So the question we must tackle is, what is C?

Let $\sigma_2 : Q \to H$ be a section such that

$$f_2(q, q') = \sigma_2(q)\sigma_2(q')\sigma_2(qq')^{-1}. \tag{2.7}$$

Then σ_2 is of the form

$$\sigma_2 : Q \to H, \quad (a, b) \mapsto \begin{pmatrix} 1 & a & \alpha(a, b) \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix},$$

for some function $\alpha : Q \to K$. Computing the right-hand side of (2.7), we have

$$\begin{aligned} f_2(q, q') &= \begin{pmatrix} 1 & a & \alpha(a, b) \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a' & \alpha(a', b') \\ 0 & 1 & b' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a + a' & \alpha(a + a', b + b') \\ 0 & 1 & b + b' \\ 0 & 0 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 1 & 0 & \alpha(a, b) + \alpha(a', b') - \alpha(a + a', b + b') + ab' \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Therefore we would like to know for which C can we find an $\alpha : Q \to K$ such that

$$C(ab' - a'b) = f(q,q') = \alpha(a,b) + \alpha(a',b') - \alpha(a+a', b+b') + ab'.$$

Rearranging terms, we have

$$(C-1)ab' - Ca'b = \alpha(a,b) + \alpha(a',b') - \alpha(a+a', b+b').$$

Since the right-hand side is symmetric in $q = (a,b)$ and $q' = (a',b')$, then necessarily $C - 1 = -C$, and therefore we conclude that $C = \frac{1}{2}$. One can easily compute that

$$\alpha(a,b) = \frac{1}{2}ab,$$

and hence

$$\sigma_2 : Q \to H, \quad (a,b) \mapsto \begin{pmatrix} 1 & a & \frac{1}{2}ab \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

is the section that induces a 2-cocycle that is invariant under the action of $\mathrm{SL}_2(K)$.

From the above, we may write H as the set $\sigma_2(Q) \times Z$, together with the multiplication rule

$$(\sigma_2(q), z)(\sigma_2(q'), z') = (\sigma_2(qq'), zz' f_2(q,q')).$$

Then the action of $\mathrm{SL}_2(K)$ on H is

$$g \cdot (\sigma_2(q), z) = (\sigma_2(g \cdot q), z) \quad \text{for } g \in \mathrm{SL}_2(K),\, q \in Q,\, z \in Z. \tag{2.8}$$

This is exactly the same formula as (2.5), but it is immensely more illuminating and pleasant to look at! We have therefore completed our goal of extending the standard action on Q and the trivial action on Z to an action on H.

Remark. We can show that f_1 and f_2 differ only by a 2-coboundary of Q with respect to Z. Indeed, letting

$$\omega : Q \times Q \to Z, \quad (q,q') \mapsto \alpha(q) + \alpha(q') - \alpha(qq'),$$

where, as before,

$$\alpha : Q \to K, \quad (a,b) \mapsto \frac{1}{2}ab,$$

we have, for $q = (a,b)$ and $q' = (a',b')$,

$$\omega(q,q') = \frac{1}{2}ab + \frac{1}{2}a'b' - \frac{1}{2}(a+a')(b+b') = -\frac{1}{2}ab' - \frac{1}{2}a'b,$$

and finally,

$$f_2(q,q') = f_1(q,q') + \omega(q,q'). \tag{2.9}$$

Hence what we have done here is unravel the group cohomological interpretation of the Heisenberg group. ◊

2.3 The Heisenberg Representation

We will now define the representation which will be the springboard of this paper. For an irreducible character $\psi : \mathrm{Z}(\mathrm{Heis}(K)) \to \mathbb{C}$, define a representation $(\pi_\psi, \mathscr{L}^2(K))$ of $\mathrm{Heis}(K)$ by the following action on $f(x) \in \mathscr{L}^2(K)$:

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \cdot f(x) := \psi(-bx + c) f(x - a). \tag{2.10}$$

In particular, we have

$$\begin{aligned} \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot f(x) &= f(x-a), \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \cdot f(x) &= \psi(-bx) f(x), \\ \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot f(x) &= \psi(c) f(x). \end{aligned} \tag{2.11}$$

It is straightforward to check that (2.10) defines an action. Indeed,

$$\begin{aligned} \left(\begin{smallmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{smallmatrix}\right) \left(\left(\begin{smallmatrix} 1 & a' & c' \\ 0 & 1 & b' \\ 0 & 0 & 1 \end{smallmatrix}\right) \cdot f(x) \right) &= \left(\begin{smallmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{smallmatrix}\right) \cdot \psi(-b'x + c') f(x - a') \\ &= \psi(-bx + c)\psi(-b'(x-a) + c') f(x - a' - a) \\ &= \psi(-(b+b')x + c + c' + ab') f(x - (a + a')) \\ &= \left(\begin{smallmatrix} 1 & a+a' & c+c'+ab' \\ 0 & 1 & b+b' \\ 0 & 0 & 1 \end{smallmatrix}\right) \cdot f(x) \\ &= \left(\left(\begin{smallmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & a' & c' \\ 0 & 1 & b' \\ 0 & 0 & 1 \end{smallmatrix}\right) \right) \cdot f(x). \end{aligned}$$

Therefore the group action in (2.10) defines a representation

$$\pi_\psi : \mathrm{Heis}(K) \to \mathrm{GL}(\mathscr{L}^2(K)), \quad A \mapsto T_A, \tag{2.12}$$

where $T_A(f) := A \cdot f$.

Before proceeding, we need to specify what we mean by $\mathscr{L}^2(K)$. As we will only discuss the cases $K = \mathbb{F}_p$ and $K = \mathbb{R}$, we will only discuss what we mean by $\mathscr{L}^2(K)$ in these situations.

- For $K=\mathbb{F}_p$, $\mathscr{L}^2(K)$ has an obvious meaning. By definition,

$$\mathscr{L}^2(K) := \left\{ f : K \to \mathbb{C} : \int_K |f(x)|^2 \, dx < \infty \right\}, \tag{2.13}$$

  and for $K=\mathbb{F}_p$, we take the discrete measure, and the above becomes

$$\mathscr{L}^2(\mathbb{F}_p) = \left\{ f : \mathbb{F}_p \to \mathbb{C} : \sum_{x \in \mathbb{F}_p} |f(x)|^2 < \infty \right\}.$$

  But since $\mathbb{F}_p$ is finite, then we in fact have

$$\mathscr{L}^2(\mathbb{F}_p) = \{ f : \mathbb{F}_p \to \mathbb{C} \}.$$

  This has a natural structure as a p-dimensional vector space over $\mathbb{C}$.

- For $K=\mathbb{R}$, to make sense of (2.13), we need to pick a measure. We will use the Lebesgue measure; call it $d\mu$. The measure space $\mathscr{L}^2(\mathbb{R})$ is a Hilbert space with respect to the inner product

$$\langle f, g \rangle := \int_{\mathbb{R}} f(x)\overline{g(x)} \, d\mu. \tag{2.14}$$

  In this paper, when we write $\mathscr{L}^2(\mathbb{R})$, we will always mean *the space of square-integrable* $\mathbb{C}$*-valued functions on* $\mathbb{R}$ *with respect to the Lebesgue measure*. That is,

$$\mathscr{L}^2(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{C} : \int_{\mathbb{R}} |f(x)|^2 \, d\mu < \infty \right\}.$$

Remark. We will often not care about the entire space $\mathrm{GL}(\mathscr{L}^2(\mathbb{R}))$. This space is extremely large and is no particularly well-behaved. As our representations will be unitary representations (that is, the action of the representation preserves a non-degenerate Hermitian form), then our group will act by operators in

$$\mathrm{Isom}(\mathscr{L}^2(\mathbb{R})) := \left\{ \phi : \mathscr{L}^2(\mathbb{R}) \to \mathscr{L}^2(\mathbb{R}) : \langle \phi(f), \phi(g) \rangle = \langle f, g \rangle \right\},$$

where the norm is the $\mathscr{L}^2$ norm obtained from the inner product formula in (2.14). To discuss unitary representations, we may replace $\mathrm{GL}(\mathscr{L}^2(\mathbb{R}))$ with $\mathrm{Isom}(\mathscr{L}^2(\mathbb{R}))$. ♢

Remark. It is easy to verify that the Heisenberg representation defined in equation (2.12) preserves the $\mathscr{L}^2$ inner product defined in (2.14). ♢

3 The Finite-Field Model

In this section, we will explicitly construct the Weil representation of $\mathrm{SL}_2(\mathbb{F}_p)\mathbb{F}_p$. Modulo the analytic problems that arise in passing from the theory of finite-dimensional representations to the theory of infinite-dimensional unitary representations, our work in this section will extend almost word-for-word to the construction of the Weil representation of $\mathrm{SL}_2(\mathbb{R})$.

3.1 The Representation Theory of $\mathrm{Heis}(\mathbb{F}_p)$

By using arithmetic properties of the dimension of the representations of finite groups and the fact that $\mathrm{Heis}(\mathbb{F}_p)$ is a non-abelian group of order p^3, we can deduce that there are:

(a) p^2 representations of dimension 1, and

(b) $p-1$ representations of dimension p.

The one-dimensional representations are obtained from inflating the one-dimensional representations of $Q \cong \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$, so in particular, in these p^2 representations, the center Z acts trivially. Now let ψ be a nontrivial irreducible character of $Z \cong \mathbb{Z}/p\mathbb{Z}$. The induced representation $\mathrm{Ind}_Z^G(\psi)$ is a representation of dimension $[G:Z] = p^2$, and by the semisimplicity of the group algebra $\mathbb{C}[H]$ viewed as a module over itself, we know that $\mathrm{Ind}_Z^G(\chi)$ breaks up into a direct sum of irreducible representations. By Clifford theory, all irreducible representations that occur inside $\mathrm{Ind}_Z^G(\psi)$ must occur with the same multiplicity. From this, we know that $\mathrm{Ind}_Z^G(\psi)$ is just p copies of some p-dimensional irreducible representation. This construction gives us $p-1$ necessarily non-isomorphic representations of H, each distinguished by the action of $Z = \mathrm{Z}(H)$.

The above can be summarized in the following proposition.

Proposition 3.1.

(a) The one-dimensional representations of H are obtained by inflating the linear representations of $Q = H/Z$.

(b) For each nontrivial character ψ of $Z = \mathrm{Z}(H)$, there exists a unique, up to isomorphism, irreducible representation of H such that Z acts by ψ. Each of these representations is of dimension p, and hence ρ is a p-dimensional irreducible representation of H, it is completely determined by its restriction $\rho|_Z$.

Remark. Proposition 3.1(b) is the finite-field equivalent of the Stone-von Neumann theorem from functional analysis. ◊

The relevance of the above to our discussion is the following. Since $\mathscr{L}^2(\mathbb{F}_p)$ is a p-dimensional $\mathbb{C}$-vector space, then the representation $(\pi_\psi, \mathscr{L}^2(\mathbb{F}_p))$ we defined in Section 2 is p-dimensional. As described in (2.11), the center $Z = \left\{ \left(\begin{smallmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right) : b \in \mathbb{F}_p \right\}$ acts nontrivially if ψ is a nontrivial character, and hence by the proposition, necessarily π_ψ is irreducible.

3.2 Constructing a Map $\mathrm{SL}_2(\mathbb{F}_p) \to \mathrm{PGL}_p(\mathbb{C})$

Now, for any $g \in \mathrm{SL}_2(\mathbb{F}_p)$, the composition $\pi_\psi \circ g$ is a representation of $\mathrm{Heis}(\mathbb{F}_p)$. Since precomposition by the action of $\mathrm{SL}_2(\mathbb{F}_p)$ does not change the action of Z on $\mathscr{L}^2(\mathbb{F}_p)$ (since $\mathrm{SL}_2(\mathbb{F}_p)$ acts trivially on Z), then by the discussion in Section 3.1, as representations, $\pi_\psi \circ g \cong \pi_\psi$. By Schur's lemma, $\mathrm{Aut}_{\mathbb{C}[H]}(\pi_\psi) = \mathbb{C}^\times$, and hence for isomorphisms

$$\Phi_1, \Phi_2 : (\pi_\psi, \mathscr{L}^2(\mathbb{F}_p)) \to (\pi_\psi \circ g, \mathscr{L}^2(\mathbb{F}_p)),$$

the composition

$$\Phi_1^{-1} \circ \Phi_2 : (\pi_\psi, \mathscr{L}^2(\mathbb{F}_p)) \to (\pi_\psi, \mathscr{L}^2(\mathbb{F}_p))$$

is a $\mathbb{C}[H]$-automorphism of $(\pi_\psi, \mathscr{L}^2(\mathbb{F}_p))$, and hence

$$\Phi_1^{-1} \circ \Phi_2 = \lambda I, \quad \text{for some } \lambda \in \mathbb{C}^\times.$$

So the intertwining operators Φ_1 and Φ_2 must only differ by a scalar, and this shows that the intertwining operator between π_ψ and $\pi_\psi \circ g$ is uniquely defined in $\mathrm{PGL}_p(\mathbb{C})$. We therefore have a well-defined map

$$\rho : \mathrm{SL}_2(\mathbb{F}_p) \to \mathrm{PGL}_p(\mathbb{C}). \tag{3.1}$$

We will ultimately show, in Section 3.5, that ρ lifts to a map $\widetilde{\rho} : \mathrm{SL}_2(\mathbb{F}_p) \to \mathrm{GL}_p(\mathbb{C})$.

We may compute explicitly the projective representation $\rho : \mathrm{SL}_2(\mathbb{F}_p) \to \mathrm{PGL}_p(\mathbb{C})$. It can be shown that $\mathrm{SL}_2(\mathbb{F}_p)$ is generated by

$$s := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad t := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \tag{3.2}$$

(In fact, these two elements also generate $\mathrm{SL}_2(\mathbb{Z})$.) We will compute the images of the two generating elements in (3.2).

Since $\mathrm{SL}_2(\mathbb{F}_p)$ acts trivially on $Z = \mathrm{Z}(H)$, then all we need to do is to find a map $\varphi_g : \mathscr{L}^2(\mathbb{F}_p) \to \mathscr{L}^2(\mathbb{F}_p)$, for $g = s, t$, such that the following square commutes for $X = \left(\begin{smallmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{smallmatrix}\right)$:

$$\begin{array}{ccc}
\mathscr{L}^2(\mathbb{F}_p) & \xrightarrow{\pi_\psi(X)} & \mathscr{L}^2(\mathbb{F}_p) \\
\Big\downarrow \varphi_g & & \Big\downarrow \varphi_g \\
\mathscr{L}^2(\mathbb{F}_p) & \xrightarrow[\pi_\psi(g\cdot X)]{} & \mathscr{L}^2(\mathbb{F}_p)
\end{array}$$

Recall here that φ_g is only unique up to a scalar. In the computations to come, we will compute $\varphi_g \in \mathrm{GL}_p(\mathbb{C})$, which will give us an element in $\mathrm{PGL}_p(\mathbb{C})$ via the surjection in the short exact sequence

$$0 \to \mathbb{C}^\times \to \mathrm{GL}_p(\mathbb{C}) \to \mathrm{PGL}_p(\mathbb{C}) \to 0.$$

For convenience, we will write

$$A := \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}.$$

We begin with computations for $s = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. In this situation,

$$s \cdot \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -a \\ 0 & 0 & 1 \end{pmatrix}, \quad s \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & b & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

and hence

$$\pi_\psi(A) : f(x) \mapsto f(x-a), \qquad \pi_\psi(B) : f(x) \mapsto \psi(-bx) f(x) \tag{3.3}$$

$$\pi_\psi \circ s(A) : f(x) \mapsto \psi(ax) f(x), \qquad \pi_\psi \circ s(B) : f(x) \mapsto f(x-b). \tag{3.4}$$

This means that we are looking for a function $\varphi_s : \mathscr{L}^2(\mathbb{F}_p) \to \mathscr{L}^2(\mathbb{F}_p)$ such that multiplying by a character gets swapped with translation. The first guess would be something analogous to the Fourier transform, since this is the function

$$\hat{\ } : \mathscr{L}^2(\mathbb{R}) \to \mathscr{L}^2(\mathbb{R}), \quad f(x) \mapsto \hat{f}(\xi) := \int_{-\infty}^{\infty} e^{-2\pi i x \xi} f(x)\, dx,$$

and it is easy to compute that this function satisfies the properties:

$$\hat{h}(\xi) = e^{-2\pi i x_0 \xi} \hat{f}(\xi), \qquad \text{where } h(x) = f(x - x_0), \tag{3.5}$$

$$\hat{h}(\xi) = \hat{f}(\xi - \xi_0), \qquad \text{where } h(x) = e^{2\pi i x \xi_0} f(x). \tag{3.6}$$

Here, $e^{2\pi i x\xi} =: \chi(x)$ is a character of the real numbers viewed as an additive group, which will denote by $\mathbb{R}^+$. This plays the role of ψ in our situation. To form a finite-field equivalent of the Fourier transform, we define the following, which we will also call $\hat{}$ to emphasize the analogy between $\mathbb{F}_p$ and $\mathbb{R}$. Let

$$\hat{} : \mathscr{L}^2(\mathbb{F}_p) \to \mathscr{L}^2(\mathbb{F}_p), \quad f(x) \mapsto \hat{f}(\xi) := \sum_{x \in \mathbb{F}_p} \psi(\xi x) f(x).$$

We will prove that this function has properties that are exactly the finite-field formulations of the properties listed in equations (3.5) and (3.6). For $h(x) = f(x - x_0)$,

$$\begin{aligned} \hat{h}(\xi) &= \sum_{x \in \mathbb{F}_p} \psi(\xi x) f(x - x_0) \\ &= \sum_{x \in \mathbb{F}_p} \psi(\xi(x + x_0)) f(x) \\ &= \psi(\xi x_0) \sum_{x \in \mathbb{F}_p} \psi(\xi x) f(x) \\ &= \psi(\xi x_0) \hat{f}(\xi). \end{aligned}$$

For $h(x) = \psi(-\xi_0 x) f(x)$,

$$\begin{aligned} \hat{h}(\xi) &= \sum_{x \in \mathbb{F}_p} \psi(\xi x) \psi(-\xi_0 x) f(x) \\ &= \sum_{x \in \mathbb{F}_p} \psi((\xi - \xi_0) x) f(x) \\ &= \hat{f}(\xi - \xi_0). \end{aligned}$$

Therefore

$$\hat{h}(\xi) = \psi(x_0 \xi) \hat{f}(\xi), \qquad \text{where } h(x) = f(x - x_0), \tag{3.7}$$

$$\hat{h}(\xi) = \hat{f}(\xi - \xi_0), \qquad \text{where } h(x) = \psi(-\xi_0 x) f(x). \tag{3.8}$$

and these are the finite-field equivalents of (3.5) and (3.6).

Looking back at equations (3.3) and (3.4), we see that we have actually found our desired function $\varphi_s : \mathscr{L}^2(\mathbb{F}_p) \to \mathscr{L}^2(\mathbb{F}_p)$. To be precise, set

$$\varphi_s : \mathscr{L}^2(\mathbb{F}_p) \to \mathscr{L}^2(\mathbb{F}_p), \quad f(x) \mapsto \hat{f}(x),$$

and then indeed, by (3.7), we have

$$f(x) \xmapsto{\pi_\psi(A)} f(x - a) \xmapsto{\varphi_s} \psi(ax) \hat{f}(x),$$

$$f(x) \xmapsto{\varphi_s} \hat{f}(x) \xmapsto{\pi_\psi \circ g(A)} \psi(ax) \hat{f}(x),$$

and by (3.8), we have

$$f(x) \xmapsto{\pi_\psi(B)} \psi(-bx)f(x) \xmapsto{\varphi_s} \hat{f}(x-b),$$
$$f(x) \xmapsto{\varphi_s} \hat{f}(x) \xmapsto{\pi_\psi \circ s(B)} \hat{f}(x-b).$$

This proves exactly what we wanted to prove, and so in fact, our first intuition of constructing something analogous to the Fourier transform was correct!

We may also write this in terms of a matrix in $\mathrm{GL}_p(\mathbb{C}) = \mathrm{GL}(\mathscr{L}^2(\mathbb{F}_p))$. We will write this matrix with respect to the basis

$$\left\{f_i(x) := \delta_{ix} : i \in \mathbb{F}_p\right\}, \quad \text{where } \delta_{ix} = \begin{cases} 1 & \text{if } x = i, \\ 0 & \text{otherwise.} \end{cases} \tag{3.9}$$

Then

$$\varphi_s = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \psi(1) & \psi(2) & \cdots & \psi(p-1) \\ 1 & \psi(1)^2 & \psi(2)^2 & \cdots & \psi(p-1)^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \psi(1)^{p-1} & \psi(2)^{p-1} & \cdots & \psi(p-1)^{p-1} \end{pmatrix}.$$

The determinant of ϕ_s is nonzero since ϕ_s is a Vandermonde matrix.

Remark. Note that (3.7) and (3.8) are not exactly the same as (3.5) and (3.6). Indeed, they differ by a sign in terms of which one switches the sign in the transition between function translation and scaling by a character. We certainly could have defined our finite-field analogue of the Fourier transform so that the aforementioned equations agreed exactly. This would be

$$\text{F.T.} : \mathscr{L}^2(\mathbb{F}_p) \to \mathscr{L}^2(\mathbb{F}_p), \quad f(x) \mapsto F(\xi) := \sum_{x \in \mathbb{F}_p} \psi(-\xi x) f(x).$$

This operator gives, up to a scalar, the image of $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. ◊

Now we move to compute φ_t for $t = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$. In this situation,

$$t \cdot \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a & \frac{1}{2}a^2 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix}, \quad t \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence

$$\pi_\psi(A) : f(x) \mapsto f(x-a), \qquad \pi_\psi(B) : f(x) \mapsto \psi(-bx)f(x), \tag{3.10}$$
$$\pi_\psi \circ t(A) : f(x) \mapsto \psi(-ax + \tfrac{1}{2}a^2) f(x-a), \qquad \pi_\psi \circ t(B) : f(x) \mapsto \psi(-bx)f(x). \tag{3.11}$$

Comparing $\pi_\psi(A)$ and $\pi_\psi \circ t(A)$, we can expect to have some sort of twisting by $\psi(x^2)$. Consider

$$\phi_c : \mathscr{L}^2(\mathbb{F}_p) \to \mathscr{L}^2(\mathbb{F}_p), \quad f(x) \mapsto \psi(cx^2)f(x).$$

Then we have

$$\begin{aligned} f(x) &\xmapsto{\pi_\psi(A)} f(x-a) \xmapsto{\phi_c} \psi(cx^2)f(x-a), \\ f(x) &\xmapsto{\phi_c} \psi(cx^2)f(x) \xmapsto{\pi_\psi \circ t(A)} \psi(-\tfrac{1}{2}a^2+ax)\psi(c(x-a)^2)f(x-a). \end{aligned}$$

Equating the right-hand side of the two compositions and using the fact that ψ is a linear character and hence a homomorphism, we get

$$cx^2 = -\frac{1}{2}a^2 + ax + c(x^2 - 2ax + a^2).$$

Massaging this equation, we obtain

$$0 = -\frac{1}{2}a^2 + ax + 2c\left(-\frac{1}{2}a^2 + ax\right),$$

which allows us to conclude that

$$c = -\frac{1}{2},$$

and hence

$$\varphi_t(f(x)) = \phi_{-\frac{1}{2}}(f(x)) = \psi(-\tfrac{1}{2}x^2)f(x).$$

Using the basis described in (3.9), we obtain the following matrix formulation of φ_t:

$$\begin{aligned} \varphi(t) &= \operatorname{diag}\left(1, \psi\left(-\frac{1^2}{2}\right), \psi\left(-\frac{2^2}{2}\right), \cdots, \psi\left(-\frac{(p-1)^2}{2}\right)\right) \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \psi(-\frac{1^2}{2}) & 0 & 0 & \cdots & 0 \\ 0 & 0 & \psi(-\frac{2^2}{2}) & 0 & \cdots & 0 \\ 0 & 0 & 0 & \psi(-\frac{3^2}{2}) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \psi(-\frac{(p-1)^2}{2}) \end{pmatrix} \end{aligned}$$

3.3 Lifting Projective Representations

Ultimately, we would like to lift the map $\rho : \mathrm{SL}_2(\mathbb{F}_p) \to \mathrm{PGL}_p(\mathbb{C})$ to a map $\widetilde{\rho} : \mathrm{SL}_2(\mathbb{F}_p) \to \mathrm{GL}_p(\mathbb{C})$. In representation theory, the problem of lifting projective representations of a group G to a linear representation of G is one that has been seriously studied in the past, and is still studied today. The major breakthrough in this area was Schur's work on something called the "Schur Multiplier." To remain honest to the goal of making this paper as explicit and approachable as possible, we will explore this topic exactly as the verb suggests: by exploration, examination, and ultimately, discovery.

This section will have a very similar flavor to that of Section 2.1. More precisely, much of our discussion will essentially be an unravelling of group cohomology. As this topic holds for any group G, and nothing is gained by specifying to the case when $G = \mathrm{SL}_2(\mathbb{F}_p)$, we will not limit ourselves to a specific group and instead let G be any one of your favorite groups.

Let $\rho : G \to \mathrm{PGL}_n(\mathbb{C})$ be a group homomorphism. If we can lift this representation, then this means that for any section $\sigma : \mathrm{PGL}_n(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$ of the surjection $\mathrm{GL}_n(\mathbb{C}) \to \mathrm{PGL}_n(\mathbb{C})$, there exists a function $c : G \times G \to \mathbb{C}^\times$ such that

$$L(gh) = c(g,h)L(g)L(h), \quad \text{for } g,h \in G, \tag{3.12}$$

where $L := \sigma \circ \rho : G \to \mathrm{GL}_n(\mathbb{C})$. By this rule, we obtain

$$c(g_1, g_2g_3)L(g_1)L(g_2g_3) = L(g_1g_2g_3) = c(g_1g_2, g_3)L(g_1g_2)L(g_3).$$

Expanding $L(g_2g_3)$ on the left-hand side and $L(g_1g_2)$ on the right-hand side according to (3.12), we obtain

$$c(g_1, g_2g_3)L(g_1)(c(g_2, g_3)L(g_2)L(g_3)) = c(g_1g_2, g_3)(c(g_1, g_2)L(g_1)L(g_2))L(g_3).$$

Hence we may conclude that

$$c(g_1, g_2g_3)c(g_2, g_3) = c(g_1g_2, g_3)c(g_1, g_2).$$

But this is exactly the condition for a 2-cocycle of G with respect to the abelian group $\mathbb{C}^\times$ (with a trivial group action), so indeed c is a 2-cocycle.

In the above, notice the dependence of the 2-cocycle $c : G \times G \to \mathbb{C}^\times$ on the choice of section σ. For a different section $\sigma' : \mathrm{PGL}_n(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$, we obtain a different cocycle $c' : G \times G \to \mathbb{C}^\times$ satisfying an analogous equation to (3.12). Explicitly, we have

$$L'(gh) = c'(g,h)L'(g)L'(h), \quad \text{for } g,h \in G, \tag{3.13}$$

where $L' := \sigma' \circ \rho$. Now, since, for any $g \in G$, $\sigma(g)$ and $\sigma'(g)$ can differ only by a (nonzero) scalar, we may write

$$L'(g) = f(g)L(g), \quad \text{for some } f : G \to \mathbb{C}^\times.$$

Substituting this into (3.13), we get

$$f(gh)L(gh) = c'(g,h)f(g)L(g)f(h)L(h).$$

It follows from (3.12) that

$$c'(g,h) = f(gh)f(g)^{-1}f(h)^{-1}c(g,h).$$

But notice that the function

$$F : G \times G \to \mathbb{C}^\times, \quad (g,h) \mapsto f(gh)f(g)^{-1}f(h)^{-1}$$

defines a 2-coboundary of G with respect to the abelian group $\mathbb{C}^\times$ (again with a trivial group action), and hence we have shown that changing the choice of a section of $\mathrm{GL}_n(\mathbb{C}) \to \mathrm{PGL}_n(\mathbb{C})$ only changes the induced 2-cocycle $c : G \times G \to \mathbb{C}^\times$ by a 2-coboundary. In particular, we have shown that any lift L of $G \to \mathrm{PGL}_n(\mathbb{C})$ determines a unique class in the second cohomology $\mathrm{H}^2(G, \mathbb{C}^\times)$. Because there is a bijective correspondence between $\mathrm{H}^2(G, \mathbb{C}^\times)$ and extensions of G by $\mathbb{C}^\times$ (see Section 2.1), this means that there exists a short exact sequence

$$0 \to \mathbb{C}^\times \to \widetilde{G} \to G \to 0,$$

fitting in the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{C}^\times & \longrightarrow & \widetilde{G} & \longrightarrow & G & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1_{\mathbb{C}^\times}} & & \downarrow{\scriptstyle \widetilde{\rho}} & & \downarrow{\scriptstyle \rho} & & \\
0 & \longrightarrow & \mathbb{C}^\times & \longrightarrow & \mathrm{GL}_n(\mathbb{C}) & \longrightarrow & \mathrm{PGL}_n(\mathbb{C}) & \longrightarrow & 0
\end{array} \tag{3.14}$$

Remark. A more algebraic way of obtaining the above commutative diagram is by taking $\widetilde{G}$ to be the pullback of $\rho : G \to \mathrm{PGL}_n(\mathbb{C})$ and $p : \mathrm{GL}_n(\mathbb{C}) \to \mathrm{PGL}_n(\mathbb{C})$. Explicitly, we let

$$\widetilde{G} := G \times_{\mathrm{PGL}_n(\mathbb{C})} \mathrm{GL}_n(\mathbb{C}) = \{(g, A) \in G \times \mathrm{GL}_n(\mathbb{C}) : \rho(g) = p(A)\}. \tag{3.15}$$

Letting $\widetilde{p}$ and $\widetilde{\rho}$ be the projection of $\widetilde{G}$ onto the G component and the $\mathrm{GL}_n(\mathbb{C})$ component, respectively, we obtain the following commuting diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker(\widetilde{p}) & \longrightarrow & \widetilde{G} & \xrightarrow{\widetilde{p}} & G & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \widetilde{\rho}} & & \downarrow{\scriptstyle \rho} & & \\
0 & \longrightarrow & \mathbb{C}^\times & \longrightarrow & \mathrm{GL}_n(\mathbb{C}) & \xrightarrow{p} & \mathrm{PGL}_n(\mathbb{C}) & \longrightarrow & 0
\end{array}$$

In the above, $\ker(\widetilde{p})$ is taken to be the kernel of the map $\widetilde{p} : \widetilde{G} \to G$, and the vertical map $\ker(\widetilde{p}) \to \mathbb{C}^\times$ is the restriction of $\widetilde{\rho}$. Now,

$$\ker(\widetilde{p}) = \left\{(g, A) \in \widetilde{G} : g = 1\right\},$$

and since $\widetilde{G}$ is a fibered product (see (3.15)), then this forces $A \in \mathrm{GL}_n(\mathbb{C})$ to be such that the image of A in $\mathrm{PGL}_n(\mathbb{C})$ is 1. Hence we have an isomorphism

$$\ker(\widetilde{p}) = \{(g, A) : g = 1, A = \lambda I_n, \text{ for some } \lambda \in \mathbb{C}^\times\} \to \mathbb{C}^\times, \quad (1, \lambda I) \mapsto \lambda$$

From the fact that $\widetilde{\rho}$ is the projection onto the $\mathrm{GL}_n(\mathbb{C})$ component of $G \times_{\mathrm{PGL}_n(\mathbb{C})} \mathrm{GL}_n(\mathbb{C})$, it follows that the vertical map $\ker(\widetilde{p}) \to \mathbb{C}^\times$ is given by $(1, \lambda 1) \mapsto \lambda$, and hence viewing $\ker(\widetilde{p})$ as $\mathbb{C}^\times$, we obtain exactly the diagram in (5.18). ♢

Remark. The map $\widetilde{G} \to G$ splits if and only if the corresponding cocycle class in $\mathrm{H}^2(G, \mathbb{C}^\times)$ is the trivial class. ♢

Returning briefly to the case when $G = \mathrm{SL}_2(\mathbb{F}_p)$, recall that we want to lift our projective representation to a linear representation *of* G, not just to an extension of G. To prove this, it would be sufficient to show that every extension of G by $\mathbb{C}^\times$ splits, or equivalently, $\mathrm{H}^2(G, \mathbb{C}^\times) = 0$.

Group cohomology is a tricky thing to compute. Because of its complexity, it is good to have multiple interpretations of the same object. We return to the case when G is any finite group to discuss this.

We call upon the Universal Coefficient Theorem.

Theorem 3.2 (Universal Coefficient Theorem)**.** *Let A be any abelian group with a trivial G-action. Then the following sequence is exact:*

$$0 \to \mathrm{Ext}^1(\mathrm{H}_{n-1}(G, \mathbb{Z}), A) \to \mathrm{H}^n(G, A) \to \mathrm{Hom}(\mathrm{H}_n(G, \mathbb{Z}), A) \to 0.$$

Proof. See Hatcher's *Algebraic Topology* Theorem 3.2. The map $\mathrm{H}^n(G, A) \to \mathrm{Hom}(\mathrm{H}_n(G, \mathbb{Z}), A)$ is described on page 191. □

We are only concerned with the specialization of the Universal Coefficient Theorem to $n = 2$. Since $\mathbb{C}^\times$ is an injective $\mathbb{Z}$-module (indeed, it is a divisible group),

$$\mathrm{Ext}^1(\mathrm{H}_1(G, \mathbb{Z}), \mathbb{C}^\times) = 0,$$

and hence for $A = \mathbb{C}^\times$, the Universal Coefficient Theorem gives us

$$\mathrm{H}^2(G, \mathbb{C}^\times) \cong \mathrm{Hom}(\mathrm{H}_2(G, \mathbb{Z}), \mathbb{C}^\times). \tag{3.16}$$

The second homology group $\mathrm{H}_2(G, \mathbb{Z})$ is called the *Schur Multiplier*. It is in the sense of (3.16) that studying $\mathrm{H}_2(G, \mathbb{Z})$ is "enough" to understand how to lift projective representations of G.

Remark. Since $\mathrm{H}_2(G,\mathbb{Z})$ is a finite abelian group,

$$\mathrm{Hom}(\mathrm{H}_2(G,\mathbb{Z}),\mathbb{C}^\times) \cong \mathrm{H}_2(G,\mathbb{Z}), \tag{3.17}$$

and hence we may conclude that

$$\mathrm{H}^2(G,\mathbb{C}^\times) \cong \mathrm{H}_2(G,\mathbb{Z}). \tag{3.18}$$

But the isomorphism (3.17) is highly non-canonical, which is why we have pushed this statement to a remark. ◊

3.4 The Schur Multiplier of $\mathrm{SL}_2(\mathbb{F}_p)$

The goal of this section is to prove the following theorem.

Theorem 3.3. *For odd primes p, the Schur multiplier of $\mathrm{SL}_2(\mathbb{F}_p)$ is trivial. That is,*

$$\mathrm{H}_2(\mathrm{SL}_2(\mathbb{F}_p),\mathbb{Z}) = 0.$$

This relies on a series of propositions.

Proposition 3.4. *Let G be a group and $H \subseteq G$ a subgroup with finite index in G. If $[G:H]$ is coprime to ℓ for some prime ℓ, then the map*

$$\mathrm{H}_n(G,\mathbb{Z})_\ell \to \mathrm{H}_n(H,\mathbb{Z})_\ell$$

is injective. Here, $\mathrm{H}_n(G,\mathbb{Z})_\ell$ denotes the ℓ-component of the finite abelian group $\mathrm{H}_n(G,\mathbb{Z})$.

Proof. We just need that the composition

$$\mathrm{H}_n(G,\mathbb{Z}) \to \mathrm{H}_n(H,\mathbb{Z}) \to \mathrm{H}_n(G,\mathbb{Z})$$

is the map given by multiplication by $[G:H]$. This can be shown by viewing homology as the derived functor of taking coinvariants. This is done in detail in Chapter 6 of Weibel's *An Introduction to Homological Algebra* [Wei94]. □

We will apply Proposition 3.4 to the case when $G = \mathrm{SL}_2(\mathbb{F}_p)$ and H is an ℓ-Sylow subgroup.

Proposition 3.5. *For ℓ an odd prime,*

$$\mathrm{H}_2(\mathrm{SL}_2(\mathbb{F}_p),\mathbb{Z})_\ell = 0.$$

We first need the following lemma.

Lemma. *For $q = p^n$,*

$$|\mathrm{SL}_2(\mathbb{F}_q)| = q(q-1)(q+1).$$

Proof of Lemma. Pick a basis v_1, v_2 of $\mathbb{F}_q^{\oplus 2}$. Let T be an element of $\mathrm{SL}_2(\mathbb{F}_q)$ and assume that

$$T(v_1) = \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in \mathbb{F}_q^{\oplus 2}.$$

Necessarily x and y are not both zero, since T is invertible. Furthermore, since T is invertible, the image of v_2 must be linearly independent to (x, y). There are $q^2 - q$ vectors linearly independent to (x, y). But since $T \in \mathrm{SL}_2(\mathbb{F}_q)$, then for any vector (x', y') linearly independent to (x, y), there exists a constant $c \in \mathbb{F}_q$ such that the transformation taking $v_1 \mapsto (x, y)$ and $v_2 \mapsto c(x', y')$ is an element of $\mathrm{SL}_2(\mathbb{F}_q)$. In this sense, picking a vector linearly independent from (x', y') only determines the line spanned by the image of v_2. The determinant condition then determines the exact image of v_2. Therefore for a given image of v_1, we have $\frac{p^2-p}{p-1}$ possible images of v_2. Since there are $p^2 - 1$ possible images of v_1, then we obtain

$$|\mathrm{SL}_2(\mathbb{F}_q)| = (p^2 - 1)\left(\frac{p^2 - p}{p - 1}\right) = p(p-1)(p+1),$$

as desired. □

Proof of Proposition 3.5. Since ℓ is an odd prime, ℓ divides at most one of p, $p-1$, and $p+1$, and ℓ is the highest power of ℓ that divides $p(p-1)(p+1)$. Therefore, using the lemma and Cauchy's theorem (if $p \,\big|\, |G|$, then G has an element of order p), we may conclude that $\mathrm{SL}_2(\mathbb{F}_p)$ contains a cyclic subgroup of size ℓ. By the Sylow theorems, the ℓ-Sylow subgroups of G are all conjugate to each other. Hence we may conclude that an ℓ-Sylow of $\mathrm{SL}_2(\mathbb{F}_p)$ is cyclic for any odd prime ℓ.

The integral homology of a cyclic group is easy to compute. It is left to the reader to show

$$\mathrm{H}_n(\mathbb{Z}/\ell\mathbb{Z}, \mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{for } n = 0 \\ \mathbb{Z}/\ell\mathbb{Z}, & \text{for } n = 1, 3, 5, \ldots \\ 0, & \text{for } n = 2, 4, 6, \ldots \end{cases}$$

In particular, we have

$$\mathrm{H}_2(\mathbb{Z}/\ell\mathbb{Z}, \mathbb{Z})_\ell = 0,$$

and by Proposition 3.4,

$$\mathrm{H}_2(\mathrm{SL}_2(\mathbb{F}_p), \mathbb{Z})_\ell = 0.$$ □

Proposition 3.6. *The 2-component of the finite abelian group $\mathrm{H}_2(\mathrm{SL}_2(\mathbb{F}_p))$ is trivial. That is,*

$$\mathrm{H}_2(\mathrm{SL}_2(\mathbb{F}_p), \mathbb{Z})_2 = 0.$$

Proof. We omit this proof as it is tangential to the arc of the Weil representation story. For the interested reader, see Steve Mitchell's course notes on Equivariant Cohomology of Finite Group Actions. In *Some Background Material* [Mit], he gives the proof that

$$H_2(SL_2(\mathbb{F}_p), \mathbb{F}_2) = 0.$$

It follows that the 2-component of the finite abelian group $H_2(SL_2(\mathbb{F}_p), \mathbb{Z})$ is trivial. $\square$

3.5 The Weil Representation of $SL_2(\mathbb{F}_p)$

In the previous section, we showed

$$H_2(SL_2(\mathbb{F}_p), \mathbb{Z}) = 0.$$

By (3.16), this means that

$$H^2(SL_2(\mathbb{F}_p), \mathbb{C}^\times) \cong \mathrm{Hom}(H_2(SL_2(\mathbb{F}_p), \mathbb{Z}), \mathbb{C}^\times) = 0.$$

Therefore every central extension of $SL_2(\mathbb{F}_p)$ by $\mathbb{C}^\times$ splits. The following proposition follows.

Proposition 3.7. *Every (finite-dimensional) projective representation of* $SL_2(\mathbb{F}_p)$ *lifts to a linear representation of* $SL_2(\mathbb{F}_p)$.

Proof. Let $G = SL_2(\mathbb{F}_p)$ and let $\rho : G \to PGL_n(\mathbb{C})$ be a group homomorphism. Then we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{C}^\times & \longrightarrow & \widetilde{G} & \xrightarrow{p} & G & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1_{\mathbb{C}^\times}} & & \downarrow{\scriptstyle \widetilde{\rho}} & & \downarrow{\scriptstyle \rho} & & \\ 0 & \longrightarrow & \mathbb{C}^\times & \longrightarrow & GL_n(\mathbb{C}) & \longrightarrow & PGL_n(\mathbb{C}) & \longrightarrow & 0 \end{array} \tag{3.19}$$

Since $H^2(G, \mathbb{C}^\times) = 0$, then $p : \widetilde{G} \to G$ splits. That is, there exists a group homomorphism $s : G \to \widetilde{G}$ such that $p \circ s = 1_G$. (Note that this just means that p has a section that is also a group homomorphism.) Then the composition $\widetilde{\rho} \circ s : G \to GL_n(\mathbb{C})$ is a lift of the projective representation $G \to PGL_n(\mathbb{C})$, and this completes the proof. $\square$

Remark. We may relax the hypotheses of Proposition 3.7 in several ways. First, we may replace $SL_2(\mathbb{F}_p)$ with any group G such that $H^2(G, \mathbb{C}^\times) = 0$. Second, the given projective representation does not need to be finite-dimensional. In both generalizations, the same proof holds. $\lozenge$

By Proposition 3.7, the projective Weil representation of $SL_2(\mathbb{F}_p)$ lifts to a linear representation of $SL_2(\mathbb{F}_p)$. This is exactly the Weil representation of $SL_2(\mathbb{F}_p)$, and this concludes our discussion of the finite-field analogue of the Weil representation of $SL_2(\mathbb{R})$. In the sections to follow, we will use our experience in constructing the finite-field model of the Weil representation to help guide us in our discussion of the real case.

4 The Real Case

We now would like to tell the story for when $K = \mathbb{R}$, using our experience with the $\mathbb{F}_p$ case as an example.

As our only assumption in Section 2 was that the characteristic of our field was not 2, then the structure of the real Heisenberg group Heis($\mathbb{R}$) is exactly the structure of the finite Heisenberg group Heis($\mathbb{F}_p$) for p an odd prime. In particular, from Section 2, we know, in a very explicit way, how $\mathrm{SL}_2(\mathbb{R})$ acts on Heis($\mathbb{R}$). In the following subsections, we will discuss how to make sense of Section 3 when we replace the finite field $\mathbb{F}_p$ with the real numbers $\mathbb{R}$.

4.1 The Representation Theory of Heis($\mathbb{R}$)

Here we will discuss the Stone-von Neumann theorem.

If (ρ_1, V_1) and (ρ_2, V_2) are unitary representations, we say that they are *unitarily equivalent* if the intertwining operators between V_1 and V_2 are unitary. That is, if Φ_h satisfies

$$\Phi_h \rho_1(h) = \rho_2(h)\Phi_h,$$

then

$$\Phi_h \Phi_h^* = \Phi_h^* \Phi_h = 1.$$

Theorem 4.1. *For a non-trivial irreducible character $\psi : \mathrm{Z}(\mathrm{Heis}(\mathbb{R})) \to \mathbb{C}$, there exists a unique, up to unitary equivalence, infinite-dimensional irreducible unitary representation of* Heis($\mathbb{R}$) *such that* $\mathrm{Z}(\mathrm{Heis}(\mathbb{R}))$ *acts by ψ.*

Proof. We refer the reader to Amritanshu Prasad's paper *An Easy Proof of the Stone-von Neumann Theorem* [Praa] for a proof. □

We would like to apply the Stone-von Neumann theorem to the Heisenberg representation (see Section 2.12). We need the following proposition.

Proposition 4.2. *The Heisenberg representation π_ψ defined on the representation space $\mathscr{L}^2(\mathbb{R})$ is unitary and irreducible.*

Proof. To show that the Heisenberg representation is unitary, it suffices to check that

$$A := \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad C := \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

act on $\mathscr{L}^2(\mathbb{R})$ as unitary operators. The Lebesgue measure $d\mu$ is translation in variant, so $\pi_\psi(A)$ is a unitary operator. An irreducible character of $\mathrm{Z}(\mathrm{Heis}(\mathbb{R})) = \mathbb{R}$ is of the form $\exp(ikx)$ for some $k \in \mathbb{R}$, and since $|\psi| = 1$, then it follows that $\pi_\psi(B)$ and $\pi_\psi(C)$ are unitary operators.

In Section 5.3, we will prove that the derived Heisenberg algebra representation is irreducible. It will follow that π_ψ is irreducible. Indeed, if π_ψ was not irreducible, then its derived representation must also not be irreducible, which contradicts Theorem 5.2.

This completes the proof. $\square$

Now, since $\mathrm{SL}_2(\mathbb{R})$ acts trivially on the center of $\mathrm{Heis}(\mathbb{R})$, then the precomposition of π_ψ by the $\mathrm{SL}_2(\mathbb{R})$-action is an irreducible representation wherein $\mathrm{Z}(\mathrm{Heis}(\mathbb{R}))$ acts the same way in π_ψ and $\pi_\psi \circ g$ for any $g \in \mathrm{SL}_2(\mathbb{R})$. By Theorem 4.1, these two representations must be unitarily equivalent. This segues nicely into the next section.

4.2 Constructing a Map $\mathrm{SL}_2(\mathbb{R}) \to \mathrm{PGL}(\mathscr{L}^2(\mathbb{R}))$

For any $g \in \mathrm{SL}_2(\mathbb{R})$, the composition $\rho_\psi \circ g$ is a representation of $\mathrm{Heis}(\mathbb{R})$. Since precomposition by the action of $\mathrm{SL}_2(\mathbb{R})$ does not change the action of Z on $\mathscr{L}^2(\mathbb{R})$ (indeed, $\mathrm{SL}_2(\mathbb{R})$ acts trivially on Z), then by the discussion in Section 4.1, as representations, $\pi_\psi \circ g \cong \pi_\psi$. Since Schur's lemma holds for compact groups, then we have, as in the finite-field case, $\mathrm{Aut}_{\mathbb{C}[H]}(\pi_\psi) = \mathbb{C}^\times$. Hence if Φ_1, Φ_2 are isomorphisms of the $\mathbb{C}[H]$-modules $\pi_\psi \circ g$ and π_ψ, they must differ only by a scalar multiple. Therefore we have a well-defined map

$$\rho : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{PGL}(\mathscr{L}^2(\mathbb{R})). \tag{4.1}$$

Unlike the finite case when we could lift the obtained projective representation of $\mathrm{SL}_2(\mathbb{F}_p)$ to a linear representation of $\mathrm{SL}_2(\mathbb{F}_p)$, in the real case, this projective representation of $\mathrm{SL}_2(\mathbb{R})$ lifts to a linear representation of a double cover of $\mathrm{SL}_2(\mathbb{R})$. We will denote this double cover of $\mathrm{SL}_2(\mathbb{R})$ by $\widetilde{\mathrm{SL}}_2(\mathbb{R})$.

Before getting to this issue, however, we will describe explicitly the projective representation $\rho : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{PGL}(\mathscr{L}^2(\mathbb{R}))$.

It can be shown that $\mathrm{SL}_2(\mathbb{R})$ is generated by

$$s := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad v(u) := \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}, \quad \text{and} \quad d(t) := \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}. \tag{4.2}$$

Remark. Note that the notation in (4.2) is non-standard. For a more standard set of generators, see page 209 of Lang's $\mathrm{SL}_2(\mathbb{R})$ [Lan75]. $\Diamond$

We would like to find a map $\Phi_g : \mathscr{L}^2(\mathbb{R}) \to \mathscr{L}^2(\mathbb{R})$ for $g = s, t(a), v(b)$ such that the following square commutes for $X = A, B$:

$$\begin{array}{ccc} \mathscr{L}^2(\mathbb{R}) & \xrightarrow{\pi_\psi(X)} & \mathscr{L}^2(\mathbb{R}) \\ \Big\downarrow \Phi_g & & \Big\downarrow \Phi_g \\ \mathscr{L}^2(\mathbb{R}) & \xrightarrow[\pi_\psi(g\cdot X)]{} & \mathscr{L}^2(\mathbb{R}) \end{array} \tag{4.3}$$

Recall that, by Schur's lemma, Φ_g is only unique up to a scalar. In the following computations, we will compute $\Phi_g \in \mathrm{GL}(\mathscr{L}^2(\mathbb{R}))$, which will give us an element in $\mathrm{PGL}(\mathscr{L}^2(\mathbb{R}))$ via the surjection in the short exact sequence

$$0 \to \mathbb{C}^\times \to \mathrm{GL}(\mathscr{L}^2(\mathbb{R})) \to \mathrm{PGL}(\mathscr{L}^2(\mathbb{R})) \to 0.$$

If we look back at Section 3.2, we will see that our computations can be easily extended to the real case. Because of this, we will focus our discussion on the $\mathbb{R}$-analogues of the operators found in Section 3.2, rather than computing everything from scratch as we did in finite-field case.

As in Section 3.2, for convenience of notation, we will write

$$A := \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}.$$

For $s = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, we are looking for a function $\Phi_s : \mathscr{L}^2(\mathbb{R}) \to \mathscr{L}^2(\mathbb{R})$ such that multiplying by a character gets swapped with translation (see (3.3) and (3.4)). In the finite-field case, we discussed the motivation from Fourier analysis, and in the real case, this motivation becomes exactly the operator we want. Define

$$\hat{}_\psi : \mathscr{L}^2(\mathbb{R}) \to \mathscr{L}^2(\mathbb{R}), \quad f(x) \mapsto \hat{f}(\xi) := \int_{-\infty}^{\infty} \psi(\xi x) f(x)\, dx. \tag{4.4}$$

We may verify that

$$\hat{h}_\psi(\xi) = \psi(b\xi) f \hat{f}_\psi(\xi), \qquad \text{where } h(x) = f(x+b), \tag{4.5}$$

$$\hat{h}_\psi(\xi) = \hat{f}(\xi - a), \qquad \text{where } h(x) = \psi(ax) f(x). \tag{4.6}$$

Hence

$$\Phi_s : \mathscr{L}^2(\mathbb{R}) \to \mathscr{L}^2(\mathbb{R}), f(x) \mapsto \hat{f}(x). \tag{4.7}$$

The computation for $v(u) = \left(\begin{smallmatrix} 1 & 0 \\ u & 1 \end{smallmatrix}\right)$ is very similar to the case of $v(1)$ in the finite-field case. As none of our computations in the finite-field case relied on the fact that we were working in $\mathbb{F}_p$, we may essentially replace all instances of $\mathbb{F}_p$ with $\mathbb{R}$. Drawing from this, we have

$$\Phi_{v(1)} : \mathscr{L}^2(\mathbb{R}) \to \mathscr{L}^2(\mathbb{R}), \quad f(x) \mapsto \psi\left(-\frac{1}{2}x^2\right) f(x).$$

Bootstrapping from this case, for $u \in \mathbb{R}$, the intertwining operator is (up to a scalar multiple, as usual),

$$\Phi_{v(u)} : \mathscr{L}^2(\mathbb{R}) \to \mathscr{L}^2(\mathbb{R}), \quad f(x) \mapsto \psi\left(-\frac{u}{2}x^2\right) f(x). \tag{4.8}$$

We can check (4.8) by direct computation of the compositions in the commutative square (4.3). Since $v(u)$ acts trivially on B and $\Phi_{v(u)}$ commutes with $\pi_\psi(B)$, then all we have left to do is to verify that $\Phi_{v(1)}$ intertwines the operators $\pi_\psi(A)$ and $\pi_\psi \circ v(u)(A)$. From the formulas in Section 2.2 (in particular, see (2.5) and (2.8)), we have

$$v(u) \cdot A = \begin{pmatrix} 1 & a & \frac{1}{2}ua^2 \\ 0 & 1 & ua \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence

$$\begin{aligned} f(x) &\xmapsto{\pi_\psi(A)} f(x-a) \xmapsto{\Phi_{v(u)}} \psi\left(-\frac{u}{2}x^2\right) f(x-a), \\ f(x) &\xmapsto{\Phi_{v(u)}} \psi\left(-\frac{u}{2}x^2\right) f(x-a) \xmapsto{\pi_\psi \circ v(u)(A)} \psi\left(uax + \frac{1}{2}ua^2\right) \psi\left(-\frac{u}{2}(x-a)^2\right) f(x-a) \\ &\qquad = \psi\left(-\frac{u}{2}x^2\right) f(x-a). \end{aligned}$$

For $d(t) = \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right)$, we have

$$d(t) \cdot \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & at & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad d(t) \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & at^{-1} \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence,

$$\begin{aligned} &\pi_\psi(A) : f(x) \mapsto f(x-a), && \pi_\psi(B) : f(x) \mapsto \psi(-bx) f(x), \\ &\pi_\psi \circ d(t)(A) : f(x) \mapsto f(x-at), && \pi_\psi \circ d(t)(B) : f(x) \mapsto \psi(-bt^{-1}x) f(x). \end{aligned}$$

Comparing $\pi_\psi(A)$ and the composition $\pi_\psi \circ d(t)(A)$, a reasonable guess is the linear operator $f(x) \mapsto f(t^{-1}x)$. It turns out that this operator is indeed an intertwining operator between π_ψ and $\pi_\psi \circ d(t)$. Set

$$\Phi_{d(t)} : \mathscr{L}^2(\mathbb{R}) \to \mathscr{L}^2(\mathbb{R}), \quad f(x) \mapsto f(t^{-1}x). \tag{4.9}$$

With this, we may compute the compositions in the commutative square (4.3). For $A = \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

$$f(x) \xmapsto{\pi_\psi(A)} f(x-a) \xmapsto{\Phi_{d(t)}} f(t^{-1}x - a),$$
$$f(x) \xmapsto{\Phi_{d(t)}} f(t^{-1}x) \xmapsto{\pi_\psi \circ d(t)(A)} f(t^{-1}(x-at)) = f(t^{-1}x - a),$$

and for $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$,

$$f(x) \xmapsto{\pi_\psi(B)} \psi(-bx)f(x) \xmapsto{\Phi_{d(t)}} \psi(-bt^{-1}x)f(t^{-1}x),$$
$$f(x) \xmapsto{\Phi_{d(t)}} f(t^{-1}x) \xmapsto{\pi_\psi \circ d(t)(B)} \psi(-bt^{-1}x)f(t^{-1}x).$$

In summary, quoting equations (4.7), (4.8), and (4.9), we have

$$\begin{aligned} \Phi_s(f(x)) &= \hat{f}(x) = \int_{-\infty}^{\infty} \psi(\xi x) f(\xi) d\xi, \\ \Phi_{v(u)}(f(x)) &= \psi\left(-\frac{u}{2}x^2\right) f(x), \\ \Phi_{d(t)}(f(x)) &= f(t^{-1}x). \end{aligned} \tag{4.10}$$

This defines a projective representation

$$\rho : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{PGL}(\mathscr{L}^2(\mathbb{R})), \quad g \mapsto [\Phi_g], \tag{4.11}$$

where $[\Phi_g]$ denotes the image of $\Phi_g \in \mathrm{GL}(\mathscr{L}^2(\mathbb{R}))$ under the surjection $\mathrm{GL}(\mathscr{L}^2(\mathbb{R})) \to \mathrm{PGL}(\mathscr{L}^2(\mathbb{R}))$. This is the projective Weil representation for $\mathrm{SL}_2(\mathbb{R})$. In the next section, we will discuss lifting this projective representation to a linear representation.

4.3 The Weil Representation for $\widetilde{\mathrm{SL}}_2(\mathbb{R})$

Recall that in Section 3.3, we discussed the Schur multiplier and its role in lifting projective representations. In the finite-field model, we were lucky in that the Schur multiplier of $\mathrm{SL}_2(\mathbb{F}_p)$ was trivial, as we showed in Section 3.4. It followed from this fact (see Theorem 3.3) that every central extension of $\mathrm{SL}_2(\mathbb{F}_p)$ by $\mathbb{C}^\times$ splits, and therefore the projective Weil representation for $\mathrm{SL}_2(\mathbb{F}_p)$ can be lifted to a linear representation of $\mathrm{SL}_2(\mathbb{F}_p)$.

In the real case—that is, in the case of $\mathrm{SL}_2(\mathbb{R})$—we are less fortunate, as the Schur multiplier of $\mathrm{SL}_2(\mathbb{R})$ is nontrivial. One proof of this nontriviality is that the projective Weil representation defined in (4.11) and (4.10) does not lift to a linear representation of $\mathrm{SL}_2(\mathbb{R})$. There are, of course, more direct ways to see that it is nontrivial. For instance, $\mathrm{SL}_2(\mathbb{R})$ is homotopy equivalent to its maximal compact subgroup $\mathrm{SO}_2(\mathbb{R}) \cong S^1$, and hence $\pi_1(\mathrm{SL}_2(\mathbb{R})) \cong \mathbb{Z}$.

So how do we obtain the linear Weil representation for $\mathrm{SL}_2(\mathbb{R})$ from the projective representation ρ described in Section 4.2? One solution is to first pass to the Lie algebra analogue of the Weil representation by applying the Lie derivative to the projective representation ρ, and then to exponentiate the obtained derived representation to obtain a representation of a cover of $\mathrm{SL}_2(\mathbb{R})$. The reason that we obtain a linear Lie algebra representation from applying the Lie derivative to a projective representation is discussed in Section 5, and the Lie derivative is discussed in detail in Section 5.1. Once we have obtained this representation of $\mathfrak{sl}(2) = \mathfrak{sl}_2(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$, the complexification of the Lie algebra of $\mathrm{SL}_2(\mathbb{R})$, what we have left to do is to exponentiate the representation.

Theorem 4.3. *The Weil representation of* $\mathfrak{sl}(2)$ *exponentiates to a representation of the metaplectic group* $\mathrm{Mp}_2(\mathbb{R})$.

This theorem is known as the Shale-Weil theorem. For the proof, we refer the interested reader to Lang [Lan75], Howe [How88], Weil [Wei64], and Shale [Sha62]. After we have computed the derived Weil representation, we will give the exponentiation of this representation of $\mathfrak{sl}(2)$. This is given explicitly in Section 5.5.

5 The Lie Algebra $\mathfrak{sl}_2(\mathbb{R})$

An equivalent formulation of the Weil representation is through the complexified Lie algebra

$$\mathfrak{sl}(2) := \mathfrak{sl}_2(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}.$$

Recall from Section 4 that the Weil representation for $\mathrm{SL}_2(\mathbb{R})$ is actually a representation of $\widetilde{\mathrm{SL}}_2(\mathbb{R})$, a double cover of $\mathrm{SL}_2(\mathbb{R})$. In the Lie algebra case, the Weil representation is in fact defined for $\mathfrak{sl}(2)$ itself. There are two primary ways to see this.

(1) For a finite-dimensional $\mathbb{C}$-vector space V, given a projective representation $G \to \mathrm{PGL}(V)$, we may differentiate to obtain a Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{pgl}(V)$. But PGL and SL have the same Lie algebra, and hence this Lie algebra homomorphism is a map $\mathfrak{g} \to \mathrm{End}(V)$, which defines a representation of the Lie algebra $\mathfrak{g}$.

(2) By a construction of the Weil representation through a Heisenberg algebra construction, analogous to the constructions we have explained in detail in Sections 3 and 4, we obtain a projective representation of $\mathfrak{sl}(2)$. We may pull this back to a representation of a central extension of $\mathfrak{sl}(2)$, but because $\mathfrak{sl}(2)$ is completely reducible, every central extension of $\mathfrak{sl}(2)$ splits. Therefore the projective representation of $\mathfrak{sl}(2)$ gives rise to a linear representation of $\mathfrak{sl}(2)$.

The first point is presented primarily for the purpose of motivating why the Weil representation for $\mathfrak{sl}(2)$ is an actual representation of $\mathfrak{sl}(2)$ itself. Indeed, this first point does not literally apply to our situation as V is infinite-dimensional in our set-up and the correct notion of "$\mathfrak{pgl}(V)$" for an infinite-dimensional V will not be discussed here.

We will instead concentrate our efforts on the second point. In Sections 5.1 and 5.4, we will discuss how to construct the Weil representation through the Heisenberg algebra by passing the story told in Section 4 over to the Lie algebra context.

Once we have told this story, we will then turn our efforts to unpacking the structure of the Weil representation as a $\mathfrak{sl}(2)$-module. This will be done in Sections 5.5 and 5.6, and will lead nicely into Section **??** of this paper (in particular, the discussion on the connection to quantum mechanics), wherein we will discuss the applications of the Weil representation in other fields.

5.1 The Lie Derivative

The key to passing between our $\mathrm{SL}_2(\mathbb{R})$ story and our $\mathfrak{sl}(2)$ story lies exactly in the process of obtaining a Lie algebra action from a Lie group action. Because we will not benefit from specifying to a specific Lie group-Lie algebra pair, our discussion in this section will be in complete generality.

Let G be a Lie group and let $\mathfrak{g}$ be its Lie algebra. Recall that there is a bijective correspondence between one-parameter subgroups

$$\alpha_X : \mathbb{R} \to G, \quad t \mapsto \exp(tX), \quad \text{for } X \in \mathfrak{g},$$

and elements of $\mathfrak{g}$, given precisely by

$$\mathfrak{g} \to \{\text{one-parameter subgroups of } G\}, \quad X \mapsto \alpha_X.$$

Let V be a $\mathbb{C}$-vector space with a smooth (that is, differentiable) G-action. We define a $\mathfrak{g}$-action on V by the formula

$$X \cdot v := \frac{d}{dt}(\exp(tX) \cdot v)\Big|_{t=0} = \lim_{t \to 0} \frac{\exp(tX) \cdot v - v}{t}. \tag{5.1}$$

One may check by hand that this defines a Lie algebra action. We leave this straightforward computation to the reader. The rule in (5.1) is known as the *Lie derivative.* We will use the Lie derivative repeatedly throughout this section, in particular in Section 5.4, where we convert our Lie group scenario to a Lie algebra scenario by obtaining the Lie algebra actions induced by our Lie group actions.

Of course, the definition in (5.1) only makes sense if the G-action on V is smooth. That is, if the map

$$G \to V, \quad g \mapsto g \cdot v, \tag{5.2}$$

is smooth for all $v \in V$. Suppose that we relaxed the condition that the G-action on V is smooth. If $v \in V$ is a vector satisfying (5.2), then we will say that v is a *smooth vector*. The smooth vectors of V clearly form a subspace of V, which we will denote by

$$V^\infty := \{v \in V : g \mapsto g \cdot v \text{ is a smooth map } G \to V\}.$$

From our discussion so far, we can already see that the subspace V^∞ plays an extremely important role in the relationship between Lie group representations and Lie algebra representations. Indeed, the following proposition is true.

Proposition 5.1. *Let V^∞ be the subspace of smooth vectors of a complex G-representation V.*

(i) V^∞ is a dense subspace of V.

(ii) V^∞ is G-invariant. It follows that V^∞ is also $\mathfrak{g}$-invariant.

Proof. This is not a difficult proof, but it requires some set-up that strays from the arc of this exposition. Because of this, we omit the proof and refer the reader to pages 18 and 19 of Howe and Tan's *Non-Abelian Harmonic Analysis* [HT92]. □

In words, Proposition 5.1 says that in passing from V to V^∞, we gain a $\mathfrak{g}$-action without losing the G-action. So we actually get *more* information when we pass to a dense subspace of V.

5.2 The Heisenberg Algebra

In order to construct the Lie algebra analogue of the story we have told in Sections 3 and 4, we first need to discuss the Lie algebra of the Heisenberg group. We will call it the Heisenberg algebra and denote it by

$$\mathfrak{heis}(\mathbb{R}) := \left\{ \begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} : a, b, c, \in \mathbb{R} \right\}.$$

We will work with the complexified Heisenberg algebra

$$\mathfrak{heis} := \mathfrak{heis}(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}.$$

Now we would like to understand the Heisenberg algebra equivalent of the Heisenberg representation π_ψ we introduced in Section 2.3.

So far, the representation space we have been working with is the Hilbert space $\mathscr{L}^2(\mathbb{R})$. The first question we must answer is, what is the space of smooth vectors of $\mathscr{L}^2(\mathbb{R})$ with respect to the actions of $\mathrm{Heis}(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{R})$? The answer is the Schwartz space $\mathscr{S}(\mathbb{R})$.

Recall from equation (2.10) that the Heisenberg representation is defined to be

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \cdot f(x) := \psi(-bx + c) f(x - a).$$

The Lie algebra of the Heisenberg group is

$$\mathfrak{heis}(\mathbb{R}) := \left\{ \begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} : a, b, c \in \mathbb{R} \right\}.$$

We will work with the complexified Heisenberg algebra

$$\mathfrak{heis} := \mathfrak{heis}(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$$

Because the map

$$\mathbb{C}^{\oplus 3} \to \mathfrak{heis}, \quad (a, b, c) \mapsto \begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}$$

defines an isomorphism of vector spaces, to simplify notation, we will use the vector notation on the left-hand side of the map to denote its image on the right-hand side.

We have

$$\exp\left(t\begin{pmatrix}0 & a & c\\ 0 & 0 & b\\ 0 & 0 & 0\end{pmatrix}\right) = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} + t\begin{pmatrix}0 & a & c\\ 0 & 0 & b\\ 0 & 0 & 0\end{pmatrix} + \frac{t^2}{2}\begin{pmatrix}0 & 0 & ab\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} = \begin{pmatrix}1 & at & ct+\frac{abt^2}{2}\\ 0 & 1 & bt\\ 0 & 0 & 1\end{pmatrix}.$$

Then, differentiating the representation of the Heisenberg group, we have

$$\begin{aligned}
\begin{pmatrix}0 & a & c\\ 0 & 0 & b\\ 0 & 0 & 0\end{pmatrix}\cdot f(x) &= \frac{d}{dt}\exp\left(t\begin{pmatrix}0 & a & c\\ 0 & 0 & b\\ 0 & 0 & 0\end{pmatrix}\right)\cdot f(x)\Big|_{t=0}\\
&= \frac{d}{dt}\begin{pmatrix}1 & at & ct+\frac{abt^2}{2}\\ 0 & 1 & bt\\ 0 & 0 & 1\end{pmatrix}\cdot f(x)\Big|_{t=0}\\
&= \frac{d}{dt}\psi(-btx+ct+\frac{abt^2}{2})f(x-at)\Big|_{t=0}\\
&= (-a\psi(-btx+ct+\frac{abt^2}{2})\frac{d}{dx}f(x-at)\\
&\qquad + (-bx+c+ab)\frac{d}{dt}(\psi(-btx+ct+\frac{abt^2}{2}))f(x-at))\Big|_{t=0}\\
&= -a\frac{d}{dx}f(x)+(-bx+c+ab)\psi'(0)f(x).
\end{aligned}$$

It is a straightforward computation to check that

$$\mathrm{L}\pi_\psi : \mathfrak{heis}(\mathbb{R}) \to \mathrm{End}(\mathscr{S}(\mathbb{R})), \qquad \begin{pmatrix}0 & a & c\\ 0 & 0 & b\\ 0 & 0 & 0\end{pmatrix} \mapsto -a\frac{d}{dx} - b\psi'(0)x + c\psi'(0),$$

defines a homomorphism of Lie algebras. Explicitly, this means that $\mathrm{L}\pi_\psi$ is a linear map that respects the Lie bracket

$$[X,Y] = XY - YX.$$

That is,

$$\mathrm{L}\pi_\psi([g,h]) = [\mathrm{L}\pi_\psi(g), \mathrm{L}\pi_\psi(h)].$$

Indeed,

$$
\begin{aligned}
[\mathrm{L}\,\pi_\psi(a,b,c),\mathrm{L}\,\pi_\psi(a',b',c')] &= \left(-a\frac{d}{dx} - b\psi'(0)x + c\psi'(0)\right)\left(-a'\frac{d}{dx} - b'\psi'(0)x + c'\psi'(0)\right) \\
&\quad - \left(-a'\frac{d}{dx} - b'\psi'(0)x + c'\psi'(0)\right)\left(-a\frac{d}{dx} - b\psi'(0)x + c\psi'(0)\right) \\
&= ab'\psi'(0)\frac{d}{dx}x + a'b\psi'(0)x\frac{d}{dx} - a'b\psi'(0)\frac{d}{dx}x - ab'\psi'(0)x\frac{d}{dx} \\
&= (ab' - a'b)\psi'(0) \\
&= \mathrm{L}\,\pi_\psi([(a,b,c),(a',b',c')]).
\end{aligned}
$$

Remark. The fact that $\mathrm{L}\,\pi_\psi$ is a Lie algebra homomorphism actually comes for free, since it is the derived representation of π_ψ. $\Diamond$

5.3 The Irreducibility of the Heisenberg Algebra Representation

In this section, we will prove the following theorem.

Theorem 5.2. *The Heisenberg algebra representation* $(\mathrm{L}\,\pi_\psi, \mathscr{S}(\mathbb{R}))$ *is irreducible.*

As an immediate consequence, we have the following.

Corollary 5.3. *The Heisenberg representation* $(\pi_\psi, \mathscr{L}^2(\mathbb{R}))$ *is irreducible.*

It follows directly from our discussion in Section 5.2 that the Heisenberg algebra $\mathfrak{heis}$ is generated, as a Lie algebra, by the operators

$$
\begin{aligned}
P &:= \pi_\psi(-1,0,0) = \frac{d}{dx}, \\
Q &:= \pi_\psi(0,-1/\psi'(0),0) = x, \\
I &:= \pi_\psi(0,0,1/\psi'(0)) = 1.
\end{aligned}
$$

The Lie bracket relations are

$$
[P,Q] = I, \quad [P,I] = [Q,I] = 0.
$$

In this proof, we will work with a different basis of $\mathfrak{heis}$, namely

$$
\begin{aligned}
a &:= P + Q, \\
a^+ &:= -P + Q, \\
I &:= 1.
\end{aligned} \tag{5.3}
$$

The Lie bracket relations are then

$$
\begin{aligned}
[a,a^+] &= [P+Q,-P+Q] = [Q,-P]+[P,Q] = 2,\\
[a,I] &= 0,\\
[a^+,I] &= 0.
\end{aligned}
\tag{5.4}
$$

Before proceeding with the proof of Theorem 5.2, we will analyze the operators a and a^+.

5.3.1 Hermite Functions

Recall from the previous section that

$$a := x + \frac{d}{dx}, \quad a^+ := x - \frac{d}{dx}.$$

We analyze some of the basic properties of a and a^+ here.

(i) Recall from equation (5.4) that

$$[a,a^+] = 2.$$

(ii) Observe that

$$a^+ = a^*, \tag{5.5}$$

where a^* denotes the adjoint of the operator a with respect to the inner product $\langle f,g\rangle = \int_{\mathbb{R}} f(x)\overline{g(x)}\,dx$. Indeed,

$$
\begin{aligned}
\langle af,g\rangle &= \int_{\mathbb{R}} (xf(x) - \tfrac{d}{dx}f(x))\overline{g(x)}\,dx\\
&= \int_{\mathbb{R}} xf(x)\overline{g(x)} - \overline{g(x)}\tfrac{d}{dx}f(x) - f(x)\tfrac{d}{dx}\overline{g(x)} + f(x)\tfrac{d}{dx}\overline{g(x)}\,dx\\
&= \int_{\mathbb{R}} xf(x)\overline{g(x)} - \tfrac{d}{dx}f(x)\overline{g(x)} + f(x)\tfrac{d}{dx}\overline{g(x)}\,dx\\
&= \int_{\mathbb{R}} f(x)a^+(\overline{g(x)})\,dx - \int_{\mathbb{R}} \tfrac{d}{dx}f(x)\overline{g(x)}\,dx\\
&= \langle f,a^+g\rangle.
\end{aligned}
$$

(iii) Also,

$$[a,(a^+)^j] = 2j(a^+)^{j-1}, \tag{5.6}$$

since

$$\begin{aligned}[a,(a^+)^j] &= a(a^+)^j - (a^+)^j \\ &= aa^+(a^+)^{j-1} - (a^+)^{j-1}a^+a \\ &= [a,a^+](a^+)^{j-1} + a^+[a,(a^+)^{j-1}] \\ &= 2(a^+)^{j-1} + a^+[a,(a^+)^{j-1}] \\ &= 2j(a^+)^{j-1}.\end{aligned}$$

Now that we have some properties of the operators a and a^+ under our belt, we will construct a basis of $\mathscr{S}(\mathbb{R})$ that behaves nicely under these operators. Let

$$v_0 := e^{-x^2/2}.$$

So $v_0 \in \mathscr{S}(\mathbb{R})$ and

$$av_0 = \left(x + \tfrac{d}{dx}\right)e^{-x^2/2} = xe^{-x^2/2} + e^{-x^2/2}(-x) = 0. \tag{5.7}$$

Now set

$$v_j := (a^+)^j v_0.$$

Then

$$av_j = a(a^+)^j v_0 = [a,(a^+)^j]v_0 + (a^+)^j av_0 = 2j(a^+)^{j-1} = 2jv_{j-1}. \tag{5.8}$$

We may compute the inner products of the v_i's and v_j's. We have

$$\langle v_i, v_j\rangle = 2^j j!\delta_{ij}\sqrt{\pi}. \tag{5.9}$$

Indeed,

$$\langle v_0, v_0\rangle = \int_{\mathbb{R}} e^{-x^2/2}\overline{e^{-x^2/2}}\,dx = \int_{\mathbb{R}} e^{-x^2}\,dx = \sqrt{\pi},$$

and so for $i \leq j$, by repeatedly reducing the indices using (5.5), (5.7), and (5.8), we have

$$\begin{aligned}\langle v_i, v_j\rangle &= \langle (a^+)^j v_{i-j}, v_j\rangle = \langle v_{i-j}, a^j v_j\rangle = \langle v_{i-j}, 2^j j! v_{j-1} v_0\rangle \\ &= \begin{cases} 2^j j!\langle v_0, v_0\rangle = 2^j j! v_{j-1}\sqrt{\pi}, & \text{if } i = j, \\ 2^j j!\langle v_{i-j-1}, av_0\rangle = 0, & \text{if } i > j.\end{cases}\end{aligned}$$

Hence from (5.9), we see that the v_i's are orthogonal and $\{(2^j j!\sqrt{\pi})^{-1/2}v_j\} \subseteq \mathscr{S}(\mathbb{R})$ forms an orthonormal set.

From the construction of the v_j's as applying the operator $a^+ = x - \frac{d}{dx}$ successively to $v_0 = e^{-x^2/2}$, then we may write

$$v_j = P_j(x)e^{-x^2/2}, \qquad P_j(x) \in \mathbb{R}[x],\ \deg(P_j(x)) = j. \tag{5.10}$$

In (5.10), we call v_j a *Hermite function* and $P_j(x)$ a *Hermite polynomial*.

Note. We actually have an explicit formula for the Hermite polynomials. It relies on the observation that

$$-e^{x^2/2}\tfrac{d}{dx}\big(e^{-x^2/2}f(x)\big) = -e^{x^2/2}\big(-xe^{-x^2/2}f(x) + e^{-x^2/2}\tfrac{d}{dx}f(x)\big) = a^+ f(x).$$

Hence by induction, we have

$$P_j(x) = (-1)^j e^{x^2}\tfrac{d^j}{dx^j}(e^{-x^2}). \tag{5.11}$$

The formula in (5.11) is known as the *Rodriguez Formula*.

It turns out that not only do the Hermite functions form an orthogonal set in $\mathscr{S}(\mathbb{R})$, but they form a *basis* of $\mathscr{S}(\mathbb{R})$ and hence of $\mathscr{L}^2(\mathbb{R})$. Here, we mean that the $\mathbb{C}$-span of the Hermite functions is dense in $\mathscr{S}(\mathbb{R})$, which is dense in $\mathscr{L}^2(\mathbb{R})$. We formulize this into a proposition.

Proposition 5.4. *The Hermite functions form an orthogonal basis of $\mathscr{S}(\mathbb{R})$.*

Proof. We only give an outline of a proof here. First note that $\mathscr{S}(\mathbb{R})$ is invariant under translation and dilation. Then we show that the $\mathbb{R}$-span of the Hermite polynomials is $\mathscr{P}(\mathbb{R})$, the set of complex polynomial functions on $\mathbb{R}$, and that $\mathscr{P}(\mathbb{R})e^{-x^2/2}$ is dense in $\mathscr{S}(\mathbb{R})$ by Taylor polynomial approximations. □

5.3.2 The Proof of Irreducibility

Proof of Theorem 5.2. Recall from (5.3) that 1, a, and a^+ form a basis of the Heisenberg algebra. From our analysis in Section 5.3.1, we have the following picture:

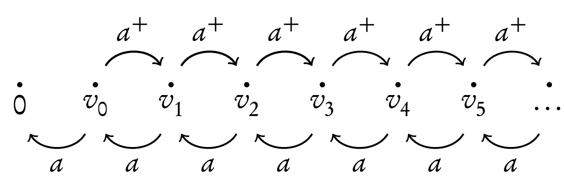

Figure 1: Action of a and a^+ on the v_j

Since the Hermite functions form an orthonormal basis of $\mathscr{S}(\mathbb{R})$, it follows from Figure 1 that the Heisenberg algebra representation on the Schwartz space $\mathscr{S}(\mathbb{R})$ is irreducible. □

Remark. In the literature, a and a^+ are called "ladder operators." Figure 1 is exactly the motivation for this terminology. $\lozenge$

5.4 The Lie Algebra Analogue

In this section, we will construct the Lie algebra analogue of the story we have told in Sections 3 and 4. That is, we will construct the Weil representation through an analysis of the representation theory of the Heisenberg group, converting all of our Lie group notions into their Lie algebra analogues via the methods discussed in Section 5.1.

Recall that in the construction of the Weil representation for $\mathrm{SL}_2(\mathbb{R})$, we used the Stone-von Neumann theorem to obtain a projective representation of $\mathrm{SL}_2(\mathbb{R})$ through the intertwining operators of π_ψ precomposed with the action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathrm{Heis}(\mathbb{R})$. Writing the condition described in the commutative diagram in (4.3) in terms of the Lie algebras $\mathfrak{sl}(2)$ and $\mathfrak{heis}(\mathbb{R})$, we have

$$\Phi_{\exp(sx)}\pi_\psi(\exp(ty)) = \pi_\psi(\exp(ty)\cdot\exp(sx)\Phi_{\exp(sx)}, \tag{5.12}$$

for $s,t\in\mathbb{R}$, $x\in\mathfrak{sl}(2)$, $y\in\mathfrak{heis}(\mathbb{R})$.

Remark. Recall that Φ_g for $g\in\mathrm{SL}_2(\mathbb{R})$ is only determined up to a constant factor. In this discussion, when we write Φ_g, we will mean $C\Phi_g$ for any $C\in\mathbb{C}^\times$. $\lozenge$

From the remark, we have, necessarily, $\Phi^{-1}_{\exp(sx)} = \Phi_{\exp(-sx)}$. Hence equation (5.12) becomes

$$\Phi_{\exp(sx)}\pi_\psi(\exp(ty))\Phi_{\exp(-sx)} = \pi_\psi(\exp(ty)\cdot\exp(sx)).$$

Differentiating both sides both with respect to s and t and evaluating at $s=t=0$, we obtain

$$\begin{aligned}
\frac{d}{ds}\frac{d}{dt}\pi_\psi(\exp(ty)\cdot\exp(sx))\bigg|_{s=t=0} &= \frac{d}{ds}\frac{d}{dt}\left(\Phi_{\exp(sx)}\pi_\psi(\exp(ty))\Phi_{\exp(-sx)}\right)\bigg|_{s=t=0}\\
&= \frac{d}{ds}\left(\Phi_{\exp(sx)}\,\mathrm{L}\,\pi_\psi(y)\Phi_{\exp(-sx)}\right)\bigg|_{s=0}\\
&= \Bigg(\left(\frac{d}{ds}\Phi_{\exp(sx)}\right)\mathrm{L}\,\pi_\psi(y)\Phi_{\exp(-sx)}\\
&\qquad - \Phi_{\exp(sx)}\,\mathrm{L}\,\pi_\psi(y)\frac{d}{ds}\Phi_{\exp(sx)}\Bigg)\bigg|_{s=0}\\
&= \mathrm{L}\,\Phi_x\,\mathrm{L}\,\pi_\psi(y) - \mathrm{L}\,\pi_\psi(y)\,\mathrm{L}\,\Phi_x\\
&= \left[\mathrm{L}\,\Phi_x, \mathrm{L}\,\pi_\psi(y)\right].
\end{aligned}$$

Summarizing, we have

$$\mathrm{L}\,\pi_\psi(x\cdot y) = [\mathrm{L}\,\Phi_x, \mathrm{L}\,\pi_\psi(y)]. \tag{5.13}$$

We will show through direct computation that $\mathrm{L}\Phi_x$ exists for each $x \in \mathfrak{sl}(2)$. Furthermore, $\mathrm{L}\Phi_x$ is unique up to a scalar difference. Indeed, if $\mathrm{L}\Phi'_x$ also satisfies equation (5.13), then we have

$$[\mathrm{L}\Phi_x, \mathrm{L}\pi_\psi(y)] = [\mathrm{L}\Phi'_x, \mathrm{L}\pi_\psi(y)].$$

Rearranging terms, we have

$$(\mathrm{L}\Phi_x - \mathrm{L}\Phi'_x)\mathrm{L}\pi_\psi(y) = \mathrm{L}\pi_\psi(y)(\mathrm{L}\Phi_x - \mathrm{L}\Phi'_x),$$

and since π_ψ is irreducible, then by Schur's lemma, we may conclude that

$$\mathrm{L}\phi_x - \mathrm{L}\Phi'_x = \lambda I, \quad \text{for some } \lambda \in \mathbb{C}.$$

We will now compute (up to a scalar difference) $\mathrm{L}\Phi_x$ for $x = e, f, h$, where

$$e^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad e^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We can easily write down the action of e^-, e^+, and h on $\mathfrak{heis}$, and the operators obtained after precomposing $\mathrm{L}\pi_\psi$ with each of these elements of $\mathfrak{sl}(2)$:

$$\begin{aligned} e^- \cdot (a,b,c) &= (0,a,0), & \mathrm{L}\pi_\psi \circ e^-(a,b,c) &= -a\psi'(0)x, \\ e^+ \cdot (a,b,c) &= (b,0,0), & \mathrm{L}\pi_\psi \circ e^+(a,b,c) &= -b\frac{d}{dx}, \\ h \cdot (a,b,c) &= (a,-b,0) & \mathrm{L}\pi_\psi \circ h(a,b,c) &= -a\frac{d}{dx} + b\psi'(0)x. \end{aligned}$$

Comparing the above operators to

$$\mathrm{L}\pi_\psi(a,b,c) = -a\frac{d}{dx} - b\psi'(0)x + c\psi'(0),$$

we may observe that what we are looking for are operators that:

- commute with d/dx but not with x (as in the case of e^-),
- commute with x but not with d/dx (as in the case of e^+), and
- swap x and d/dx (as in the case of h).

Hence we are likely looking at d^2/dx^2, x^2, and $x \cdot d/dx$, respectively. Computing, we get

$$\begin{aligned}
[x^2, \mathrm{L}\pi_\psi(a,b,c)] &= 2a\frac{d}{dx}, \\
[d^2/dx^2, \mathrm{L}\pi_\psi(a,b,c)] &= -2b\psi'(0)\frac{d}{dx}, \\
[x\cdot d/dx, \mathrm{L}\pi_\psi(a,b,c)] &= a\frac{d}{dx} - b\psi'(0)x.
\end{aligned}$$

Note that our computations here only determine $\mathrm{L}\Phi_g$ up to a constant difference. Therefore, from the above, all we may conclude is that

$$\begin{aligned}
\mathrm{L}\Phi_{e^-} &= -\frac{1}{2}x^2 + C_{e^-}, \\
\mathrm{L}\Phi_{e^+} &= \frac{1}{2\psi'(0)}\frac{d^2}{dx^2} + C_{e^+}, \\
\mathrm{L}\Phi_h &= x\frac{d}{dx} + C_h,
\end{aligned}$$

for some constants $C_{e^-}, C_{e^+}, C_h \in \mathbb{C}$. In order to pin down these constants, we must use the Lie bracket relations of $\mathfrak{sl}(2)$.

The Lie bracket relations of $\mathfrak{sl}(2)$ can be computed easily:

$$\begin{aligned}
[e^-, e^+] &= e^-e^+ - e^+e^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -h, \\
[e^-, h] &= e^-h - he^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = 2e^-, \\
[e^+, h] &= e^+h - he^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = -2e^+.
\end{aligned}$$

Because we would like the map $\mathfrak{sl}(2) \to \mathrm{End}(\mathscr{S}(\mathbb{R}))/\mathbb{C}^\times$ defined by $g \mapsto \mathrm{L}\Phi_g$ to be a Lie algebra homomorphism, then necessarily we must have, up to some scalar multiple,

$$\left[\mathrm{L}\Phi_{e^-}, \mathrm{L}\Phi_{e^+}\right] = -\mathrm{L}\Phi_h, \tag{5.14}$$

$$\left[\mathrm{L}\Phi_{e^-}, \mathrm{L}\Phi_h\right] = 2\mathrm{L}\Phi_{e^-}, \tag{5.15}$$

$$\left[\mathrm{L}\Phi_{e^+}, \mathrm{L}\Phi_h\right] = -2\mathrm{L}\Phi_{e^+}. \tag{5.16}$$

From (5.14), we have

$$\begin{aligned} \mathrm{L}\Phi_h &= \left[\mathrm{L}\Phi_{e^+}, \mathrm{L}\Phi_{e^-}\right] \\ &= \left(x^2 + C_{e^+}\right)\left(\frac{d^2}{dx^2} + C_{e^-}\right) - \left(\frac{d^2}{dx^2} + C_{e^-}\right)\left(x^2 + C_{e^+}\right) \\ &= -4\left(x\frac{d}{dx} + \frac{1}{2}\right). \end{aligned}$$

Therefore

$$C_h = \frac{1}{2}.$$

By similar computations, we obtain from (5.15) and (5.16) that

$$C_{e^-} = 0 = C_{e^+}.$$

Hence we may conclude that, up to a scalar multiple,

$$\begin{aligned} \mathrm{L}\Phi_{e^-} &= -\frac{1}{2}x^2, \\ \mathrm{L}\Phi_{e^+} &= \frac{1}{2\psi'(0)}\frac{d^2}{dx^2}, \\ \mathrm{L}\Phi_h &= x\frac{d}{dx} + \frac{1}{2}. \end{aligned} \tag{5.17}$$

This therefore defines a projective representation

$$\mathrm{L}\rho : \mathfrak{sl}(2) \to \mathrm{End}(\mathscr{S}(\mathbb{R}))/\mathbb{C}^\times, \quad g \mapsto [\mathrm{L}\Phi_g],$$

where $[\mathrm{L}\Phi_g]$ denotes the equivalence class in $\mathrm{End}(\mathscr{S}(\mathbb{R}))/\mathbb{C}$ represented by $\mathrm{L}\Phi_g$.

Now the question becomes, Can we lift this projective representation to a linear representation? The question is, of course, yes. Indeed, letting $\widetilde{\mathfrak{sl}}(2)$ be the pull back of $\mathrm{L}\rho : \mathfrak{sl}(2) \to \mathrm{End}(\mathscr{S}(\mathbb{R}))/\mathbb{C}$ and the surjection $p : \mathrm{End}(\mathscr{S}(\mathbb{R})) \to \mathrm{End}(\mathscr{S}(\mathbb{R}))/\mathbb{C}$, we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{C} & \longrightarrow & \widetilde{\mathfrak{sl}}(2) & \xrightarrow{\;p\;} & \mathfrak{sl}(2) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1_{\mathbb{C}}} & & \downarrow{\scriptstyle \widetilde{\mathrm{L}\rho}} & & \downarrow{\scriptstyle \mathrm{L}\rho} & & \\ 0 & \longrightarrow & \mathbb{C} & \longrightarrow & \mathrm{End}(\mathscr{S}(\mathbb{R})) & \longrightarrow & \mathrm{End}(\mathscr{S}(\mathbb{R}))/\mathbb{C} & \longrightarrow & 0 \end{array} \tag{5.18}$$

From this perspective, we have a central extension of $\mathfrak{sl}(2)$ by $\mathbb{C}$, and we would like to understand these objects. We have the following proposition.

Proposition 5.5. *Every central extension of* $\mathfrak{sl}(2)$ *by* $\mathbb{C}$ *splits.*

This is in fact a direct consequence of the following theorem due to Whitehead.

Theorem 5.6. *If* $\mathfrak{g}$ *is a semisimple Lie algebra, then every central extension of* $\mathfrak{g}$ *by* $\mathbb{C}$ *splits. Equivalently,*

$$\mathrm{H}^2(\mathfrak{g}, \mathbb{C}) = 0.$$

Proof. Consider the short exact sequence of Lie algebras

$$0 \to \mathbb{C} \to \mathfrak{e} \xrightarrow{p} \mathfrak{g} \to 0, \tag{5.19}$$

with $\mathbb{C}$ central in $\mathfrak{e}$. The Lie algebra $\mathfrak{e}$ can be endowed with a $\mathfrak{g}$ structure in the following way: for $x \in \mathfrak{g}$ and $y \in \mathfrak{e}$, define the action of x on y to be

$$x \cdot y := [\widetilde{x}, y],$$

where $\widetilde{x}$ is any preimage of x under the surjection p. This action is well-defined since the $\ker(p)$ is central in $\mathfrak{e}$.

Since $\mathfrak{g}$ is semisimple, $\mathfrak{e}$ splits into a direct sum of $\mathfrak{g}$-modules. In particular, since $\mathfrak{e}$ contains a copy of the trivial representation by assumption, this means that there exists a $\mathfrak{g}$-module homomorphism $\sigma : \mathfrak{g} \mapsto \mathfrak{e}$ splitting $\mathfrak{e}$ as $\mathfrak{g} \oplus \mathbb{C}$ as $\mathfrak{g}$-modules. Explicitly, we may set

$$\sigma(x) := \widetilde{x}.$$

This is a Lie algebra homomorphism and hence necessarily $\sigma(\mathfrak{g})$ must be a Lie subalgebra of $\mathfrak{e}$. Hence the central extension in (5.19) splits, and this completes the proof. □

Remark. For a detailed discussion of Lie algebra cohomology, see Chapter 7 of Weibel's *An Introduction to Homological Algebra* [Wei94]. ◊

It turns out that the operators in (5.17) exactly define a lift of the projective representation $\mathrm{L}\rho : \mathfrak{sl}(2) \to \mathrm{End}(\mathscr{S}(\mathbb{R}))/\mathbb{C}$ to a linear representation $\widetilde{\mathrm{L}\rho} : \mathfrak{sl}(2) \to \mathrm{End}(\mathscr{S}(\mathbb{R}))$.

5.5 Defining the Weil Representation for $\mathfrak{sl}(2)$

In this section, we will define the Weil representation for an alternate basis of $\mathfrak{sl}(2)$.

The most common choice of a standard basis for $\mathfrak{sl}(2)$ is the one formed by the matrices

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad e^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

which was the basis that we used in our construction of the Weil representation for $\mathfrak{sl}(2)$ through the Heisenberg algebra representation. In this section, we will make use of another standard basis of $\mathfrak{sl}(2)$. Let

$$\begin{aligned}
\bar{k} &= i\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = i(e^- - e^+),\\
n^+ &= \frac{1}{2}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) = \frac{1}{2}(h + i(e^+ + e^-)),\\
n^- &= \frac{1}{2}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) = \frac{1}{2}(h - i(e^+ + e^-)).
\end{aligned}$$

The reason we care about the basis $\{\bar{k}, n^+, n^-\}$ rests in the observation that

$$k = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\bar{k}$$

is the infinitesimal generator of the maximal compact subgroup $\mathrm{SO}(2,\mathbb{R})$ of $\mathrm{SL}(2,\mathbb{R})$. Now, if we have a unitary representation, or a continuous quasisimple representation of $\mathrm{SL}(2,\mathbb{R})$, then k acts diagonally with eigenvalues in $i\mathbb{Z}$ (since k generates $\mathrm{SO}(2,\mathbb{R})$). So $\bar{k} = ik$ must act diagonally with eigenvalues in $\mathbb{Z}$. Hence we can expect to find standard $\mathfrak{sl}(2)$-modules with respect to this basis. It will turn out that, in the Weil representation, $\bar{k}$ acts on $\mathscr{S}(\mathbb{R})$ as the Hermite operator $x^2 - \frac{d^2}{dx^2}$. It is because of this that we would like to formulate the Weil representation with respect to the basis $\{\bar{k}, n^+, n^-\}$ instead of the basis $\{h, e^+, e^-\}$.

We begin by defining a representations ω of $\mathfrak{sl}(2)$ on the Schwartz space $\mathscr{S}(\mathbb{R})$ on $\mathbb{R}$ via the standard basis $\{h, e^+, e^-\}$ of $\mathfrak{sl}(2)$. We write them as operators on functions in $\mathscr{S}(\mathbb{R})$:

$$\omega(h) = x\frac{d}{dx} + \frac{1}{2}, \tag{5.20}$$

$$\omega(e^+) = \frac{i}{2}x^2, \tag{5.21}$$

$$\omega(e^-) = \frac{i}{2}\frac{d^2}{dx^2}. \tag{5.22}$$

Theorem 5.7 (Shale-Weil)**.** *The Weil representation for* $\mathfrak{sl}(2)$ *exponentiates to a unitary representation of* $\widetilde{\mathrm{SL}}(2,\mathbb{R})$*, the double cover of* $\mathrm{SL}(2,\mathbb{R})$*, on* $\mathscr{L}^2(\mathbb{R})$*.*

Proof. See Section 4.3 for a discussion. □

Explicitly, the $\widetilde{\mathrm{SL}}(2,\mathbb{R})$-representation induced by the above $\mathfrak{sl}(2)$-representation is defined by the following action, where $f \in \mathscr{L}^2(\mathbb{R})$ and $t \in \mathbb{R}$:

$$\omega(\exp(th))f(x) = e^{t/2}f(e^t x), \tag{5.23}$$

$$\omega(\exp(te^+))f(x) = e^{itx^2/2}f(x), \tag{5.24}$$

$$\omega(\exp(te^-)) = \text{convolution with } \tfrac{1+i}{2}(\pi t)^{-1/2}e^{itx^2/2}. \tag{5.25}$$

Note that the right-hand side of each equation is a one-parameter subgroup of G. To verify the (5.23) and (5.24), we may differentiate the one-parameter subgroups and check that their respective infinitessimal generators in $\mathfrak{sl}(2)$ matches with the left-hand side. The final equality is obtained from (5.24) by noticing that the one-parameter subgroup in (5.25) is the Fourier transform of the one-parameter subgroup in (5.24).

The operator of particular interest to us is

$$2\omega(\hbar) = 2i(\omega(e^-) - \omega(e^+) = x^2 - \frac{d^2}{dx^2}.$$

This is known as the *Hermite operator*. The Hermite operator has many applications in analysis as well as physics.

Recall that in Section 5.3.1, we studied the operators

$$a = x + \frac{d}{dx} \quad \text{and} \quad a^+ = x - \frac{d}{dx}.$$

Note that the Hermite operator is the product of these to operators; that is,

$$2\omega(\hbar) = aa^+.$$

Hence our study of the operators a and a^+ give us a basis of $\omega(\hbar)$-eigenvectors of the $\mathbb{C}$-vector space $\mathscr{S}(\mathbb{R})$.

5.6 The Module Structure of the Weil Representation

In Section 5.3.1, we studied the relationship between Hermite functions and the Heisenberg algebra representation. The close ties between Heisenberg algebra representation and the Weil representation for $\mathfrak{sl}(2)$ suggest a relationship between the operators a and a^+ and the Weil representation. We write down this relationship explicitly.

From the definitions, we have

$$x = \frac{1}{2}(a + a^+), \tag{5.26}$$

$$\frac{d}{dx} = \frac{1}{2}(a - a^+), \tag{5.27}$$

and hence we may express the Weil representation in terms of the operators a and a^+:

$$\omega(e^+) = \frac{i}{8}(a + a^+)^2, \tag{5.28}$$

$$\omega(e^-) = \frac{i}{8}(a - a^+)^2, \tag{5.29}$$

$$\omega(h) = \frac{1}{4}(a^2 - (a^+)^2) + \frac{1}{2}. \tag{5.30}$$

In terms of the standard basis $n^+, n^-, \hbar$,

$$\omega(n^+) = \frac{1}{4}(-(a^+)^2 + 1), \tag{5.31}$$

$$\omega(n^-) = \frac{1}{4}(a^2 + 1), \tag{5.32}$$

$$\omega(\hbar) = \frac{1}{4}(aa^+ + a^+a). \tag{5.33}$$

Recall from Section 5.3.1 that we defined a basis $\{v_j : j \in \mathbb{Z}_{\geq 0}\}$ of the Schwartz space $\mathscr{S}(\mathbb{R})$ consisting of Hermite functions. The Hermite functions are defined iteratively by setting

$$v_0 := e^{-x^2/2} \quad \text{and} \quad v_j := (a^+)^j v_0.$$

As we saw in Section 5.3.1, the operators a and a^+ play an important role on the Hermite functions since

$$av_j = 2jv_{j-1}. \tag{5.34}$$

It follows that the Hermite functions v_j are eigenfunctions with respect to the operators aa^+ and a^+a. The aa^+-eigenvalue of v_j is $2(j+1)$ and the a^+a-eigenvalue of v_j is $2j$.

Hence from (5.33), we see that $\hbar$ acts on the Hermite function v_j by scalar multiplication:

$$\omega(\hbar)v_j = \frac{1}{4}(aa^+ + a^+a)v_j = \frac{1}{4}(2(j+1) + 2j) = \left(j + \frac{1}{2}\right)v_j.$$

From (5.31) and (5.32), we see that n^+ acts on the Hermite functions by sending v_j to v_{j+2} and n^- acts on the Hermite functions by sending v_j to v_{j-2}. Drawing this out in a picture similar to Figure 1, we obtain Figure 2. This allows us to visualize the Weil representation of $\mathfrak{sl}(2)$ in terms of a $\hbar$-eigenbasis of $\mathscr{S}(\mathbb{R})$.

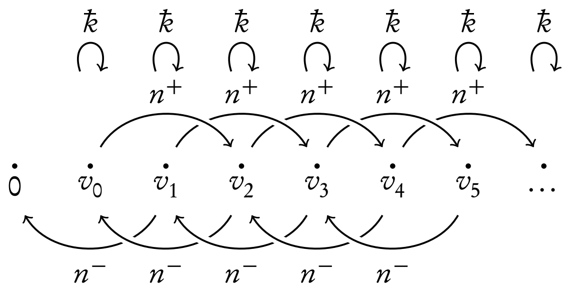

Figure 2: The Weil Representation

From the picture, the structure of the Weil representation $(\rho, \mathscr{S}(\mathbb{R}))$ as a $\mathfrak{sl}(2)$-module becomes almost immediately clear.

Proposition 5.8. *The Weil representation of* $\mathfrak{sl}(2)$ *is a direct sum of two lowest weight modules:*

$$\mathscr{S}(\mathbb{R}) = V_{1/2} \oplus V_{3/2},$$

where V_λ *denotes a lowest weight module of lowest weight* λ*.*

Proof. It suffices to show that the subspaces spanned by $\{v_{2j}\}_{j\in\mathbb{Z}_{\geq 0}}$ and $\{v_{2j+1}\}_{j\in\mathbb{Z}_{\geq 0}}$ are invariant under the action of $\mathfrak{sl}(2)$ and are in fact irreducible as $\mathfrak{sl}(2)$-modules. Indeed, the span of $\{v_{2j}\}_{j\in\mathbb{Z}_{\geq 0}}$ (respectively, $\{v_{2j+1}\}_{j\in\mathbb{Z}_{\geq 0}}$) is a lowest weight modules of lowest weight $1/2$ (respectively, $3/2$).

From the actions of $\hbar$, n^+, and n^- as described in equations (5.33), (5.31), and (5.32), respectively, we see that the described subspaces of $\mathscr{S}(\mathbb{R})$ must be $\mathfrak{sl}(2)$-invariant and must contain no $\mathfrak{sl}(2)$-invariant subspaces. □

We end with an illustration of the decomposition of the Weil representation into lowest weight modules.

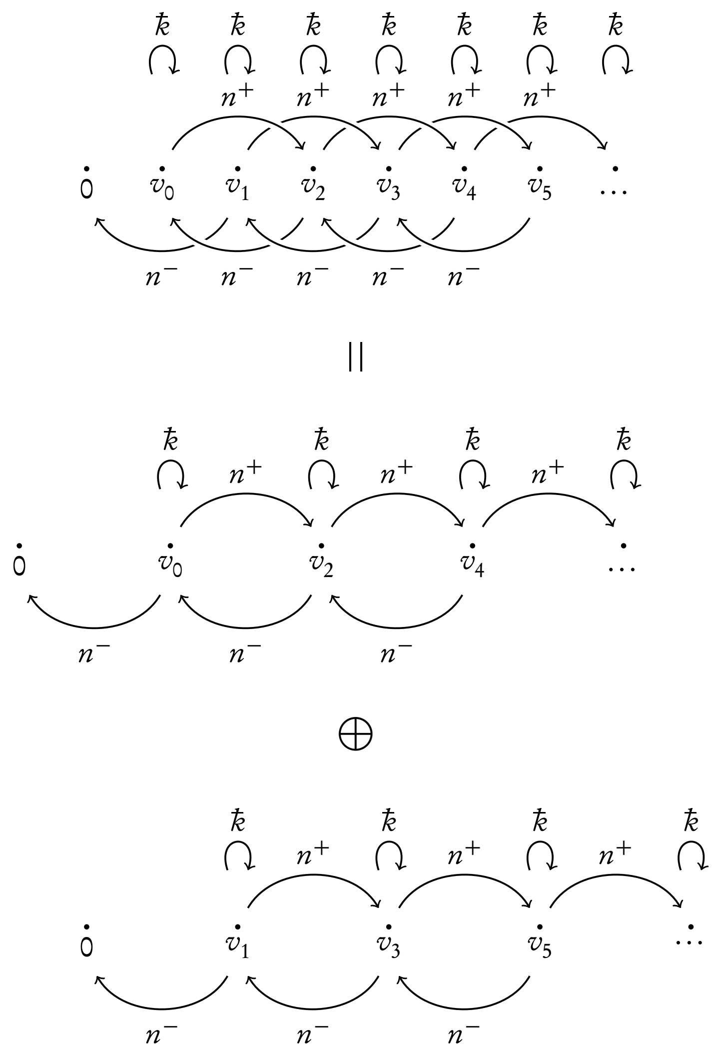

Figure 3: Decomposition of the Weil Representation into Lowest Weight Modules

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References and Bibliography

Articles and Papers

[CE48] Claude Chevalley and Samuel Eilenberg. Cohomology Theory of Lie groups and Lie Algebras. *Trans. Amer. Math. Soc.*, 63:85–124, 1948.

[DeB] Jonathan DeBacker. Some Notes of the Representation Theory of Reductive p-adic Groups.

[GH07] Shamgar Gurevich and Ronny Hadani. The Geometric Weil Representation. *Selecta Math. (N.S.)*, 13(3):465–481, 2007.

[GH09] Shamgar Gurevich and Ronny Hadani. Quantization of Symplectic Vector Spaces over Finite Fields. *J. Symplectic Geom.*, 7(4):475–502, 2009.

[How88] Roger Howe. The Oscillator Semigroup. In *The mathematical heritage of Hermann Weyl (Durham, NC, 1987)*, volume 48 of *Proc. Sympos. Pure Math.*, pages 61–132. Amer. Math. Soc., Providence, RI, 1988.

[Mit] Steve Mitchell. Some Background Material.

[Nek] Jan Nekovar. Higher Heisenberg Lie Algebras and Metaplectic Representations.

[Praa] Amritanshu Prasad. An Easy Proof of the Stone-von Neumann Theorem.

[Prab] Amritanshu Prasad. $\mathrm{GL}_2(\mathbf{F}_p)$.

[Pra09] Amritanshu Prasad. On character values and decomposition of the Weil representation associated to a finite abelian group. *J. Anal.*, 17:73–85, 2009.

[Ros] Jonathan Rosenberg. A Selective History of the Stone-von Neumann Theorem.

[Sha62] David Shale. Linear Symmetries of Free Boson Fields. *Trans. Amer. Math. Soc.*, 103:149–167, 1962.

[Wei64] André Weil. Sur Certains Groupes d'Opérateurs Unitaires. *Acta Math.*, 111:143–211, 1964.

Books

[BtD85] Theodor Bröcker and Tammo tom Dieck. *Representations of Compact Lie Groups*, volume 98 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1985.

[DF04] David S. Dummit and Richard M. Foote. *Abstract Algebra*. John Wiley & Sons Inc., Hoboken, NJ, third edition, 2004.

[Gri04] David J. Griffiths. *An Introduction to Quantum Mechanics*. Pearson [Prentice Hall], New Jersey, 2004.

[HT92] Roger Howe and Eng-Chye Tan. *Nonabelian Harmonic Analysis*. Universitext. Springer-Verlag, New York, 1992. Applications of $SL(2, \mathbf{R})$.

[Kar87] Gregory Karpilovsky. *The Schur Multiplier*, volume 2 of *London Mathematical Society Monographs. New Series*. The Clarendon Press Oxford University Press, New York, 1987.

[Lan75] Serge Lang. $\mathrm{SL}_2(\mathbf{R})$. Addison-Wesley Publishing Co., Reading, Mass.-London-Amsterdam, 1975.

[Lan02] Serge Lang. *Algebra*, volume 211 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, third edition, 2002.

[Rot79] Joseph J. Rotman. *An Introduction to Homological Algebra*, volume 85 of *Pure and Applied Mathematics*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1979.

[Wei94] Charles A. Weibel. *An Introduction to Homological Algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.

Colophon

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