The Story of $\mathfrak{sl}(2,\mathbb{C})$ and its Representations

or

Watch Charlotte Multiply 2×2 Matrices with Her Left Hand[1]

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[1]Due to a pinched radial nerve in my right hand, I was unable to write with my right hand for the majority of PROMYS 2012. As a result, I learned to write left-handed and gave this lecture left-handed. Yay for forced ambidexterity!

1 Introduction

This is a write-up of a talk I gave for the Lie Theory PROMYS counselor seminar. What we will do here is to classify all finite-dimensional representations of $\mathfrak{sl}(2,\mathbb{C}) := \{A \in \mathfrak{gl}(2,\mathbb{C}) : \mathrm{tr}(A) = 0\}$ and to explicitly give a transversal for the isomorphism classes of these representations.

Recall from the first few lectures that the representation theory of compact groups strongly resembles that of finite groups—the existence of the Haar measure allows us to replace the summations we see in the character theory of finite groups with integration with respect to the Haar measure. This allows us to make sense of Maschke's theorem of the semisimplicity of $\mathbb{C}[G]$ modules when G is a compact group. (Here, $\mathbb{C}[G]$ is the group algebra.) Explicitly, this means that for any $\mathbb{C}[G]$ module V (that is, any complex representation V of G), if W is a nontrivial submodule of V, there exists a submodule U such that $V \cong W \oplus U$. Therefore to understand the (finite-dimensional) representation theory of G, it suffices to understand the *irreducible* representations of G.

While the title of this talk suggests a focus on $\mathfrak{sl}(2,\mathbb{C})$, our actual starting point is the compact three-dimensional real Lie group $\mathrm{SU}(2) := \{A \in \mathrm{GL}(2,\mathbb{C})\colon AA^* = 1, \det(A) = 1\}$, where A^* denotes the conjugate transpose of A. We will compute the Lie algebra $\mathfrak{su}(2)$ of $\mathrm{SU}(2)$, complexify $\mathfrak{su}(2)$ to get $\mathfrak{sl}(2,\mathbb{C})$, the Lie algebra of $\mathrm{SL}(2,\mathbb{C})$, and since we may pass easily between the Lie group picture and the Lie algebra picture in this case, the problem of classifying all finite-dimensional irreducible representations of $\mathrm{SU}(2)$ is the same problem as classifying all finite-dimensional representations of $\mathfrak{sl}(2,\mathbb{C})$. For convenience of notation, we will write $\mathfrak{sl}(2) := \mathfrak{sl}(2,\mathbb{C})$.

We will construct a transversal for the irreducible representations of $\mathfrak{sl}(2)$ by considering the complex vector spaces of homogenous polynomials of degree n in x and y. With this explicit construction, we will be able to talk about the representation theory of this semisimple Lie algebra in extremely concrete terms, while simultaneously introducing the language of Lie theory. In this jargon, what we will do in the following lecture is to prove that every finite-dimensional representation of $\mathfrak{sl}(2)$ is a highest weight module of weight $n \in \mathbb{N} \cup \{0\}$, and that for every $n \in \mathbb{N} \cup \{0\}$, there exists a unique (up to isomorphism) finite-dimensional representation of $\mathfrak{sl}(2)$ of weight n. It will turn out that the highest weight module of weight n has dimension $n+1$ as a complex vector space.

2 SU(2), $\mathfrak{su}(2)$, SL(2), and $\mathfrak{sl}(2)$

Recall that

$$\mathrm{SU}(2) := \{A \in \mathrm{GL}(2,\mathbb{C})\colon AA^* = 1, \det(A) = 1\},$$

where A^* denotes the conjugate transpose of A. In other terms, SU(2) is the set of area-preserving (and orientation-preserving) linear transformations that preserve a Hermitian form.

Remark. SU(2) is only a *real* Lie group. In particular, it has no complex manifold structure! Hence $\mathfrak{su}(2)$ is only a real Lie algebra (and again, in particular, is not a complex vector space). $\Diamond$

The Lie algebra $\mathfrak{su}(2)$ of the compact real Lie group SU(2) is

$$\mathfrak{su}(2) := \{A \in \mathfrak{gl}(2, \mathbb{C}) \colon A + A^* = 0, \operatorname{tr}(A) = 0\}. \tag{1}$$

One may check explicitly that this is indeed the Lie algebra of SU(2). Indeed, first recall that for any Lie group G and its Lie algebra $\mathfrak{g}$, we have that $\mathfrak{g}$ is the tangent space of G at the identity and that we have a surjective map

$$\exp : \mathfrak{g} \to G, \quad A \mapsto \exp(A).$$

Hence for any $A \in \mathfrak{su}(2)$, we have $\exp(tA) \in \mathrm{SU}(2)$ for any $t \in \mathbb{R}$. That is,

$$\exp(tA)\exp(tA)^* = 1 \qquad \text{and} \qquad \det(\exp(tA)) = 1.$$

Since $\mathfrak{su}(2)$ is the tangent space of SU(2) at the identity, then we only need to keep track of the first-order terms. That is,

$$(I + tA)(\overline{I + tA})^{\mathrm{t}} = 1 \qquad \text{and} \qquad \det(I + tA) = 1.$$

The first equation forces the condition that $A + A^* = 0$ and the second condition forces the condition that $\operatorname{tr}(A) = 0$. This verifies the defining conditions of $\mathfrak{su}(2)$ as stated in Equation (1).

One may verify that an $\mathbb{R}$-basis of $\mathfrak{su}(2)$ is

$$\begin{aligned} X &:= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \\ Y &:= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \\ Z &:= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \end{aligned} \tag{2}$$

We may explicitly multiply these matrices together to better understand their relationship between each other:

$$\begin{aligned}
XY &= \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) = -Z, \\
YX &= \left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) = Z, \\
XZ &= \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right) = Y, \\
ZX &= \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & -i \\ -i & 0 \end{smallmatrix}\right) = -Y, \\
YZ &= \left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} -i & 0 \\ 0 & i \end{smallmatrix}\right) = -X, \\
ZY &= \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right) = X.
\end{aligned}$$

Notice also that

$$X^2 = Y^2 = Z^2 = -1.$$

Hence we have

$$\begin{aligned}
YX &= -XY = Z, \\
XZ &= -ZX = Y, \\
ZY &= -YZ = X.
\end{aligned}$$

Notice the resemblence to the pure quaternions $\mathbb{H}_p := \{ix + jy + kz : i, j, k \in \mathbb{R}\}$. In fact, if we consider $\mathbb{H}_p$ as a three-dimensional Lie algebra with the Lie bracket $[a, b] = ab - ba$ for $a, b \in \mathbb{H}_p$, then

$$\begin{aligned}
\mathfrak{su}(2) &\to \mathbb{H}_p, \\
X &\mapsto i \\
Z &\mapsto j \\
Y &\mapsto k
\end{aligned}$$

explicitly defines an isomorphism of Lie algebras.

One may check that

$$\mathfrak{su}(2) = \left\{ \left(\begin{smallmatrix} ix & -\overline{\beta} \\ \beta & -ix \end{smallmatrix}\right) : x \in \mathbb{R}, \beta \in \mathbb{C} \right\}.$$

In a computation similar to that of Equation (1), we may check that

$$\mathfrak{sl}(2) = \{A \in \mathfrak{gl}(2, \mathbb{C}) : \operatorname{tr}(A) = 0\}.$$

From this, we may check that

$$\mathfrak{sl}(2) = \mathfrak{su}(2) \oplus i\, \mathfrak{su}(2). \tag{3}$$

Therefore the following map of Lie algebras is an isomorphism:

$$\begin{aligned} \mathfrak{su}(2) \otimes_{\mathbb{R}} \mathbb{C} &\to \mathfrak{sl}(2) \\ A \otimes \lambda &\mapsto \lambda A. \end{aligned}$$

From here on, we will primarily work with the Lie algebra $\mathfrak{sl}(2)$. We will also work with the following standard basis of $\mathfrak{sl}(2)$:

$$\begin{aligned} h &:= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\ e &:= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \\ f &:= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{4}$$

Recalling that the Lie bracket is defined to be

$$[A,B] = AB - BA,$$

we may compute the Lie bracket of h, e, f by computing the matrix products. We then obtain the Lie bracket relations

$$\begin{aligned} [h,e] &= -2e, \\ [h,f] &= 2f, \\ [e,f] &= -h. \end{aligned} \tag{5}$$

The element h is chosen because it is diagonal, and the elements e and f are chosen because they are, as shown in Equation (5), eigenvectors of the adjoint action of h on $\mathfrak{sl}(2)$.

Definition 1. Let V be a representation of $\mathfrak{sl}(2)$. If $x \in V$ satisfies

$$h \cdot x = \lambda x$$

for some $\lambda \in \mathbb{C}$, then we will say that x is a *weight vector of weight* λ or an *element of weight* λ. The subspace of weight vectors of weight λ is called the *weight space of* λ.

In this language, we say that e is an element of weight -2 and f is an element of weight 2.

3 Derived Representations

To get started, we must first make precise what it means to pass from the world of Lie groups to the world of Lie algebras. Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. Recall that $\mathfrak{g}$ is the tangent space of G at the identity and that the exponential map is a surjective map $\mathfrak{g} \to G$. This motivates the following definition.

Definition 2. Let V be a set with a smooth G-action. Then the corresponding *derived action* of $\mathfrak{g}$ on V is defined to be

$$X \cdot v := \frac{d}{dt}(\exp(tX) \cdot v)\bigg|_{t=0}, \tag{6}$$

where $X \in \mathfrak{g}$ and $v \in V$.

Remark. The term "derived action" is nonstandard. ♢

Remark. One may check that Equation (6) indeed defines a Lie algebra action. That is, for $X, Y \in \mathfrak{g}$,

$$[X, Y] \cdot v = X(Y \cdot v) - Y(X \cdot v).$$

This is a straightforward verification. ♢

In the case that V is a complex vector space, Definition 2 allows us to obtain a representation of $\mathfrak{g}$ from a representation of G. This representation is called the *derived representation.*

In the next couple of sections, we will compute the derived action for different representations of SL(2).

4 The Derived Standard Representation

The most natural thing for SL(2) to act on is $\mathbb{C}^2$. Indeed, it acts on $\mathbb{C}^2$ by matrix multiplication:

$$A \cdot (x, y) := A \begin{pmatrix} x \\ y \end{pmatrix}, \quad \text{where } A \in \mathrm{SL}(2), (x, y) \in \mathbb{C}^2. \tag{7}$$

Using the formula in Equation (6), we may compute the induced derived representation of $\mathfrak{sl}(2)$. For $X \in \mathfrak{sl}(2)$,

$$\begin{aligned} X \cdot v &= \frac{d}{dt}(\exp(tX) \cdot v)\bigg|_{t=0} \\ &= \frac{d}{dt}(\exp(tX)v)\bigg|_{t=0} \\ &= \frac{d}{dt}\left(v + tXv + \frac{t^2}{2}X^2 v + \cdots\right)\bigg|_{t=0} \\ &= Xv. \end{aligned}$$

So the derived standard representation is given by matrix multiplication. This seems quite reasonable: the Lie algebra representation induced by the standard Lie group representation is exactly the standard Lie algebra representation.

5 Homogeneous Polynomials in Two Variables

What else can SL(2) act on naturally? Well, in the standard representation, we had SL(2) act on $(x,y)\in\mathbb{C}^2$ via matrix multiplication. The next natural thing we can let SL(2) act on is $\mathbb{C}[x,y]$. Define the action to be

$$A\cdot f(x,y):=f(A^{-1}\cdot(x,y)),\quad \text{where } A\in \mathrm{SL}(2) \text{ and } f\in\mathbb{C}[x,y]. \tag{8}$$

Here, $A^{-1}\cdot(x,y)$ denotes the action of A^{-1} on (x,y) in the standard representation described in Section 7. It is a direct computation to check that Equation (8) defines a group action.

We would now like to compute the induced derived action. To understand how $\mathfrak{sl}(2)$ acts on $\mathbb{C}[x,y]$, it is sufficient to understand how $\mathfrak{sl}(2)$ acts on a monomial $x^k y^{n-k}$. Furthermore, since we know that e,f,h as defined in Equation (4) defines a basis of $\mathfrak{sl}(2)$, it is sufficient to compoute the derived action of e, f and h on $x^k y^{n-k}$. Using the formula in Equation (6) again, we have

$$\begin{aligned}
e\cdot x^k y^{n-k} &= \frac{d}{dt}\left(\exp(te)\cdot x^k y^{n-k}\right)\Big|_{t=0}\\
&= \frac{d}{dt}\left(\left(\begin{smallmatrix}1&0\\t&1\end{smallmatrix}\right)\cdot x^k y^{n-k}\right)\Big|_{t=0}\\
&= \frac{d}{dt}\left(x^k(-xt+y)^{n-k}\right)\Big|_{t=0}\\
&= (x^k)(n-k)(-x)(-xt+y)^{n-k-1}\Big|_{t=0}\\
&= -(n-k)x^{k+1}y^{n-(k+1)}.
\end{aligned}$$

Hence as an operator on $\mathbb{C}[x,y]$, $e=\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)$ acts as $-x\frac{d}{dy}$. From a very similar computation, we obtain that as an operator on $\mathbb{C}[x,y]$, $f=\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$ acts as $-y\frac{d}{dx}$. From the Lie bracket relations in Equation (5), we know that

$$h=[f,e]=fe-ef,$$

so $h=\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$ acts on $\mathbb{C}[x,y]$ by

$$y\frac{d}{dx}x\frac{d}{dy}-x\frac{d}{dy}y\frac{d}{dx}=yx\frac{d}{dx}\frac{d}{dy}+y\frac{d}{dy}-xy\frac{d}{dy}\frac{d}{dx}-x\frac{d}{dx}=y\frac{d}{dy}-x\frac{d}{dx}.$$

We may think of $\mathfrak{sl}(2)$ as the operator algebra generated by the above operators on the polynomial ring $\mathbb{C}[x,y]$. That is,

$$\begin{aligned} e &= -x\frac{d}{dy} \\ f &= -y\frac{d}{dx} \\ h &= y\frac{d}{dy} - x\frac{d}{dx}. \end{aligned} \tag{9}$$

A key observation is that each of these operators sends a monomial of degree n to another monomial of degree n. In particular, this means that we may decompose $\mathbb{C}[x,y]$ into complex vector spaces of homogenous polynomials. Let

$$V_n := \{\text{degree-}n \text{ homogeneous polynomials in } x \text{ and } y\} \subseteq \mathbb{C}[x,y].$$

It is clear that the degree n monomials $x^k y^{n-k}$ for $k = 0, \dots, n$ form a basis of V_n. Hence

$$\dim(V_n) = n + 1.$$

Since V_n is preserved by the action of e, f, and h, we know that it must be invariant under the action of $\mathfrak{sl}(2)$. Hence V_n is an $(n+1)$-dimensional representation of $\mathfrak{sl}(2)$.

6 Finite-dimensional Irreducible Representations of $\mathfrak{sl}(2)$

In this section, we will analyze the structure of the $\mathfrak{sl}(2)$-module V_n. For convenience, let us write

$$w_k := x^k y^{n-k}. \tag{10}$$

Then from the equations in (9), we have

$$\begin{aligned} e w_k &= (n-k) w_{k+1}, \\ f w_k &= k w_{k-1}, \\ h w_k &= \left(y\frac{d}{dy} - x\frac{d}{dx} \right) w_k \\ &= y(n-k)(x^k y^{n-k-1}) - x(k)(x^{k-1} y^{n-k}) \\ &= (n-2k) w_k. \end{aligned} \tag{11}$$

From this, we see that e and f operate by shifting the weight spaces around. This can be seen in the above explicit computation, but this in fact follows from the Lie bracket operations of $\mathfrak{sl}(2)$. The following proposition highlights the important ideas in this phenomenon.

Proposition 3. *Let V be a representation of $\mathfrak{sl}(2)$. If $w \in V$ is an element of weight $\lambda \in \mathbb{C}$, then*

(i) ew is an element of weight $\lambda - 2$, and

(ii) fw is an element of weight $\lambda + 2$.

Proof. We have

$$hew = [h,e]w + ehw = -2ew + ehw = (\lambda - 2)ew,$$

which proves (i) and

$$hfw = [h,f]w + fhw = 2fw + 2fhw = (\lambda + 2)fw,$$

which proves (ii). □

Motivated by Proposition 3, we call e a *lowering operator* and f a *raising operator*. Putting all of this information together, we obtain the following picture of V_n:

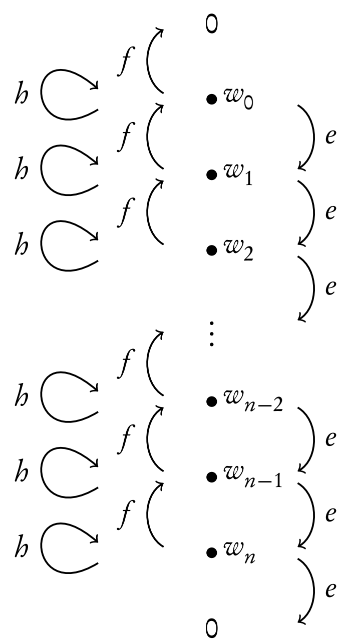

Figure 1: Ladder Operators on V_n

This picture is one that shows up everywhere in the representation theory of Lie algebras. In $\mathfrak{sl}(2)$, the picture is simple and clear, and one can really see explicitly how everything fits together.

At the same time, $\mathfrak{sl}(2)$ is complicated enough that its story strongly resembles the story of $\mathfrak{g}$ for any (semisimple) Lie algebra $\mathfrak{g}$.

From the picture, we may prove that V_n is irreducible.

Remark. This is a miraculous thing that it is so easy to prove the irreducibility of these representations. Indeed, in the Lie group setting, some analytic work must be done if one would want to proceed via checking that the inner product of the character of V_n with itself is 1. ◊

Proposition 4. *V_n is irreducible as a $\mathfrak{sl}(2)$-representation.*

Proof. Let $W \subset V_n$ be a nonzero $\mathfrak{sl}(2)$-invariant subspace of V_n. Pick a nonzero vector $w \in W$ and write

$$w = \alpha_0 w_0 + \alpha_1 w_0 + \cdots + \alpha_n w_n, \quad \text{for } \alpha_i \in \mathbb{C}.$$

Let k be the greatest i such that $\alpha_i \neq 0$. Then

$$f^k w = c\alpha_k w_k,$$

for some nonzero constant c. (One may compute that $c = k!$.) But this means that $w_k \in W$ and from Figure 1, we can see that we must have all the w_i in W since we can just apply the raising and lowering operators f and e to w_k. Therefore $W = V_n$, and this completes the proof that V_n is irreducible. □

The irreducibility of V_n as a SL(2)-representation (and also as a SU(2)-representation) follows directly.

Corollary 5. *V_n is irreducible as a* SL(2)*-representation.*

7 Highest Weight Modules

So far, we have constructed an irreducible representation of $\mathfrak{sl}(2)$ of dimension n for every $n \in \mathbb{N}$. What is remarkable is that these turn out to be *all* of the representations of $\mathfrak{sl}(2)$. In order to prove this result, we need to introduce some new language.

Definition 6. Let V be a representation of $\mathfrak{sl}(2)$. We say that a nonzero element $v \in V$ is *primitive* if it is a weight vector and if $fv = 0$.

Proposition 7. *Every finite-dimensional representation V of $\mathfrak{sl}(2)$ has a primitive element.*

Proof. We know that there is some nonzero weight vector $v \in V$. By Proposition 3, we know that for any nonzero weight vector v such that $fv \neq 0$, v and fv must be linearly independent (as they are elements of different weights). Since V is finite-dimensional, there must exist some $k \in \mathbb{N}$ such that $f^k v = 0$ and $f^{k-1}v \neq 0$. Then $f^{k-1}v$ is a primitive element. □

Given an irreducible finite-dimensional representation V and a primitive element v, we may define

$$w_{-1} := 0,$$
$$w_0 := v,$$
$$w_k := \frac{(-1)^k}{k!} e^k w_0.$$

The constant term of $\frac{(-1)^k}{k!}$ comes from the extra constants that accumulate from applying the operator $-x\frac{d}{dy}$ to our space V_n of homogeneous polynomials of degree n in x and y. These w_k's are defined to exactly model the w_k's that we defined in the previous sections. Using only Proposition 3, we may generate Figure 1 for any irreducible finite-dimensional representation V. What remains to be shown, then, is that the highest weight of an irreducible finite-dimensional representation V must be a natural number n.

Proposition 8. *Let λ be the highest weight of an irreducible finite-dimensional $\mathfrak{sl}(2)$-representation V. Then $\lambda \in \mathbb{N}$.*

Proof. Let $v \in V$ be a primitive element. Necessarily v is an element of weight λ. We know that there exists an n such that $e^{n+1}v = 0$ but $e^n v \neq 0$ and that the collection of e^i for $i = 0, \ldots, n$ forms a basis of V. We also know that $e^i v$ is an element of weight $\lambda - 2i$. Now, using the commutator relation $h = [f, e]$ repeatedly, we have the following string of equalities:

$$\begin{aligned}
f e^{n+1} v &= [f, e]e^n v + e f e^n v \\
&= h e^n v + e f e^n v \\
&= (\lambda - 2n)e^n v + e((\lambda - 2(n-1))e^{n-1} v + e f e^{n-1} v) \\
&= [(\lambda - 2n) + (\lambda - 2(n-1)) + \cdots (\lambda - 2(1))(\lambda)]\, e^n v + e^{n+1} f v \\
&= \left[(n+1)\lambda - 2\left(\sum_{k=1}^{n} k \right) \right] e^n \\
&= ((n+1)\lambda - n(n+1))e^n v \\
&= (n+1)(\lambda - n)e^n v.
\end{aligned}$$

But $e^{n+1}v = 0$ by assumption and $e^n \neq 0$, and therefore we must have that $\lambda = n$. In particular, this shows that if V is a representation of $\mathfrak{sl}(2)$ of dimension $n+1$, it is a highest weight module of highest weight n. If V is the trivial representation, then it has highest weight 0. This completes the proof. □

Remark. We have implicitly shown that irreducible finite-dimensional representations of the same dimension are isomorphic as they have the same highest weight.

8 Conclusion

What we have shown in this paper is the following. In Section 7, we showed that every finite-dimensional representation of $\mathfrak{sl}(2)$ is a highest weight module of weight λ and that $\lambda \in \mathbb{N}$. Since we have also shown that the highest weight of an irreducible representation uniquely determines the representation (up to isomorphism).

In Sections 5 and 6, we constructed a representation of dimension n for every $n \in \mathbb{N}$. In our construction, we also had that these were highest weight modules of highest weight $n-1$, though we did not say it in this language. Therefore, the complex vector spaces of homogenous polynomials of degree n in x and y, which we denoted by V_n, exactly form a transversal for the isomorphism classes of the finite-dimensional irreducible representations of $\mathfrak{sl}(2)$! So we have completed our goal!