REPRESENTATION THEORY OF FINITE GROUPS

CHARLOTTE CHAN

Contents

Introduction: Why These Notes Exist

These are notes that I will be typing up over the course of the Fall 2010 semester. They are based off the reading course given by Dr. Pter Hermann through the Budapest Semesters in Mathematics program. Additional comments have been added by me for my own benefit, as I am writing this for the sole purpose of learning the material better. For this course, the textbook for reading and reference will be Martin Isaacs' *Character Theory of Finite Groups*. We will cover about half of the book over the course of this semester. It is (according to Professor Hermann) a readable book, so it would be appropriate for this (planned-to-be) reading course.

1. Lecture: 10 September 2010

This first lecture will be an approach from an elementary perspective, That is, we will not use the language of modules during this discussion.

We begin by defining a representation. We will give two definitions: one from the perspective of a finite vector space V over $\mathbb{C}$ and one from the perspective of $\mathbb{C}^n$. We will denote these (A) and (B), respectively, and throughout this lecture, we will give different approaches to the theory. (It may seem that these two approaches are equivalent, but what is worth noting is the (A) gives the theory in a basis-free way, whereas (B) gives the theory in a way that seems natural and tangible by way of linear algebra.)

Definition 1.1. A *representation* is a homomorphism $f : G \to \mathrm{GL}(V)$ (resp. $f : G \to \mathrm{GL}_n(\mathbb{C})$) where V is a finite vector space over $\mathbb{C}$.

In this course, we will only examine the case when G is finite. Now consider the notion of an invariant subspace, which leads naturally into the notion of an irreducible representation.

Definition 1.2. A *G-invariant subspace in V* is a subspace $W \le V$ (resp. $W \le \mathbb{C}^n$) such that, for all g, W is $f(g)$-invariant (resp. W is invariant under the action of elements of $\mathrm{GL}_n(\mathbb{C})$ defined by matrix multiplication).

Definition 1.3. A representation f (resp. $\tilde{f}$) is *irreducible* if only 0 and V (resp. 0 and $\mathbb{C}^n$) are the only invariant subspaces V (resp. $\mathbb{C}^n$).

We now look at the notion of a (direct) sum of representations.

Definition 1.4. (A) If $f_1 : G \to \mathrm{GL}(V_1)$ and $f_2 : G \to \mathrm{GL}(V_2)$ are two representations, then the direct sum of f_1 and f_2 gives rise to a representation $f : G \to \mathrm{GL}(V_1 \oplus V_2)$.

(B) If $\tilde{f}_1 : G \to \mathrm{GL}_{n_1}(\mathbb{C})$ and $\tilde{f}_2 : G \to \mathrm{GL}_{n_2}(\mathbb{C})$ are two representations, then the direct sum of $\tilde{f}_1$ and $\tilde{f}_2$ gives rise to a representation $\tilde{f} : G \to \mathrm{GL}_{n_1+n_2}(\mathbb{C})$ that sends $g \mapsto \begin{pmatrix} \tilde{f}_1(g) & 0 \\ 0 & \tilde{f}_2(g) \end{pmatrix}$.

But how does this relate to the notion of irreducible representations? We first need a notion of equivalence between representations, and then we move into Maschke's theorem.

Definition 1.5. (A) For f_1, f_2 defined as before, we say that they are *equivalent*, denoted $f_1 \sim f_2$ if there exists a linear isomorphism $\beta : V_1 \to V_2$ such that, for any $g \in G, v \in V_1$, we have $(v^{f_2(g)})^\beta = (v^\beta)^{f_1(g)}$. (The notation $v^{f(g)}$ denotes the action of $f(g)$ on v. So in words, this says that if we act by $f_2(g)$ first and then by β, we get the same thing as if we act by β first and then by $f_1(g)$.) So $f_1(g) \circ \beta = \beta \circ f_2(g)$, or, $f_2(g) = \beta^{-1} \circ f_1(g) \circ \beta$.

(B) For $\tilde{f}_1, \tilde{f}_2$ defined as before, we say that they are *equivalent*, denoted $\tilde{f}_1 \sim \tilde{f}_2$ if there exists an invertible matrix $M \in \mathbb{C}^{n_2 \times n_1}$ such that, for any $g \in G$, $\tilde{f}_1(g)M = M\tilde{f}_2(g)$, or, $\tilde{f}_2(g) = M^{-1}\tilde{f}_1(g)M$. (Notice that the invertibility of M implies that $n_2 = n_1$.)

Recall from linear algebra the notion of a projection map.

Definition 1.6. A map $\pi \in \mathrm{Hom}(V)$ is a *projection* if $\pi^2 = \pi$. (So $\pi|_{Im\pi} = id_{Im\pi}$.)

Notice that π has the following property:

Lemma 1.1. *If $\pi \in \mathrm{Hom}(V)$ is a projection, then $V = \ker \pi \oplus \pi(V)$.*

Proof. Assume $c \in \ker \pi \cap \pi(V)$. $c \in \ker \pi$ implies that $\pi(c) = 0$ while $c \in \pi(V)$ implies $\pi(c) = c$, so $c = 0$. □

Now we can state Maschke's theorem.

Theorem 1.1. *(Maschke) Every representation is equivalent to a direct sum of (finitely many) irreducible representations.*

Proof. (via (A)) Let W be an invariant subspace in V with respect to the representation $f : G \to \mathrm{GL}(V)$. We want to show that there exists a $U \le V$ such that $V = W \oplus U$, where U is also invariant. We see that this is true by first considering a projection π with $\pi(V) = W$. (We can construct such a π by decomposing $V = W \oplus U'$, where π acts trivially on W and then kills U'. Keep in mind that U' must not necessarily be invariant.) Now we define a new function, π^* and prove that it is a projection. Let

$$\pi^* = \frac{1}{|G|} \sum_{g \in G} f^{-1}(g) \pi f(g).$$

Looking at this, we can see that if $w \in W$, then $\pi^*(w) = w$, so $W \subseteq \pi^*(W)$. We also have that $\pi^*(W) \subseteq W$. Now let $U = \ker \pi^*$. What remains to be shown is that U is invariant. Pick $h \in G$. Then $f^{-1}(h)\pi^* f(h) = \frac{1}{|G|} \sum_{g \in G} f^{-1}(h) f^{-1}(g) \pi f(g) f(g) = \pi^*$. Hence $\pi^* f(h) = f(h)\pi^*$. This means that, for $u \in U$, we have $\pi^* f(h)(u) = f(h)(\pi^*(u)) = f(h)(0) = 0$, so U is invariant. This completes the proof as we have found an invariant U such that $W = V \oplus U$. □

Remark. It is worth noting that Maschke's Theorem certainly holds for all vector spaces. However, if we consider a general $F[G]$-module, it holds only for finite groups G and fields F whose characteristic does not divide G. Since we are only dealing with finite groups G and the field $\mathbb{C}$ right now, we need not worry about the instances when the theorem fails.

Theorem 1.2. *(Schur's Lemma) (1) Let $f_1 : G \to \mathrm{GL}_{n_1}(\mathbb{C})$ and $f_2 : G \to \mathrm{GL}_{n_1}(\mathbb{C})$ be irreducible representations, and let $M \in \mathbb{C}^{n_2 \times n_1}$ be such that $M f_1(g) = f_2(g) M$. If $f_1 \not\sim f_2$, then $M = 0$. (2) Let $f : G \to \mathrm{GL}_n(\mathbb{C})$ be an irredicible representation. If $M \in \mathbb{C}^{n \times n}$ is such that $M f(g) = f(g) M$ then $M = \lambda I_n$. (Note that we do not require M to be invertible.)*

Proof of (1). $\ker M$ is invariant w.r.t. f_1 since $z \in \ker M \Rightarrow Mz = 0 \Rightarrow f_2(g)Mz = 0 \Rightarrow M f_1(g) z = 0$. Also $\mathrm{Im} M$ is invariant w.r.t. f_2 since $y \in \mathrm{Im} M \Rightarrow y = Mx$ for some $x \in \mathbb{C}^{n_2 \times n_1} \Rightarrow f_2(g) M x = M f_1(g) x \in \mathrm{Im} M$. This means that $\ker M$ and $\mathrm{Im} M$ are both trivial. If $M \neq 0$, then $n_1 = n_2$. If $n_1 \neq n_2$, then $M = 0$. This completes the proof. □

Proof of (2). There is some nonzero vector $v \in \mathbb{C}^{n \times n}$ such that $Mv = \lambda v$, $\lambda \in \mathbb{C}$. Let $M_1 = M - \lambda I$. M_1 still has the property that, for all $g \in G$, $M_1 f(g) = f(g) M_1$. The proof of (1) shows that $\ker M_1$ and $\mathrm{Im} M_1$ are invariant. By construction, $v \in \ker M_1$. Since f is irreducible, then it must be that $M_1 = 0$. Hence $M = \lambda I_n$. □

This was the end of the first lecture. Some vague ideas for homework were thrown out, including the suggestion to read Chapter 1 of the text (module approach), try the exercises from Chapter 1, and look at Chapter 2.

2. Lecture: 17 September 2010

In this lecture, we continue with the elementary approach to introducing some fundamental concepts of representation theory. We will take f_1, f_2, f to be irreducible representations of some given group G and call each of their characters χ_1, χ_2, χ, respectively.

We begin by defining a "sandwich" matrix:

$$M := \sum_{g\in G} f_2(g^{-1})Af_1(g), \text{ for } A \in \mathbb{C}^{n_2\times n_1}.$$

Notice that M satisfies $Mf_1(g) = f_2(g)M$. This is a straightforward check. Pick any $h \in G$. then

$$\begin{aligned} f_2(h^{-1})Mf_1(h) &= f_2(h^{-1})\Big(\sum_{g\in G} f_2(g^{-1})Af_1(g)\Big)f_1(h) \\ &= \sum_{g\in G} f_2((gh)^{-1})Af_1(gh) = M. \end{aligned}$$

In particular, we will investigate the case when $A = E^{i,j}$. We take a slight detour here to discuss notation. We will write $E^{i,j}$ to denote the $n\times n$ matrix with a 1 in the ith row, jth column and 0's elsewhere. For any matrix D, we will write $D_{i,j}$ to mean the entry in the ith row, jth column of D. Returning to the subject matter at hand, we can write out the definition of M and then apply properties of linear algebra, and we get the following string of equalities:

$$\begin{aligned} M_{k,l} &= \sum_{g\in G}(f_2(g^{-1})E^{i,j}f_1(g))_{k,l} = \sum_{g\in G}\sum_{t=1}^{n_2} f_2(g^{-1})_{k,t}(E^{i,j}f_1(g))_{t,l} \\ &= \sum_{g\in G}\sum_{t=1}^{n_2} f_2(g^{-1})_{k,t}\cdot\sum_{p=1}^{n_1} E^{i,j}_{t,p}f_1(g)_{p,l} \\ &= \sum_{g\in G}\sum_{t=1}^{n_2}\sum_{p=1}^{n_1} f_2(g^{-1})_{k,t}f_1(g)_{p,l}\delta_{i,t}\delta_{j,p} \\ &= \sum_{g\in G} f_2(g^{-1})_{k,i}f_1(g)_{j,l}. \end{aligned}$$

From this result and also Schur's lemma, we can conclude that

$$0 = \sum_{g\in G} f_2(g^{-1})_{k,i}f_1(g)_{j,l} \text{ for all (appropriate) } k, l.$$

Furthermore, notice that if $M = \sum_{g\in G} f(g^{-1})E^{i,j}f(g)$, then, again by Schur's lemma, we have $M = \lambda I$ so

$$\lambda\delta_{k,l} = M_{k,l} = \sum_{g\in G} f(g^{-1})_{k,i}f(g)_{j,l} \text{ for all (appropriate) } k, l.$$

Let us now introduce the notion of character.

Definition 2.1. The *character* of a representation f for G is a function $\chi : G \to \mathbb{C}$ defined as $\chi(g) := \mathrm{Tr}(f(g))$. Here, we would say that χ is the *character of G afforded by f*.

Notice that by definition, and by recalling from linear algebra that $\mathrm{Tr}(A) = \mathrm{Tr}(B^{-1}AB)$, the character χ is a *class function*, i.e. it is constant on each conjugacy class of G, i.e. for $g_1, g_2 \in G$, $g_1 \sim g_2 \Rightarrow f(g_1) \sim f(g_2)$. It is clear that we lose a lot of information by only considering the trace of the matrix corresponding to a given representation. Nevertheless, it turns out that the character of a representation still carries a lot of information about the group G. For instance, for some special small groups, it can tell us the size of all the conjugacy classes. It turns out that we can also compute λ from the character, where λ is the complex constant satisfying $M = \lambda I$ where

M is the sandwich matrix corresponding to some matrix. We begin by investigating the following:

$$\sum_{g\in G}\chi_2(g^{-1})\chi_1(g) = \sum_{g\in G}\left(\sum_{k=1}^{n_2}(f_2(g^{-1}))_{k,k}\sum_{l=1}^{n_1}(f_1(g))_{l,l}\right)$$
$$= \sum_{k,l}\sum_{g\in G} f_2(g^{-1})_{k,k}f_1(g)_{l,l}.$$

Using our previous calculation of $M_{k,l}$, where M is the $n\times n$ sandwich matrix corresponding to $E^{k,l}$, we get that

$$\sum_{g\in G}\chi_2(g^{-1})\chi_1(g) = \begin{cases}\sum_{k,l}\lambda\delta_{k,l} & \text{if } f_1\sim f_2\\ 0 & \text{otherwise.}\end{cases}$$

Say we take $f_1\sim f_2$. Then for the λ in the above identity,

$$\lambda I = M = \sum_{g\in G} f_2(g^{-1})E^{k,l}f_1(g).$$

Taking the trace of both sides and dividing by n, we get

$$\lambda = \frac{1}{n}\operatorname{Tr}(\lambda I) = \frac{1}{n}\operatorname{Tr}\left(\sum_{g\in G} f_2(g^{-1})E^{k,l}f_1(g)\right)$$
$$= \frac{1}{n}\sum_{g\in G}\operatorname{Tr}(f_2(g^{-1})E^{k,l}f_1(g)) = \frac{|G|}{n}\delta_{k,l}.$$

This may seem strange at first, as we have found that λ depends on the choice of i,j, but this is fine. See, λ is completely dependent on the choice of M, and M is completely dependent on the choice of i,j since it is the sandwich matrix corresponding to $E^{i,j}$. So, in fact, it is quite normal (expected, even!) that λ depends on i,j. Now, combining these results, we have (supposedly) hence proved the following result.

Theorem 2.1. *For irreducible representations f_1, f_2 of G with characters χ_1,χ_2, respectively, we have*

$$\sum_{g\in G}\chi_2(g^{-1})\chi_1(g) = \begin{cases}|G| & \text{if } f_1\sim f_2\\ 0 & \text{if } f_1\not\sim f_2\end{cases}$$

Proof. It is clear that the sum is 0 if $f_1\not\sim f_2$. For the case when $f_1\sim f_2$, from the previous results, we have

$$\sum_{g\in G}\chi_1(g^{-1})\chi_2(g) = \sum_{k,l}\lambda\delta_{k,l} = \sum_{k,l}\frac{|G|}{n}\delta_{k,l}^2 = |G|,$$

as desired. □

We have the following proposition, which will allow us to write $\sum_{g\in G}\chi_2(g^{-1})\chi_1(g)$ in a cleaner way.

Theorem 2.2. *For any character χ of f, an irreducible representation of G, $\chi(g^{-1}) = \overline{\chi(g)}$.*

This gives us a relationship between characters of irreducible representations that we can formalize by introducing a notion of an inner product on these characters.

Definition 2.2. Let χ_i, χ_i be the characters of G afforded by f_1, f_2, respectively. We define the inner product $\langle\chi_i,\chi_j\rangle$ in the following way:

$$\langle\chi_i,\chi_j\rangle = \frac{1}{|G|}\sum_{g\in G}\overline{\chi_2(g)}\chi_1(g).$$

From the above definition and Theorem 2.2, we see that the set of characters of irreducible representations of G forms an orthonormal set. From this, it is not so difficult to see that the equivalence of characters directly corresponds with the equivalence of representations. We consider the case of irreducible representations f_1, f_2 first.

Theorem 2.3. $f_1 \sim f_2 \Longleftrightarrow \chi_1 = \chi_2$.

Proof. If $f_1 \sim f_2$, then there is an invertible matrix T such that $f_1 = T^{-1} f_2 T$. Pick an element $g \in G$. Then, $\chi_1(g) = \mathrm{Tr}(f(g)) = \mathrm{Tr}(T^{-1} f_2(g) T) = \mathrm{Tr}(f_2(g)) = \chi_2(g)$. So $\chi_1 = \chi_2$. Conversely, assume $\chi_1 = \chi_2$. Then $\langle \chi_1, \chi_2 \rangle = \frac{1}{|G|} \sum_{g\in G} \overline{\chi_2(g)} \chi_1(g) = \frac{1}{|G|} \sum_{g\in G} |\chi_2|^2 > 0$. So in particular, $\sum_{g\in G} \overline{\chi_2(g)} \chi_1(g) \neq 0$, so $f_1 \sim f_2$. □

Now we can generalize to any two representations h_1, h_2, not necessarily irreducible, and their characters ψ_1, ψ_2, respectively.

Theorem 2.4. $h_1 \sim h_2 \Longleftrightarrow \psi_1 = \psi_2$.

Proof. ($\Rightarrow$) is obvious. For ($\Leftarrow$), we can write $\psi_1 = \sum_{i=1}^{t} m_i \chi_i$ and $\psi_2 = \sum_{i=1}^{t} q_i \chi_i$. Pick an arbitrary irreducible character χ_k. Since $\psi_1 = \psi_2$, then certainly $\langle \psi_1, \chi_k \rangle = \langle \psi_2, \chi_k \rangle$. So $m_k = q_k$. This holds for every $k = 1, \ldots, t$, so by the previous result, we have that h_1, h_2 have the same number of each irreducible submodule and hence they must be equivalent representations. □

Amazingly enough, it turns out that the set of characters of irreducible representations of G not only forms an orthonormal set, but it forms an orthonormal basis of the space of class functions of G! For ease, we introduce some notation. We will write $\mathrm{Irr}(G)$ to mean the set of all irreducible characters of G and $\mathrm{cf}(G)$ to mean the inner product space of class functions of G. We summarize by asserting the following.

Theorem 2.5. $\mathrm{Irr}(G)$ *is an orthonormal basis of* $\mathrm{cf}(G)$.

THERE ARE STILL SOME THINGS TO BE DONE TO THESE NOTES: We need to prove Theorem 2.2, which I don't know how to do it yet.

3. Lecture: 24 September 2010

We continue our discussion of basic representation and character theory from an elementary perspective. This will conclude this approach; in future weeks, we will use the language of modules and more sophisticated algebra to discuss the theory.

Recall that $\mathrm{cf}(G) = \{f : G \to \mathbb{C} \mid \text{for all } x, y \in G, f(y^{-1}xy) = f(x).\}$. Clearly, this forms a vector space over $\mathbb{C}$. It has dimension equal to the number of conjugacy classes of G (i.e. the class number of G). In fact, this is not only a vector space but it's an inner product space! We can define the inner product in the same way that we defined the inner product on characters of G. That is, if $f_1, f_2 \in \mathrm{cf}(G)$, then

$$[f_1, f_2] := \frac{1}{|G|} \sum_{g \in G} \overline{f_1(g)} f_2(g).$$

Recall from last lecture that we found that the irreducible characters of G formed an orthonormal set, since for any $\chi_1, \chi_2 \in \mathrm{Irr}(G)$, we had

$$[\chi_1, \chi_2] = \begin{cases} 1 & \text{if } \chi_1 = \chi_2 \\ 0 & \text{otherwise.} \end{cases}$$

In the previous notes, we ended with Theorem 2.5, the statement that $\mathrm{Irr}(G)$ is a basis of $\mathrm{cf}(G)$. Our main goal will be to prove this statement, also defining and introducing some things along the way. Now, notice that Theorem 2.5 is equivalent to saying that we cannot find an $f \in \mathrm{cf}(G)$ that is linearly independent to every element of $\mathrm{Irr}(G)$. This, in turn, is equivalent to the below statement:

Theorem 3.1. *Let $f \in \mathrm{cf}(G)$ be a such that $[f, \chi] = 0$ for all $\chi \in \mathrm{Irr}(G)$. Then $f = 0$.*

Proof. Given a character $\chi \in \mathrm{Irr}(G)$, let $\mathfrak{X} : G \to \mathrm{GL}_n(\mathbb{C})$ be the representation of G that affords χ. Let us define the following $n \times n$ matrix:

$$M_f := \sum_{g \in G} \overline{f(g)} \mathfrak{X}(g).$$

Notice that $\mathfrak{X}(h^{-1}) M_f \mathfrak{X}(h) = M_f$, so we can apply Schur's lemma to M_f and get that $M_f = \lambda I$ for some $\lambda \in \mathbb{C}$. Computing λ, we get

$$\lambda = \frac{1}{n} \mathrm{Tr}(M_f) = \frac{1}{n} \sum_{g \in G} \overline{f(g)} \,\mathrm{Tr}(\mathfrak{X}(g)) = \frac{1}{n} \sum_{g \in G} \overline{f(g)} \chi(g) = \frac{|G|}{n} [f, \chi] = 0.$$

Hence we can conclude that for each irreducible representation $\mathfrak{X}$, the corresponding $M_f = 0$. This implies that M_f with respect to an arbitrary representation $\mathfrak{Y}$ can be defined similarly and in fact $M_f = 0$ once again. (This follows from our previous argument since M_f is similar to a matrix in block form with each block identically 0.) We would like to somehow conclude from here that $f = 0$, but we don't have enough quite yet. We take a break from the proof here to discuss some theory and return to this proof at the end of the lecture.

If we define the action by G on $\mathrm{GL}_n(\mathbb{C})$ by multiplication by some group element, then the resulting representation is called the *regular representation of G*. Since multiplication by some group element simply permutes all the group elements (as seen in a Cayley table), M_f with respect to the regular representation will be a permutation matrix.

Now let Ω be a set of size n. Then if π is a permutation on Ω, the corresponding matrix P is defined to be

$$P_{i,j} = \begin{cases} 1 & \text{if } j = i^{\pi} \\ 0 & \text{otherwise.} \end{cases}$$

In fact, these matrices are multiplicative! We see this in the following way.

Let π, ν be permutations on Ω with corresponding matrices P_π, P_ν. Then we claim that $P_{\pi\nu} = P_\pi \cdot P_\nu$. (Note that functions are applied on the right. That is, when we write $\pi\nu$, we mean "do π first and then do ν.") On the left side, the k, lth entry will be 1 when $l = k^{\pi\nu}$, by definition of the permutation matrix. On the right side, the k, lth entry will be $\sum_{i=1}^{n} (P_\pi)_{k,i} \cdot (P_\nu)_{i,l}$. Now

$(P_\pi)_{k,i} = 1$ when $i = k^\pi$ and $(P_\nu)_{i,l} = 1$ when $l = i^\nu$, so it follows that the sum is 1 when $l = k^{\pi\nu}$ and 0 otherwise. It follows then that $P_{\pi\nu} = P_\pi \cdot P_\nu$.

From this, we see that multiplication by the permutation action on Ω corresponds to multiplication by the permutation matrix. In this way, we see that if $G = \Omega$, then the action of G of multiplication by some group element can be equivalently viewed as the multiplication of a permutation matrix defined in the way we chose the map $\pi \to P_\pi$. Hence we get the following definition.

Definition 3.1. The *regular representation* $\mathfrak{R}$ *of* G is the representation defined by the action of G by multiplication by some group element, i.e. G acts on $\mathrm{GL}_{|G|}(\mathbb{C})$ by multiplication by a permutation matrix. So the representation $\mathfrak{R} : G \to \mathrm{GL}_{|G|}(\mathbb{C})$ is defined by $g \mapsto \mathfrak{R}(g)$, where $(\mathfrak{R}(g))_{h,t} = 1$ if and only if $t = hg$, where $h, g, t \in G$. ($\mathfrak{R}(g)$ is 0 elsewhere.)

We now return to the proof of Theorem 3.1.

Proof of Theorem 3.1 continued. We ended before with the comment that M_f, defined with respect to any representation, gives $M_f = 0$. In particular, we can define M_f with respect to the regular representation $\mathfrak{R}$. Since $0 = M_f$, then certainly $0 = e \cdot M_f$, where e is the row vector with a 1 in the e-corresponding slot. Then we have

$$0 = e \cdot M_f = e \cdot \sum_{g \in G} \overline{f(g)} \mathfrak{R}(g) = \sum_{g \in G} \overline{f(g)} \cdot e \cdot \mathfrak{R}(g).$$

Now, $e \cdot \mathfrak{R}(g)$ picks out the e-corresponding row of $\mathfrak{R}(g)$, which has a 1 in the g-corresponding slot and 0's elsewhere. Since $\{e \cdot \mathfrak{R}(g) \mid g \in G\}$ forms a linearly independent set, then it must follow that $\overline{f(g)} = 0$ for all $g \in G$. Therefore $f = 0$, as desired. □

This concludes today's lecture. The remaining time was spent discussing problems from Chapter 2 of Isaacs' book. Problem 2.1 and 2.6 were submitted and all the assigned Chapter 2 problems will be due next Friday. The problems are: 2.1, 2.6, 2.8, 2.9, 2.10 (more combinatorial), 2.13 (hard), 2.16, 2.17.

4. Lecture: 1 October 2010

This lecture covers the first half of Chapter 3 of Isaacs' book and also brings in a small bit of Chapter 2. More importantly, by the end of this lecture, we will have (basically) proved Burnside's $p^a q^b$ theorem. So essentially, we are building up the machinery to the result. We begin by discussing algebraic integers.

Fact. A complex number α is an algebraic integer if $\mathbb{Z}[\alpha]$ is a finitely generated additive abelian group.

Proof. □

Now take any two algebraic integers α, β. We want to show that $\alpha \pm \beta$ is again an algebraic integer. To do this, consider $\mathbb{Z}[\alpha, \beta] = \{g(\alpha, \beta) : g \in \mathbb{Z}[x, y]\}$. We can show that this is a finitely generated additive abelian group. We do this by taking a basis $\{\alpha_i\}$ of $\mathbb{Z}[\alpha]$ and a basis $\{\beta_j\}$ of $\mathbb{Z}[\beta]$ and taking the products of elements in the basis, we get $\{\alpha_i \beta_j\}$, a generating set for $\mathbb{Z}[\alpha, \beta]$. Since $\mathbb{Z}[\alpha \pm \beta]$ is a subgroup of the finitely generated additive subgroup $\mathbb{Z}[\alpha, \beta]$, then it must also be finitely generated. Hence $\alpha \pm \beta$ is an algebraic integer.

Applying this to character theory, and remembering that $\chi(g) = \sum \omega$ for nth roots of unity ω, we get that $\chi(g)$ is the finite sum of algebraic integers and is hence itself an algebraic integer. This is certainly an important result, but the main thing that we are interested in is the fact that for $\chi \in \operatorname{Irr}(G), a \in G, a \in K$, where K is the conjugacy class containing a, then $\frac{\chi(a)|K|}{\chi(e)}$ is an algebraic integer! This is not at all obvious, and the rest of this lecture will be dedicated to proving and analyzing the consequences of this fact.

Consider the set $\mathbb{C}[G] = \{\sum_{g \in G} \alpha_g \cdot g \mid \alpha_g \in \mathbb{C}\}$. This is a $\mathbb{C}$-vector space! It is also a ring, with addition defined in the standard vector space way, and multiplication defined in the most natural way possible, i.e.

$$\Big(\sum_{g \in G} \alpha_g \cdot g\Big)\Big(\sum_{h \in G} \beta_h \cdot h\Big) = \sum_{c \in G} \gamma_c \cdot c, \text{ where } \gamma_c := \sum_{gh = c} \alpha_g \beta_h.$$

So we have that $\mathbb{C}[G]$ is a $\mathbb{C}$-vector space and also a ring, and to finish our verification that $\mathbb{C}[G]$ is a group algebra, we only have left to show that for any $a, b \in \mathbb{C}[G], \lambda \in \mathbb{C}$, we have $\lambda \cdot (ab) = (\lambda \cdot a) \cdot b = a \cdot (\lambda \cdot b)$, which is obvious by definition.

Now consider $\mathfrak{X} : G \to \mathrm{GL}_n(\mathbb{C})$, an irreducible representation of G. We extend $\mathfrak{X}$ linearly and construct $\widehat{\mathfrak{X}} : \mathbb{C}[G] \to M_n(\mathbb{C})$, an algebra homomorphism. To be more explicit, we construct $\widehat{\mathfrak{X}}$ by

$$\widehat{\mathfrak{X}}\Big(\sum_{g \in G} \alpha_g \cdot g\Big) := \sum_{g \in G} \alpha_g \cdot \mathfrak{X}(g),$$

and from here we see that it is straightforward to verify that $\widehat{\mathfrak{X}}$ is an algebra homomorphism.

We are interested (don't ask me why) in the center of our group algebra. We denote this by $\mathrm{Z}(\mathbb{C}[G])$. It is easy to check that $\mathrm{Z}(\mathbb{C}[G])$ is a subalgebra of $\mathbb{C}[G]$. Since it is a subalgebra, then the fact that it is a subring and also a subspace comes for free.

Now take any element $z \in \mathrm{Z}(\mathbb{C}[G])$. Then $zg = gz$ for all $g \in G$, since we can view each group element as a special element of the group algebra. This means that $g^{-1}zg = z$ for all $g \in G$. We can write $z = \sum_{a \in G} \alpha_a \cdot a$, and conjugating by a group element g, we get $g^{-1}zg = \sum_{a \in G} \alpha_a \cdot g^{-1}ag$, and since $z = g^{-1}zg$, then equating both sides gives us that $\alpha_a = \alpha_{g^{-1}ag}$ for all $g, a \in G$. This means that the coefficient of g in our expression for z is constant on each conjugacy class, and hence we can write the sum as $\sum_K \gamma_K \cdot (\sum_{k \in K} k)$, where the sum runs through all the conjugacy classes K of G. Let $\omega_K = \sum_{k \in K} k$. This is called a class sum and the set of class sums forms a linearly independent set of pairwise disjoint sums. Now, since $\mathrm{Z}(\mathbb{C}[G])$ is also a ring, then it is closed under multiplication, so $\omega_K \cdot \omega_L = \sum_M r_M \cdot \omega_M$ for some $r_M \in \mathbb{C}$. But if we think about this more carefully, we notice that r_M counts the number of ways we can choose an element of K and an element of L and get a product in M, so in fact $r_M \in \mathbb{N}$. More precisely, $r_M = \{(k, l) : m = kl, k \in K, l \in L\}$.

Now, $\widehat{\mathfrak{X}}$ is an algebra homomorphism that preserves commutativity, so $\widehat{\mathfrak{X}}(\omega_K)$ commutes with $\mathfrak{X}(g)$ for all $g \in G$. By Schur's lemma, then, we have that $\widehat{\mathfrak{X}}(\omega_K) = \lambda_K \cdot I$. From this, we can get compute λ_K:

$$\lambda_K = \frac{1}{n}\operatorname{Tr}(\lambda_K I) = \frac{1}{n}\widehat{\mathfrak{X}}(\omega_K) = \frac{1}{n}\chi(K)|K| = \frac{\chi(K)|K|}{\chi(e)}.$$

$\widehat{\mathfrak{X}}$ is an algebra homomorphism, $\widehat{\mathfrak{X}}(\omega_K)\cdot\widehat{\mathfrak{X}}(\omega_L) = \sum_M r_M \cdot \widehat{\mathfrak{X}}(\omega_M)$, and expanding the equality on each side, we get $\lambda_K\lambda_L I = (\sum_M r_M \cdot \lambda_M)I$, which gives us that $\lambda_K\lambda_L = \sum_M r_{M,K,L}\lambda_M$, where here we wrote $r_M = r_{M,K,L}$ to emphasize that this value depends on the choice of the conjugacy classes K, L.

Let K be a fixed conjugacy class and let L run over all conjugacy classes to get a homogeneous system of equations, each of the form

$$0 = \sum_M (r_{M,K,L} - \delta_{M,L}\lambda_K)\lambda_M,$$

where this equation follows directly from the previous equation after moving all expressions to one side. Since not all λ_M can be zero, then this system has a nontrivial solution, which means that if A is the matrix corresponding to this system, $\det(A) = 0$. Since $A = (r_{M,K,L} - \delta_{M,L}\lambda_K)$ is an $m\times m$ matrix where m is the number of conjugacy classes of G, then $\det(A)$ is a monic polynomial over λ_K and the fact that $\det(A) = 0$ means that λ_K is a root of this monic polynomial with integer coefficients. Hence λ_K is an algebraic integer, so $\lambda_K = \frac{\chi(K)|K|}{\chi(e)}$ is an algebraic integer. Hence we have proved the following theorem

Theorem 4.1. *Let $\chi \in \operatorname{Irr}(G)$ and let K be a conjugacy class of G. Then $\frac{\chi(K)|K|}{\chi(e)}$ is an algebraic integer.*

From this, the following (surprising!) theorem is an easy consequence

Theorem 4.2. *Let $\chi \in \operatorname{Irr}(G)$ and let K be a conjugacy class of G with $g \in K$. Assume that $(\chi(1), |K|) = 1$. Then $\frac{\chi(g)}{\chi(e)}$ is an algebraic integer.*

Proof. Apply the Fundamental Theorem of Arithmetic, i.e. $(a, c) = 1, c \mid ab \Rightarrow c \mid b$. Or alternatively, solve the appropriate linear Diophantine equation. □

It seems strange that this result comes as such an easy consequence of our development of this theory, but in what we will in fact see is that $(\chi(1), |K|) = 1$ is a very strong assumption and that the conclusion $\chi(1) \mid \chi(a)$ is very restrictive. Writing $\chi(a)$ as a sum of roots of unity and examining the quotient of $\chi(a)$ and $\chi(1)$, we get

$$\gamma := \frac{\chi(a)}{\chi(e)} = \frac{\varepsilon_1 + \cdots + \varepsilon_n}{n} \Rightarrow |\gamma| \leq 1.$$

Let θ be a primitive root of unity and consider the number field $\mathbb{Q}(\theta)$. Then $\gamma \in \mathbb{Q}(\theta)$ and $\gamma^\sigma \in \mathbb{Q}(\theta)$ for any $\sigma \in \operatorname{Gal}(\mathbb{Q}(\theta)/\mathbb{Q})$. Also, $|\gamma^\sigma| \leq 1$. Now consider the polynomial

$$\prod_{\sigma\in\operatorname{Gal}(\mathbb{Q}(\theta)/\mathbb{Q})} (x - \gamma^\sigma) \in \mathbb{Q}[x].$$

Since the image of an algebraic integer under a field automorphism is again an algebraic integer, then the above polynomial must have coefficients in $\mathbb{Z}$. In particular, the product of all Galois conjugates of γ is an algebraic integer; i.e.

$$\prod_{\sigma\in\operatorname{Gal}(\mathbb{Q}(\theta)/\mathbb{Q})} \gamma^\sigma \in \mathbb{Z}.$$

Since this product has norm at most 1, then this product is either $0, 1$, or -1. If it is 0, then $\gamma = 0$. If $\gamma = 1$, then all the roots of unity ε_i must be equal. This would meant that $\mathfrak{X}(a) = \varepsilon_1 I$, where $\mathfrak{X}$ is the representation that affords χ, and hence $\mathfrak{X}(a) \in Z(\Im(\mathfrak{X}))$. In particular, this means that if G is simple, then $a = e$.

It in fact turns out that this is most of the work we have to do in order to prove Burnside's $p^a q^b$ theorem. This was not done in class, but I may add it to these notes. Recommended homework: Almost all the problems are good here. Professor Hermann also has a particular liking towards 3.11.

5. Lecture: 8 October 2010

We will skip Chapter 4 for now, perhaps returning to it later, and proceed to Chapter 5 of Isaacs' book, which introduces the concept of induced characters. We begin with a definition.

Definition 5.1. Let $H \le G$ and consider a homomorphism $f : H \to \mathbb{C}$. Define a function $\hat{f} : G \to \mathbb{C}$ such that

$$\hat{f}(g) := \begin{cases} f(g) & \text{if } g \in H \\ 0 & \text{otherwise.} \end{cases}$$

Now define a function $f^G : G \to \mathbb{C}$ to be

$$f^G(g) := \frac{1}{|H|} \sum_{x \in G} \hat{f}(x^{-1}gx).$$

Then f^G is the *induced function from H to G.*

It is easy to see that induction is linear. Also, by definition of the induced function f^G, we can see that regardless of our choice of f, f^G is a class function of G. But what we are concerned about is the case when f is a class function of H, and after the following theorem, we will focus our attention on the case when f is a character of H.

Theorem 5.1. *(Frobenius Reciprocity) Consider $H \le G$. Let $f : H \to \mathbb{C}$ is a class function on H and $\psi : G \to \mathbb{C}$ be a class function on G. Then*

$$[f^G, \psi] = [f, \psi_H]_H.$$

Proof. This is a straightforward proof. We start with the left side and get a sequence of equalities terminating with the right side.

$$\begin{aligned} [f^G, \psi] &= \frac{1}{|G|} \sum_{x \in G} \overline{f^G(x)} \cdot \psi(x) = \frac{1}{|G|} \sum_{x \in G} \left(\frac{1}{|H|} \sum_{y \in G} \overline{\hat{f}(y^{-1}xy)} \right) \cdot \psi(x) \\ &= \frac{1}{|G|} \frac{1}{|H|} \sum_{y \in G} \sum_{x \in G} \hat{f}(y^{-1}xy) \cdot \psi(x) = \frac{1}{|G|} \frac{1}{|H|} \sum_{y \in G} \sum_{x \in G} \hat{f}(x) \cdot \psi(x) \qquad (1) \\ &= \frac{1}{|H|} \sum_{x \in G} \overline{\hat{f}(x)} \cdot \psi(x) = \frac{1}{|H|} \sum_{x \in H} \overline{f(x)} \cdot \psi_H(x) = [f, \psi_H]_H, \end{aligned}$$

where the middle equality in (1) holds by noticing that we have $\hat{f}(y^{-1}xy) \cdot \psi(x) = \hat{f}(y^{-1}xy) \cdot \psi(y^{-1}xy)$ since ψ is a class function, and then changing the variable to give simply $\hat{f}(x) \cdot \psi(x)$ since x ranges over all elements of G. The equalities above prove the theorem. □

As promised, we will now examine the induction from a character of H to a character of G.

Theorem 5.2. *Let σ be a (possibly reducible) character of H. Then σ^G is a character of G. (Here we use the same notation as has been introduced.)*

Proof. It is clear that $\sigma^G \in \mathrm{cf}(G)$. Since $\mathrm{Irr}(G)$ is an orthonormal basis for the space of class functions, then we can write $\sigma^G = \sum_{\chi \in \mathrm{Irr}(G)} \lambda_\chi \cdot \chi$, where $\lambda_\chi \in \mathbb{Z}$. We can use the Frobenius reciprocity to compute λ_χ:

$$\lambda_\chi = [\chi, \sigma^G] = [\chi_H, \sigma]_H,$$

which are all non-negative integers since χ_H and σ are characters of H. □

It would be nice if the induction of an irreducible character of H turned out to be an irreducible character of G, but unfortunately, this is not true. The inverse, however, holds.

Theorem 5.3. *Let σ be a character of H. If $\sigma^G \in \mathrm{Irr}(G)$, then $\sigma \in \mathrm{Irr}(H)$.*

Proof. □

It is not clear at this point what motivated the Frobenius reciprocity law, but one suspicion is that we can use the Frobenius reciprocity to prove that the Frobenius kernel is a subgroup of a Frobenius group. We will prove this. The first question, then, is to ask, what is a Frobenius group?

Definition 5.2. A group $G \leq S_n$ is called a *Frobenius group* if (i) G is transitive, (ii) for all non-identity $a \in G$, a doesn't have 2 fixed points, (iii) G is not regular. (In Wikipedia's words, a Frobenius group is a "transitive permutation group on a finite set, such that no non-trivial element fixes more than one point and some non-trivial element fixes a point.")

These conditions are extremely restrictive. G acts on some finite set. Call it A and let its elements be indexed as $i = 1, \dots n$. Then if we let $H := G_1$, the stabilizer of 1, then for all $i \leq n$, we have $G_i \sim H$. This is since G is transitive, and so for every $i \in A$, there is an $x \in G$ such that $i = 1^x$. Hence we have

$$G_i = G_{1^x} = x^{-1}G_1x = x^{-1}Hx,$$

which verifies the claim that $G_i \sim H$. It is necessary that H is a proper subgroup of G since G is transitive, and we also have that no two stabilizers have a nontrivial intersection. If we take some $a \in G\backslash H$, and consider an element $g \in H \cap a^{-1}Ha$, then g fixes $1 \in A$ and also $1^a \in A$, so it must be the identity. From this analysis, we can write down an equivalent definition of a Frobenius group (the verification that this definition implies the first definition is not hard).

Definition 5.3. A group G is called a *Frobenius group* if there exists a nontrivial $H \lneq G$ such that for all $a \in G\backslash H$, $H \cap a^{-1}Ha = 1_G$.

We call H the *Frobenius complement* and it turns out to be unique up to conjugation. Now say we take all the distinct conjugates of the Frobenius complement H. If we take the union of all of these subgroups, what do we have left? In general, if G is finite, then $G \neq \cup$ conjugates of H. Define a set F such that $1_G \in F$ and

$$G = (F\backslash 1) \cup (\text{conjugates of } H).$$

We call F the *Frobenius kernel* and it turns out that F is a normal subgroup of G! You would think that normality is the surprising part, but in fact, if we can prove that F is a subgroup of G, normality comes for free (since the union of conjugates of H is invariant under conjugation, F must also be invariant under conjugation and hence normal). So in fact, it is amazing that F is a subgroup at all!

There is no known character-free proof that the Frobenius kernel is a subgroup, and most purely group-theoretic proofs of special cases are very complicated. We will prove here, using character theory, that the Frobenius kernel is a subgroup by proving that it is the union of kernels of characters (so normality comes for free in the proof also, not only in the construction).

Theorem 5.4. *The Frobenius kernel is a normal subgroup of a Frobenius group.*

Proof. Let H be the Frobenius component of G and F the Frobenius kernel. Take a class function $f : H \to \mathbb{C}$ with $f(1) = 0$. Then

$$f^G(g) = \frac{1}{|H|}\sum_{x \in G} \hat{f}(x^{-1}gx) = \begin{cases} 0 & \text{if } g \in F \\ f(g) & \text{if } 1 \neq g \in x^{-1}Hx, x \in G \end{cases}$$

Hence we have $(f^G)_H = f$. Now let us consider a nontrivial irreducible character $\sigma \in \mathrm{Irr}(H)$. Define a function

$$f_\sigma := \sigma(1) \cdot 1_H - \sigma.$$

Notice that f_σ is a character of H (and hence automatically a class function) and $f_\sigma(1) = 0$. This means that if we induct on f_σ from H to G, we have $(f_\sigma^G)_H = f_\sigma$ and f_σ^G vanishes outside H. Since induction is linear, we can write

$$f_\sigma^G = \sum_{\chi \in \mathrm{Irr}(G)} \lambda_\chi \chi, \lambda_\chi \in \mathbb{Z}.$$

We want to show that $\lambda_\chi \geq 0$ for all $\chi \in \mathrm{Irr}(G)$, thereby showing that f_σ^G is a character of G.

Let's first look at the easiest case possible. When $\chi = 1_G$, Frobenius reciprocity gives us

$$[1_G, f_\sigma^G] = [1_H, f_\sigma]_H = [1_H, \sigma(1)\cdot 1_H - \sigma] = [1_H, \sigma(1)\cdot 1_H] - [1_H, \sigma] = \sigma(1) - 0 = \sigma(1).$$

This is the coefficient of the trivial character when we write f_σ^G as a linear combination of irreducible characters. Hence we have

$$f_\sigma^G = \sigma(1)\cdot 1_G + \cdots \Rightarrow [f_\sigma^G, f_\sigma^G] = \sigma(1)^2 + \cdots$$

On the other hand, we can compute the inner product $f_\sigma^G, f_\sigma^G]$ a different way:

$$[f_\sigma^G, f_\sigma^G] = [(f_\sigma^G)_H, f_\sigma] = [f_\sigma, f_\sigma] = \sigma(1)^2 + 1.$$

Hence $f_\sigma^G = \sigma(1)\cdot 1_G \pm \chi_\sigma$, where $\chi_\sigma \in \mathrm{Irr}(G)$. (Here, we write the subscript σ to remind us that the choice of this character depends on the choice of σ.) Since $f_\sigma(G) = |G:H|f_\sigma(1) = 0$, then we must have the case of subtraction, i.e.

$$f_\sigma^G = \sigma(1)\cdot 1_G - \chi_\sigma.$$

Now consider some $a \in F$. Then since a is outside H and f_σ^G vanishes outside H, then by substitution, we have $\sigma(1) - \chi_\sigma(a) = 0$, so $\chi_\sigma(a) = \sigma(1) = \chi_\sigma(1)$. In particular, this implies that $a \in \ker(\chi_a)$, so $F \subseteq \ker(\chi_\sigma) \lhd G$, and since σ was chosen arbitrarily, then necessarily we have $F \subseteq \cap_{\substack{\sigma\in\mathrm{Irr}(H)\\ \sigma\neq 1_H}} \ker\chi_\sigma$. To prove the reverse inclusion, consider $x \in \cap_\sigma \ker\chi_\sigma, x \neq F$. Without loss of generality, assume $x \in H$. Then since $(f_\sigma^G)_H = f_\sigma$, we have

$$f_\sigma(x) = f_\sigma^G(x) = \sigma(1) - \chi_\sigma(x) = \sigma(1) - \chi_\sigma(1) = \sigma(1) - \sigma(1) = 0.$$

But by definition, $f_\sigma(x) = \sigma(1) - \sigma(x)$, so $x \in \ker(\sigma)$ for each $\sigma \in \mathrm{Irr}(H)$. This forces $x = 1$, but $1 \in F$, so this is a contradiction. Hence we can conclude that

$$F = \cap_{\substack{\sigma\in\mathrm{Irr}(H)\\ \sigma\neq 1_H}} \ker\chi_\sigma \lhd G.$$

□

This completes our discussion of Frobenius and also completes this lecture. The contents of Isaacs' Chapter 5 are continued in Chapter 7. We will do Chapter 6, which involves the splitting of a character and the restriction of a character to a normal subgroup. As a comment, Problem 5.19 is a particularly interesting one from this chapter.

6. Lecture: 15 October 2010

I was really confused by Professor Hermann's lecture for the first half, so I am instead going to write some notes on some of the important parts of Chapter 5 of Isaacs' book.

Towards the end of lecture, we started Chapter 6.

We introduce a definition:

Definition 6.1. Let φ be a class function of H, and let $g \in G$. We define $\varphi^g \in \mathrm{cf}(H)$ such that $\varphi^g(h) := \varphi(h^{g^{-1}}) = \varphi(ghg^{-1})$.

7. Lecture: 29 October 2010

We continue again in Chapter 6 of Isaacs' book. Recall Clifford's theorem.

Theorem 7.1 (Clifford)**.** *Let $H \lhd G$ and let $\chi \in \operatorname{Irr} G$. Let ϑ be an irreducible constituent of χ_H and suppose $\vartheta = \vartheta_1, \vartheta_2, \ldots, \vartheta_t$ are the distinct conjugates of ϑ in G. Then we have*

$$\chi_H = e \sum_{i=1}^{t} \vartheta_i,$$

where $e = [\chi_H, \vartheta]$.

Now let $T := \{g \in G : \vartheta^g = \vartheta\}$. Notice we have $H \leq T \leq G$. Furthermore, $|G : T| = t$, where t is as in the statement of Clifford's theorem; i.e. t is the number of distinct conjugates of ϑ. Now let's define two sets:

$$A := \{\psi \in \operatorname{Irr}(T) : \vartheta \subseteq \psi_H\}, \quad B := \{\chi \in \operatorname{Irr}(G) : \vartheta \subseteq \chi_H\}.$$

We have a bijection between these two maps defined by induction. Formalizing this and also stating some other properties, we have

a) For all $\psi \in A, \psi^G \in \operatorname{Irr}(G)$
b) For all $\psi \in A, \psi^G \in B$, so that we have a bijection between the sets A and B.
c) If $\psi^G = \chi$, then χ is the unique constituent of χ_T in A.
d) If $\psi^G = \chi$, then $[\psi_H, \vartheta] = [\chi_H, \vartheta]$.

Proof. There was a bit of a confusing proof here that I didn't quite understand. I should fill this in later. □

Several corollaries follow from the above discussion.

Corollary 7.1. *If χ is a primitive irreducible character of G, and $N \lhd G$, then $\chi_N = e\vartheta$ for some $\vartheta \in \operatorname{Irr}(N)$.*

Corollary 7.2. *Let χ be a primitive, faithful irreducible character of G. If A is an abelian normal subgroup of G, then $A \leq \mathrm{Z}(G)$.*

Corollary 7.3. *If G is a nilpotent group, then G is an M-group.*

We now stray slightly away from the main topic and prove the following group-theoretic proposition.

Proposition 7.1. *Let G be a nilpotent group. Then there exists a self-centralizing abelian normal subgroup. That is, there exists a normal subgroup $A \lhd G$ such that $A = \mathrm{C}_G(A)$.*

Proof. Let A be a maximal normal abelian subgroup of G. Suppose $\mathrm{C}_G(A) \gneq A$. Then $C := \mathrm{C}_G(A) \lhd G$. Let $\overline{G} := G/A, \overline{C} := C/A \lhd \overline{G}$. We know from our supposition that $\overline{C}$ is nontrivial, so there exists a subgroup $\overline{D} \leq \overline{C}$ such that $\overline{D} \lhd \overline{G}$ such that $\overline{D}$ is cyclic. (Notice that $\overline{D}$ is any cyclic subgroup of $\overline{C} \cap \mathrm{Z}(\overline{G})$.) By construction, $\overline{D} = D/A$ nontrivial, and $D \lhd G$. We know that $A \leq \mathrm{Z}(D)$, so $D/\mathrm{Z}(D)$ is cyclic. Hence D is abelian. This contradicts the maximality of A, and the desired result follows. □

Now we return to character theory.

Let $\chi \in \operatorname{Irr}(G)$ be such that $\chi = \psi^G$ for some $\psi \in \operatorname{Irr}(H)$, $H \leq G$. Let H be minimal. Then ψ is primitive, and this induces a character $\overline{\psi}$ on the quotient group $\overline{H} := H/\ker\psi$. All abelian normal subgroups of $\overline{H}$ are in $\mathrm{Z}(\overline{H})$ so there exists a self-centralizing normalizing subgroup in $\overline{H}$. Therefore $\overline{\psi}$ is a linear character, and hence ψ is linear.

We have a divisibility property generalizing the previous relation that $\chi(1) \mid [G : \mathrm{Z}(\chi)]$.

Proposition 7.2. *Let $A \lhd G$, A abelian. Then for all $\chi \in \operatorname{Irr}(G)$, we have $\chi(1) \mid [G : A]$.*

Proof. Let $\chi \in \mathrm{Irr}(G)$. If $\lambda \in \mathrm{Irr}(A)$ is a constituent of χ_A, then λ is linear. With respect to λ, we have $A \leq T \leq G$. Therefore $\chi = \psi^G$ for some $\psi \in \mathrm{Irr}(T)$ and $\lambda \subseteq \psi_A$. Here, we have $\psi_A = e\lambda$, so $A \leq \mathrm{Z}(\psi)$, which implies that $\psi(1) \mid |T : \mathrm{Z}(\psi)| \mid |T : A|$. Therefore $\chi(1) = \psi(1)|G : T| \mid |G : A|$, as desired. □

This concludes this lecture. Next week, we will discuss extendibility from a normal group to the whole group, which is somehow connected to the characters of normal subgroups that are G-invariant. We will finish Chapter 6 and then go back to discuss Chapter 4. The suggested exercises of Chapter 6 (considering how much material we have covered thus far) are the following: 6.1, 6.2, 6.4.

8. Lecture: 5 November 2010

We continue in our discussion of Chapter 6. For reference, we will discuss Theorems 6.16 - 6.18.

Theorem 8.1. *Let $N \lhd G$, and let $\varphi, \vartheta \in \mathrm{Irr}(N)$ be invariant in G. Assume also that $\varphi\vartheta \in \mathrm{Irr}(N)$ and that $\varphi = \chi_N$ for some $\chi \in \mathrm{Irr}(G)$. Let $\mathscr{S} := \{\beta \in \mathrm{Irr}(G) : [\varphi^G, \beta] \neq 0\}$, $\mathscr{T} := \{\psi \in \mathrm{Irr}(G) : [(\varphi\vartheta)^G, \psi] \neq 0\}$. Then $\beta \mapsto \beta\chi$ defines a bijection of $\mathscr{S}$ onto $\mathscr{T}$.*

Proof. We have that $(\varphi^G)_N$ is a multiple of φ and comparing degrees, we get $(\varphi^G)_N = |G : N|\varphi$. Then $(\varphi^G\chi)_N$ is a scalar multiple of $\varphi\vartheta$, so $(\varphi^G\chi)_N = |G : N|\varphi\vartheta$. Now let

$$\varphi^G = \sum_{\beta \in \mathscr{S}} n_\beta \beta, \text{ where } n_\beta > 0, n_\beta \in \mathbb{Z}.$$

Taking the inner product of this character with itself, we have

$$\sum_{\beta \in \mathscr{S}} n_\beta^2 = [\varphi^G, \varphi^G] = [(\varphi^G)_N, \varphi] = |G : N|[\varphi, \varphi].$$

Multiplying by χ, we have $\varphi^G\chi = \sum_{\beta \in \mathscr{S}} n_\beta \beta\chi$, so

$$[\varphi^G\chi, \varphi^G\chi] = \sum_{\beta_1, \beta_2, \in \mathscr{S}} n_{\beta_1} n_{\beta_2} [\beta_1\chi, \beta_2\chi] = \sum_{\beta \in \mathscr{S}} n_\beta^2 [\beta\chi, \beta\chi] + \sum_{\beta_1 \neq \beta_2} n_{\beta_1} n_{\beta_2} [\beta_1\chi, \beta_2\chi].$$

On the other hand, we have $[(\varphi\vartheta)^G, (\varphi\vartheta)^G] = |G : N| = [\varphi^G, \varphi^G]$, and since $\varphi^G\chi = (\varphi\vartheta)^G$, then we can conclude

$$\sum_{\beta \in \mathscr{S}} n_\beta^2 = [\varphi^G, \varphi^G] = [(\varphi\vartheta)^G, (\varphi\vartheta)^G] = [\varphi^G\chi, \varphi^G\chi] = \sum_{\beta \in \mathscr{S}} n_\beta^2 [\beta\chi, \beta\chi] + \sum_{\beta_1 \neq \beta_2} n_{\beta_1} n_{\beta_2} [\beta_1\chi, \beta_2\chi].$$

It follows that $n_\beta = 1$ for all $\beta \in \mathscr{S}$ and that $[\beta_1\chi, \beta_2\chi]$ if and only if $\beta_1 = \beta_2$, and so $\beta \mapsto \beta\chi$ is indeed a bijection and takes $\mathscr{S}$ into $\mathscr{T}$. □

The main purpose of this theorem is to prove the following result, which is a special case of the above.

Corollary 8.1. *Let $N \lhd G$ and let $\chi \in \mathrm{Irr}(G)$ be such that $\chi_N = \vartheta \in \mathrm{Irr}(N)$. Then the characters $\beta\chi$ for $\beta \in \mathrm{Irr}(G/N)$ are irreducible, distinct for distinct β and are all of the irreducible constituents of ϑ^G.*

Proof. We take Theorem 8.1 for the case when $\varphi = 1_N$. □

One of the many useful applications of Clifford theory is in the study of characters (and hence representations) of solvable groups. We first introduce a definition.

Definition 8.1. A *normal series* is a chain of subgroups $1 = N_0 \leq N_1 \leq N_2 \leq \cdots \leq N_k = G$ such that $N_i \lhd G$ for all $i = 1, \ldots, k$. A *chief series* is a normal series with the additional property that the quotient N_{i+1}/N_i is characteristically simple. This quotient is called the *chief factor.*

Recall from algebra that if $C \leq G$ is characteristic in $N \lhd G$, then $C \lhd G$. However, $C \lhd N \lhd G$ does not necessarily imply that $C \lhd G$. Hence characteristicness (i.e. invariance under the action of $\mathrm{Aut}(G)$) is stronger than normality. Note also that all the chief factors of a group G are abelian if and only if G is solvable. Another fact from algebra is the following: If each chief factor of G has prime order, then G is supersolvable. (Recall that G is supersolvable if each quotient group in its derived series is cyclic, and since groups of prime order are necessarily cyclic, then this follows.) Since finite nilpotent groups are supersolvable, then we have also connected nilpotence to this situation.

Now using this new language, we can prove the following theorem that is very useful in the case of solvable groups because of the note above. According to Isaacs, it is called the "going down" theorem.

Theorem 8.2. *Let K/L be an abelian chief factor of G. Let $\vartheta \in \mathrm{Irr}(K)$ be G-invariant. Then one of the following holds:*

a) $\vartheta_L \in \mathrm{Irr}(L)$*;*
b) $\vartheta_L = e\varphi$ *for some* $\varphi \in \mathrm{Irr}(L)$ *and* $e^2 = |K : L|$*;*
c) $\vartheta_L = \sum_{i=1}^{t} \varphi_i$ *where* $\varphi \in \mathrm{Irr}(L)$ *are distinct and* $t = |K : L|$*.*

Proof. Take some $\varphi \in \mathrm{Irr}(L)$ such that $\varphi \subseteq \vartheta_L$ and define $T := \mathrm{I}_G(\varphi)$. Notice that the index $|G : T|$ is the number of distinct conjugates of φ under the action of G. We have

$$0 \neq [\varphi, \vartheta_L] = [\varphi^g, \vartheta_L^g] = [\varphi^g, \vartheta_L],$$

so if φ is a constituent of ϑ_L, then all its conjugates φ^g are also. Now the number of distinct conjugates is the index $|K : T \cap K|$, so this index is the same as $|G : T|$. Hence $KT = G$. (This is a special case of the following fact: If G acts on A and $H \leq G$ acts transitively on A, then $G = HG_\alpha$, where G_α is the stabilizer for $\alpha \in G$, a group element chosen arbitrarily.)

Now, $K \cap T \lhd T$ and so $T \leq \mathrm{N}_G(K \cap T)$. Also, $K \cap T \lhd K$ so $K \leq \mathrm{N}_G(K \cap T)$. Then taking the images under the quotient map with kernel L, we have $K \cap T/L \lhd K/L$. Since K/L is an abelian chief factor, then it follows here that $K \cap T$ is K or L.

If $K \cap T = K$, then $\vartheta_L = e\varphi$ for some e. Let $\lambda \in \mathrm{Irr}(K/L)$. This is a linear character so $\lambda\vartheta \in \mathrm{Irr}(K)$, and by looking at the degree, $(\lambda\vartheta)_L = \vartheta_L = e\varphi$. If $\lambda_1\vartheta \neq \lambda_2\vartheta$ for all $\lambda_1 \neq \lambda_2$, then $e|K : L|\vartheta(1) \leq \varphi^K(1) = |K : L|\varphi(1)$. Hence $e^2\varphi(1) = e\vartheta(1) \leq \varphi(1)$, and therefore $e = 1$. So a) holds. If $\lambda_1\vartheta$ are not pairwise distinct, then there exists $\lambda \neq \mu \in \mathrm{Irr}(K/L)$ such that $\lambda\vartheta = \mu\vartheta$. We have $L \leq U := \ker(\lambda\overline{\mu}) \lneqq K$. Now, ϑ vanishes outside U (since $\lambda\vartheta - \mu\vartheta = 0$), and since ϑ is G-invariant, then it vanishes outside U^g. Hence ϑ vanishes outside the intersection $\cap_{g \in G} U^g$, but this is just L so $\vartheta|_{K \setminus L} = 0$. By a previous proposition (2.29 in Isaacs),

$$[\vartheta_L, \vartheta_L] = |K : L|[\vartheta, \vartheta] = |K : L|.$$

But since $\vartheta_L = e\varphi$, then $[\vartheta_L, \vartheta_L] = e^2$, and hence b) holds. If $K \cap T = L$, then $\vartheta_L = \sum_{i=1}^{t} \varphi_i$, and so c) holds. □

This concludes today's lecture. We will continue with Chapter 6 next lecture, which will be a make-up class on Monday morning.

9. Lecture: 8 November 2010

We continue to discuss Chapter 6. Last time, we finished with a theorem classifying the restriction of a K-character to a subgroup L, where K/L is an abelian chief factor of G. Following this theorem, we have several corollaries.

Proposition 9.1. *Let $N \lhd G$ with $|G : N| = p$, a prime. Suppose $\chi \in \mathrm{Irr}(G)$. Then either*

a) χ_N is irreducible or
b) $\chi_N = \sum_{i=1}^{p} \vartheta_i$, where ϑ_i are distinct and irreducible.

Proof. The condition that $|G : N| = p$ gives us that G/N is abelian and that there are no normal subgroups between N and G. Hence we can apply Theorem 8.2, taking $K = G, L = N$. Clearly p is not a square, and hence the second conclusion of that theorem does not apply. The result follows. □

Proposition 9.2. *Let $N \lhd G$ and suppose $|G : N| = p$, a prime. Let $\vartheta \in \mathrm{Irr}(N)$ be invariant in G. Then ϑ is extendible to G.*

Proof. Let $\chi \in \mathrm{Irr}(G)$. By Clifford's theorem, $\chi_N = e\vartheta$ for some e. Since ϑ is invariant in G, then b) from the previous proposition cannot hold. So we must have $e = 1$. □

Now we will move on to discuss M-groups. As a side note... something interesting about M-groups is that there is no characterization of them outside of character theory! Now for some new terms.

Definition 9.1. Let $N \lhd G$ and let $\chi \in \mathrm{Irr}(G)$. Then χ is a *relative M-character* with respect to N if there exists a subgroup H with $N \leq H \leq G$ and $\psi \in \mathrm{Irr}(H)$ such that $\psi^G = \chi$ and $\chi_N \in \mathrm{Irr}(N)$. (Note here that the requirement that χ_N is irreducible is what makes this a meaningful definition. Without this, all characters would be relative M-characters!) If every $\chi \in \mathrm{Irr}(G)$ is a relative M-character with respect to N, then G is a *relative M-group* with respect to N.

Remark (Taken from Isaacs). Note that $\chi \in \mathrm{Irr}(G)$ is a relative M-character with respect to 1 if and only if it is a monomial character, and G is a relative M-group with respect to 2 if and only if it is an M-group. Also, it is clear that if G is a relative M-group with respect to N, then G/N is an M-group.

Theorem 9.1. *Suppose $N \lhd G$ and G/N is solvable. Suppose, furthermore, that every chief factor of every subgroup of G/N has nonsquare order. Then G is a relative M-group with respect to N.*

Proof. Consider $\chi \in \mathrm{Irr}(G)$. If χ_N is irreducible, then we're done. Now let $K \lhd G$ be such that $N \leq K$ and K is the minimal subgroup such that $\chi_K \in \mathrm{Irr}(K)$. Then there exists and $L \geq N$ such that K/L is a chief factor. By the hypothesis of the theorem, K/L is abelian with nonsquare order. This means, by Theorem 8.2 that either $(\chi_K)_L \in \mathrm{Irr}(L)$ or that $\chi_L = \sum_{i=1}^{t} \varphi_i, \varphi \in \mathrm{Irr}(L), t = |K : L|$. The first case cannot happen by the minimality of K.

Let $T := \mathrm{I}_G(\varphi_1) \geq L$. Then $\chi = \psi^G$ for some $\psi \in \mathrm{Irr}(T)$, so we can apply the above argument replacing G with T. (Note that $T < G$ since φ_i are distinct conjugates.) Applying induction on $|G : N|$, we conclude that T is a relative M-group with respect to N and that $\psi = \vartheta^T$ for some $\vartheta \in \mathrm{Irr}(H)$ where $n \leq H \leq T$ and $\vartheta_N \in \mathrm{Irr}(N)$. We have $\chi = \psi^G = (\vartheta^T)^G = \vartheta^G$, and this completes the proof. □

(This next proof caused a lot of trouble during lecture.)

Theorem 9.2. *Let $N \lhd G$. If all Sylow subgroups of N are abelian and G is solvable and is a relative M-group with respect to N, then G is an M-group.*

Proof. Consider $\chi \in \mathrm{Irr}(G)$. Then since G is a relative M-group with respect to N, then χ must be a relative M-character with respect to N. Now choose a subgroup $H \leq G$ with $N \leq H$ and with the property that given any $\psi \in \mathrm{Irr}(H), \psi_N \in \mathrm{Irr}(N), \psi^G = \chi$. Choose $U \leq H$ to be minimal such that there exists $\vartheta \in \mathrm{Irr}(U)$ with $\vartheta^H = \psi$. Then $\vartheta^G = (\vartheta^H)^G = \psi^G = \chi$. We want to show that ϑ is linear. (Since then we've shown that for all $\chi \in \operatorname{Irr}(G)$, there exists a linear ϑ such that $\vartheta^G = \chi$, which means that every $\chi \in \operatorname{Irr}(G)$ is monomial.)

Let $M := U \cap N$. Then $(\vartheta^{NU})^H = \vartheta^H = \psi \in \operatorname{Irr}(H)$ and χ_{NU} is irreducible. (This last statement holds since χ_N is irreducible and $N \leq NU$.) This implies that $\psi_{NU} = \vartheta^{NU}$, and hence $(\vartheta^{NU})_N \in \operatorname{Irr}(N)$. Now, $(\vartheta_M)^N = (\vartheta_{U\cap N})^N = (\vartheta^{NU})_N \in \operatorname{Irr}(N)$, so $\vartheta_M \in \operatorname{Irr}(M)$. By the minimality of U, ϑ is a primitive character of U (which means, as a reminder, that there does not exist a character φ such that $\varphi^G = \vartheta$, where φ is an irreducible character of a proper subgroup of U).

Now let $K = \ker \vartheta$ and let $\overline{U} = U/K, \overline{M} = MK/K$. Then ϑ is a faithful primitive character of $\overline{U}$. Furthermore, since $\overline{M} < N/K$, then all the Sylow subgroups of $\overline{M}$ are abelian (since N has this property). Let $Z = Z(\overline{M}) \lhd \overline{U}$. So $Z \leq \overline{M}$. If $Z < \overline{M}$, then pick an $A \leq \overline{M}$ such that A/Z is a chief factor of $\overline{U}$ (we can do this since G is solvable). Then necessarily A/Z is a p-group. Now let $P \in \operatorname{Syl}_p(A)$. Then P is abelian and also $A = PZ$. So A must also be abelian. By construction, $A \lhd \overline{U}$, and by Corollary 6.13 in Isaacs (equivalently, Corollary 7.2 in these notes), $A \leq Z(\overline{U}) \leq Z(\overline{M}) = Z$, which contradicts the assumption that $A > Z$.

We conclude then that $Z = \overline{M}$, so $\overline{M}$ is abelian. We knew already that $\vartheta_M \in \operatorname{Irr}(M)$, which implies that $\vartheta_{MK} \in \operatorname{Irr}(MK)$, so $\vartheta_{MK} \in \operatorname{Irr}(\overline{M})$. Hence ϑ_{MK} is linear. We have $\vartheta(1) = \vartheta_{MK}(1) = 1$, so ϑ is a linear character. This completes the proof. □

10. Lecture: 19 November 2010

I missed lecture on November 12 because I was ill. Theorems 6.24-6.26 were covered during that lecture. We continue to discuss extendibility.

Theorem 10.1. *Let $N \lhd G$. Assume: All Sylow subgroups of N are abelian, G is solvable, and G is an M-group with respect to N. Then G is an M-group.*

Proof. Consider a character $\chi \in \mathrm{Irr}(G)$. □