Introduction to Representation Theory and First Examples

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I would first like to thank Jeremy Booher for LaTeX-ing this up during my lecture so that when it came time for me to type things up for the yearbook, it saved me in cramming to make the deadline.

Now for some representation theory! This first lecture will be a gentle one—just a few definitions and many examples. This is so that we can start of this seminar with a firm understanding of what is going on and so that when we think about representations, we have some concrete ideas of what sorts of things we're dealing with. I will basically assume nothing but linear algebra and basic group theory. First, here is an overview of what where we're going in terms of this seminar as a whole.

We will start by looking at some general facts about representations of finite groups (in characteristic 0). We will study characters, discuss some Wedderburn theory, and look at induced representations. Then we will discuss the details of the representation theory of the symmetric group Sym_n, which we will first look at from a combinatorial perspective and then move to a more algebraic perspective. The combination of these two will set us up for a final discussion of the modular representation theory of symmetric groups.

1 What Is Representation Theory?

Well, it's exactly what it sounds like: it is the theory that arises from the study of representations. So, then: what is a representation? We first fix some notation and then give a definition. Let G be a group, and K a field. All our representations will be of finite groups on finite-dimensional vector spaces V unless noted otherwise.

Definition 1. A *representation* of G is a (group) homomorphism $\rho : G \to \mathrm{GL}_n(K)$.

Recall that $\mathrm{GL}_n(K)$ is the group of invertible linear transformations of K^n, which, after picking a basis, can be identified with invertible $n \times n$ matrices with entries in K.

Alternatively, we may think of a representation in the following way:

Definition 2. A *representation* of G is a vector space V over K with a linear $G-$action. Explicitly, this means that given $g_1, g_2 \in G$, $v_1, v_2, v \in V$ and $a, b \in K$, the action satisfies

$$g_1(g_2 v) = (g_1 g_2)v \quad \text{and} \quad g(av_1 + bv_2) = agv_1 + bgv_2.$$

Given a representation as in Definition 1, we can construct the group action satisfying Definition 2. Letting $V = K^n$, $v \in V$ and $g \in G$, we may define $gv = \rho(g)v$ to get a G-action on the vector space V. Going the other direction, given an action of G on V, we may define $\rho(g) \in \mathrm{GL}_n(K)$, $n = \dim V$, to be the transformation sending $v \in V$ to gv. Hence we see that

$$\text{Definition 1} \Longleftrightarrow \text{Definition 2}.$$

We also have the notion of a character.

Definition 3. Let ρ be a representation of G on the vector space V. Then the *character* χ_V afforded by V is defined as $\chi_v(g) = \mathrm{tr}(\rho(g))$.

Note that the character is a class function: it is independent of the choice of representative for the conjugacy class of G because the trace of conjugate matrices is the same. This is essentially the fact that the trace of a linear transformation is independent of the choice of basis.

2 First Examples

Now that we have established the basic definitions, we will look at some first examples. The idea of presenting so many examples is to make sure we establish a good foundation and also to give a taste of the flavor of representation theory.

Example 1. Let G be any group, $V = K$, and $gx = x$ for all $x \in K$. This is the trivial representation, and $\chi_V(g) = 1$ for all $g \in G$.

Example 2. Let G be a finite group, $n = |G|$, and $V = K^n$. Let $G = \{g_1, \ldots, g_n\}$, and pick a basis $\{v_{g_i}\}$ indexed by elements of G. Define gv_{g_i} to be v_{gg_i}. In terms of coordinates,

$$g(x_{g_1}, x_{g_2}, \ldots, x_{g_n}) = (x_{g^{-1}g_1}, x_{g^{-1}g_2}, \ldots, x_{g^{-1}g_n}).$$

Note that there are no fixed points of the action of g unless g is the identity. Therefore $\chi_v(g)$ equals 0 unless G is the identity, in which case it is $|G|$. This is called the regular representation, and χ_V is the regular character.

For example, if $G = \mathbb{Z}_n$ then the action on K^n simply cyclically shifts the coordinates. 1 sends $(x_1, x_2, \ldots, x_n)$ to $(x_n, x_1, x_2, \ldots, x_{n-1})$. The coordinates shift the other direction because $1(v_1) = v_2$, so the coefficient of v_1, x_1, becomes the coefficient of v_2, in the second spot.

Example 3. Let $G = S_n$, $X = \{1, 2, \ldots, n\}$ and $V = K^n$. Then S_n acts on V by permuting a basis $\{v_i\}$. For $\sigma \in S_n$, $\sigma(v_i) = v_{\sigma(i)}$. Consider the matrix of a permutation. If σ fixes i, the (i, i) entry of the matrix is a 1. Otherwise the (i, i) entry is 0. Thus $\chi_v(\sigma)$ equals the number of fixed points of σ.

Definition 4. Let V be a representation of G. A *subrepresentation* is a linear subspace $W \subset V$ such that $gw \in W$ for $g \in G$, $w \in W$.

Example 4. Continuing the example of $G = S_n$ acting on K^n, what are the subrepresentations? The span of the single vector $v_1 + v_2 + \ldots + v_n$ is certainly fixed ($\sigma(v_1 + \ldots + v_n) = v_{\sigma(1)} + \ldots + v_{\sigma(n)} = v_1 + \ldots + v_n$), so we have a one dimensional subrepresentation. Note that this is the trivial representation. Denote it by W_1.

Now let $W_2 = \{\alpha_1 v_1 + \ldots + \alpha_n v_n : \alpha_1 + \ldots t + \alpha_n = 0\}$. It is invariant under the action of S_n since permuting the coordinates does not change the sum.

These are the only two subrepresentations in characteristic 0. We can show that $V = W_1 \oplus W_2$. This implies that $\chi_{W_1} + \chi_{W_2} = \chi_V$. As W_1 is trivial, $\chi_{W_2}(\sigma)$ is one less than the number of fixed points of σ.

Exercise 1. In characteristic 0, show W_1 and W_2 are the only non-trivial subrepresentations, and that $V = W_1 \oplus W_2$.

Note that the permutation representation of S_n is not the same as the regular representation of S_n. Furthermore, beware that the character of a representation is only a group homomorphism, not a homomorphism. It is only homomorphism for one dimensional representations. We will see (in Ian Frankel's lecture, which is the lecture that will follow this one) that this always happens for irreducible representations of finite Abelian groups. Then $\mathrm{GL}_1(K) \simeq K^\times$, and the trace of a representation is done through this identification. This explains why characters in number theory (for example, Dirichlet characters) are always group homomorphisms.

Definition 5. If V is a representation such that the only subrepresentations are $\{0\}$ and V, we say that V is *irreducible.*

In the previous example, W_1 and W_2 are irreducible while V is not.

Example 5. Let $G = D_n$ be the dihedral group. It is generated by two elements, r and s, where $r^n = s^2 = 1$ and $rs = sr^{-1}$. The standard representation of D_n (thought of the symmetries of an $n-gon$) is on $\mathbb{R}^2$. We let r rotates by $\frac{2\pi}{n}$ and s reflects across the $y-$axis. r and s act by the matrices

$$r : \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}, \qquad s : \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

where $\theta = \frac{2\pi}{n}$.

In particular, these matrices make is easy to calculate the characters. $\chi(r) = 2\cos(\theta)$, while $\chi(s) = 0$, and in general the character can be calculated by multiplying the matrices.

Note that this representation is irreducible, because the rotation r fixes no lines.

Example 6. Let $G = \mathrm{SU}(2) = \{A \in \mathrm{GL}_2(\mathbb{C}) : \overline{A}^t A = I_n, \det(A) = 1\}$, and let the field $K = \mathbb{C}$. (SU(2) is a three dimensional Lie group, by the way.) Let P_n be the vector space of homogeneous polynomials of degree n in variables x and y. Thus $P_0 = \mathbb{C}$, $P_1 = \{ax + by : a, b \in \mathbb{C}\}$, $P_2 = \{ax^2 + bxy + cy^2 : a, b, c \in \mathbb{C}\}$ and so forth. In general, the dimension of P_n is $n + 1$.

For $A \in G$, let it act on $p \in P_n$ via $A \cdot p(x, y) = p(A^{-1}(x, y)^t)$. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$$A \cdot p(x, y) = p\left(\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} (x, y)^t\right) = p(dx - by, -cx + ay).$$

It turns out that these irreducible, and in fact all irreducible representations of SU(2) arise in this way. This is a basic example in the theory of Lie groups. Physicists use similar representations in quantum field theory.

There is a good reason that we will usually work in characteristic 0.

Example 7. Let p be a prime, G be a finite $p-$group, and $K = \mathbb{Z}/p\mathbb{Z}$. Let V be an irreducible representation. The span of the orbit of v is a subrepresentation. G is a finite group, so as V is irreducible, it must be finite dimensional.

Let G act on $V \backslash \{0\}$, which has $p^n - 1$ elements. However, any G-orbit has size dividing the order of the group by the orbit stabilizer theorem. Thus all G-orbits have size p^a for $0 \leq a < n$. Therefore there exists at least one orbit of size one, which means that the one dimensional subspace spanned by that vector is a G invariant subspace. Therefore any representation contains a copy of the trivial representation, so the only irreducible representations of a p-group is the trivial one.

This last example is one that will come back at the end of this seminar. It turns out that this fact is true if K is any field of characteristic p, and the fact that the trivial representation is the only irreducible representation of a p-group over such a field, is an important one to keep in mind especially in the final lecture of this seminar, which will be on some basics of modular representation theory and in particular the modular representation theory of symmetric groups.