The Local Langlands Correspondence

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Abstract

These notes were primarily Live-TeXed during Prof. Mitya Boyarchenko's Topics in Algebra course *The Local Langlands Correspondence* in the Fall of 2012. Minor edits have been made by me. I apologize for any errors, mathematical or otherwise, beforehand.

The font used for this document is MinionPro. Information about installing this (free!) font can be found at `http://www.ctan.org/tex-archive/fonts/minionpro/`.

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1 5 September 2012: Two Toy Examples and First Motivations

The theme of this course will be to construct various group representations geometrically. We begin the course by going through two toy examples that will give the flavor of what we will do throughout this term.

1.1 Toy Example 1

In this example, we will construct the irreducible representations of the general linear group $\mathrm{GL}_2(\mathbb{F}_q) =: \Gamma$.

Let B be the Borel subgroup consisting of upper triangular matrices in Γ, and let T be the split torus inside Γ consisting of diagonal matrices. Note that $T \cong \mathbb{F}_q^\times \times \mathbb{F}_q^\times$.

The irreducible representations of a product group $G \times H$ are, up to isomorphism, of the form $\rho_G \otimes \rho_H$, where ρ_G and ρ_H are irreducible representations of G and H, respectively. Hence for $T \cong \mathbb{F}_q^\times \times \mathbb{F}_q^\times \cong \mathbb{Z}/(q-1)\mathbb{Z} \times \mathbb{Z}/(q-1)\mathbb{Z}$, the irreducible representations are given by $\chi_1 \otimes \chi_2$ where χ_1 and χ_2 are irreducible representations of the cyclic group $\mathbb{Z}/(q-1)\mathbb{Z}$.

Given an irreducible representation of $T = \mathbb{F}_q^\times \times \mathbb{F}_q^\times$, we may construct a linear (that is, one-dimensional) representation of B as follows. We have a natural surjection

$$B \to \mathbb{F}_q^\times \times \mathbb{F}_q^\times \cong T, \quad \left(\begin{smallmatrix} x & * \\ 0 & y \end{smallmatrix}\right) \mapsto (x, y).$$

Then using χ_1 and χ_2, we obtain a map $\mathbb{F}_q^\times \times \mathbb{F}_q^\times$ given by

$$\mathbb{F}_q^\times \times \mathbb{F}_q^\times \to \mathbb{C}^\times, \quad (x, y) \mapsto \chi_1(x)\chi_2(y).$$

Let χ be the composition

$$B \to \mathbb{F}_q^\times \times \mathbb{F}_q^\times \to \mathbb{C}^\times, \quad \left(\begin{smallmatrix} x & * \\ 0 & y \end{smallmatrix}\right) \mapsto (x, y) \mapsto \chi_1(x)\chi_2(y).$$

This is a group homomorphism $B \to \mathbb{C}^\times$ and therefore defines a linear representation of B.

Now consider the induced representation $\mathrm{Ind}_B^\Gamma(\chi)$.

Remark. The process described above can be more succinctly said. The described homomorphism χ is the *inflation* of $\chi_1 \otimes \chi_2$ from T to B.

Using this language, we can state the following fact.

Fact 1.1. *For an irreducible representation χ of T, either* $\mathrm{Ind}_B^\Gamma(\mathrm{Inf}_T^B(\chi))$ *is irreducible or is a sum of two irreducibles.*

Extending the above linearly to the entire representation ring, we obtain a functor

$$\mathscr{R}_{\mathbb{C}} : \mathscr{K}_0(T) \to \mathscr{K}_0(\Gamma), \quad V \mapsto \mathrm{Ind}_B^{\Gamma}(\mathrm{Inf}_T^B(V)).$$

This functor is called *Harish-Chandra induction*. We can alternatively think of $\mathscr{R}_{\mathbb{C}}$ as

$$\mathscr{R}_{\mathbb{C}} : \mathscr{K}_0(T) \to \mathscr{K}_0(\Gamma), \quad V \mapsto \mathbb{C}[\Gamma/U] \otimes_{\mathbb{C}[T]} V,$$

where U is the set of unipotent matrices $\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$.

Using this language, we may restate Fact 1.1 as: For an irreducible representation χ of T, either $\mathscr{R}_{\mathbb{C}}(\chi)$ is irreducible or is a sum of two irreducibles.

Definition 1.1. If V is an irreducible representation of Γ that appears in the Harish-Chandra induction of an irreducible representation of the split torus T, then V is a *principle series representation*.

Definition 1.2. If V is an irreducible representation of Γ that is not a principle series representation, then we say that V is a *cuspidal representation*.

The cuspidal representations are harder to construct. To do this, we will look at the ℓ-adic homology of a particular plane affine curve X.

Let X be the plane affine curve cut out by the polynomial $(x^q y - x y^q)^{q-1} - 1$. Note that Γ acts on X by multiplication on the coordinates. We may check explicitly that for $A \in \Gamma$, we have

$$((x')^q y' - x'(y')^q)^{q-1} - 1, \quad \text{where } \left(\begin{smallmatrix} x' \\ y' \end{smallmatrix}\right) = A \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right).$$

In this computation one sees that the exponent $q-1$ is important because $\det(A) \in \mathbb{F}_q^\times$ and hence $\det(A)^{q-1} = 1$.

Remark. One may wonder where the polynomial above comes from. It turns out that

$$(x^q y - x y^q)^{q-1} = -\prod(\alpha x + \beta y),$$

where α and β vary over the elements of $\mathbb{F}_q$ and are not both zero. The proof of this is straightforward. We prove it via a series of claims.

Claim 1. Using the notation above,

$$\prod(\alpha x + \beta y) = -(xy(x+y)\cdots(x+(q-1)y))^{q-1}.$$

Proof. This is not so hard to see. Indeed, everything of the form $(\alpha x + \beta y)$ is of the form $m(x + ny)$ for some $m \in \mathbb{F}_q^\times$. Furthermore, m varies over all elements of $\mathbb{F}_q^\times$ and it is a fact from elementary number theory that $\prod_{m \in \mathbb{F}_q^\times} m = -1$ in $\mathbb{F}_q$. (Note that $x^{q-1} - 1 = 0$ for all $x \in \mathbb{F}_q^\times$.) Hence this completes the proof. □

Claim 2. We have

$$\prod_{n \in \mathbb{F}_q^\times} (x + ny) = x^{q-1} - y^{q-1}.$$

Proof. This is also not so hard to see. Note that $x^{q-1} - y^{q-1} = 0$ whenever $x = ny$ for some $n \in \mathbb{F}_q^\times$. The result follows. □

This gives some motivation for why we want to study this particular plane affine curve X. ◇

Remark. If we were working with $\mathrm{SL}_2(\mathbb{F}_q) =: G$ instead of with $\mathrm{GL}_2(\mathbb{F}_q) = \Gamma$, then we would consider the affine algebraic variety cut out by $x^q y - x y^q$ instead. We no longer need the exponent $q - 1$ because $\det(A) = 1$ for $A \in G$, by definition.

I think there should be a more intrinsic motivation to why we consider the so-called *Drinfeld variety* when working out the representations of $\mathrm{SL}_2(\mathbb{F}_q)$, but I'm not sure at the moment. ◇

The action of Γ on X induces an action of Γ on $H^*(X)$.

Fix a prime $\ell \neq p = \mathrm{char}(\mathbb{F}_q)$ and let K be a sufficiently large algebraic extension of $\mathbb{Q}_\ell$. (A field K is *sufficiently large* if it contains the $|\Gamma|$th roots of unity.) Rouquier proved that there exists a bounded complex of $K\Gamma$-modules who's cohomology groups are, as $K\Gamma$-modules, the cohomology groups with compact support of the variety X (with coefficients in the constant sheaf K). To simplify notation, we will denote by $H^i(X)$ the $K\Gamma$-module $H^i_c(X, K)$.

Fact 1.2. *$H^1(X, \overline{\mathbb{Q}}_p)$ decomposes into a direct sum of all of the cuspidal representations of Γ, each with multiplicity 1.*

We can actually decompose $H^1(X, \overline{\mathbb{Q}}_p)$ explicitly. $\mathbb{F}_{q^2}^\times$ acts on X by scaling $(x, y) \in X$. Indeed, this is an action since we may interpret X as the affine plane curve cut out by $\prod(\alpha x + \beta y) + 1$, which gives that scaling by λ yields a factor of $\lambda^{q^2-1} = 1$ in the first term. Note that the actions of $\mathbb{F}_{q^2}^\times$ and Γ commute. Now take any group homomorphism $\chi : \mathbb{F}_{q^2}^\times \to \overline{\mathbb{Q}}_\ell^\times$. Let

$$H^1(X, \overline{\mathbb{Q}}_\ell)[\chi] := \{v \in H^1(X, \overline{\mathbb{Q}}_\ell) : t \cdot v = \chi(t)v \text{ for all } t \in \mathbb{F}_{q^2}^\times\}.$$

This is the subspace of $H^1(X, \overline{\mathbb{Q}}_\ell)$ such that the ℓ-adic linear representation χ of $\mathbb{F}_{q^2}^\times$ acts as the trivial representation.

Fact 1.3. *If $\chi(t) \neq \chi(t^q)$, then the $\overline{\mathbb{Q}}_\ell\Gamma$-module $H^1(X, \overline{\mathbb{Q}}_\ell)[\chi]$ is an irreducible cuspidal representation of Γ.*

1.1.1 Questions

1. What happens if $\chi(t) = \chi(t^q)$?
2. What happens to $H^n(X, \overline{\mathbb{Q}}_\ell)[\chi]$ for $n \neq 1$?
3. What is $\dim(H^1(X, \overline{\mathbb{Q}}_\ell))$?
4. How can we be sure that these are all the representations?
5. I want to know more about the representations. From the above construction, I would like to actually compute the character table of Γ.

1.2 Toy Example 2

This second example will have a similar flavor to Toy Example 1, though it may feel different. Also, if one wants to make some real sense out of this example, one would have to set up the example very differently. In other words, there are lots of lies in this example, but the point is to explain why formal groups appear in this area of study.

Let E be an elliptic curve over $\mathbb{C}$. Thinking of E as an algebraic group, we may define, for any $N \in \mathbb{N}$,

$$E_{N\text{-tors}} := \{x \in E : Nx = 0\} \cong (\mathbb{Z}/N\mathbb{Z})^2.$$

(Remark: Topologically, E is a product of two copies of S^1.) Now let's consider a family of elliptic curves instead of just one elliptic curve. Let

$$\pi : E \to B$$

be a smooth morphism of algebraic varieties whose fibers are elliptic curves. Then for each $N \in \mathbb{N}$, we have an *étale covering* $X_N \to B$ where a point of X_N over $b \in B$ is an isomorphism (of abelian groups)

$$(\mathbb{Z}/N\mathbb{Z})^2 \to \pi^{-1}(b)_{N\text{-tors}}.$$

Then $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ acts on X_N by acting on $(\mathbb{Z}/N\mathbb{Z})^2$; in fact, $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ is the group of deck transformations of $X_N \to B$. (This is actually true if X_N is connected. But if X_N is not connected, then we in fact have many more deck transformations than just $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$.) This then gives an action of $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ on $H^*(X_N, \mathbb{C})$.

We can look at the relationship between these actions and N. If $N_1, N_2 \in \mathbb{N}$ and $N_1 \mid N_2$, then we have a natural map $X_{N_2} \to X_{N_1}$. More explicitly, this means that the following diagram commutes:

$$\begin{array}{ccc} (\mathbb{Z}/N_2\mathbb{Z})^2 & \xrightarrow{\sim} & E_{N_2\text{-tors}} \\ \uparrow & & \uparrow \\ (\mathbb{Z}/N_1\mathbb{Z})^2 & \xrightarrow{\sim} & E_{N_1\text{-tors}} \end{array}$$

This natural map is compatible with the surjection

$$\mathrm{GL}_2(\mathbb{Z}/N_2\mathbb{Z}) \to \mathrm{GL}_2(\mathbb{Z}/N_1\mathbb{Z})$$

given by reduction modulo N_1.

Now look at $H^*(X_N, \mathbb{C})$ and get, by a pullback,

$$\begin{array}{ccc} H^*(X_{N_1}, \mathbb{C}) & \longrightarrow & H^*(X_{N_2}, \mathbb{C}) \\ \circlearrowleft & & \circlearrowleft \\ \mathrm{GL}_2(\mathbb{Z}/N_1\mathbb{Z}) & \twoheadleftarrow & \mathrm{GL}_2(\mathbb{Z}/N_2\mathbb{Z}) \end{array}$$

Now we take the direct limit of and get the space

$$\varinjlim_{N \in \mathbb{N}} H^*(X_N, \mathbb{C})$$

on which we have an action of

$$\varprojlim_{N \in \mathbb{N}} \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}) \cong \mathrm{GL}_2(\widehat{\mathbb{Z}}) \cong \prod_{p \text{ prime}} \mathrm{GL}_2(\mathbb{Z}_p).$$

Something like the above can be used to construct representations of $\mathrm{GL}_2(\mathbb{Q}_p)$. In our case, we will be replacing the role of the elliptic curves with formal groups.

Surprisingly, although this is primarily developed in the areas of number theory, representation theory, and algebraic geometry, there are some serious applications to algebraic topology by way of homotopy theory.

Some references for this class:

- For cohomology theory, look at SGA $4\frac{1}{2}$ and possibly Deligne-Lusztig's original 1976 paper. It is a long paper but in Section 2, he works out some examples.
- For the finite field stuff, look at Digne and Michel's *Representations of Finite Groups of Lie Type*.
- For $\mathbb{Q}_p$ stuff, look at Gross and Hopkins (as suggested in the syllabus).
- Also look at the paper of Drinfeld where he gives the original definition of the Lubin-Tate tower. This paper is called something like "Elliptic modules" or similar.

A paper that I found: `http://math.harvard.edu/archive/126_fall_98/papers/mahill.pdf`. This gives the construction of the irreducible representations of $\mathrm{GL}_2(\mathbb{F}_q)$ discussed in Section 1.1.

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Our plan for this term will be the following topics:

1. Background on representations
2. Examples for $\mathrm{GL}_n(\mathbb{F}_q)$ and $\mathrm{GL}_n(\mathbb{Q}_p)$
3. Formal groups
4. The Lubin-Tate tower
5. The local Langlands correspondence and Deligne-Lusztig theory

2.1 Representation Theory Background

Let G be an abstract group and let H be a subgroup of G.

Given a representation $\rho : H \to \mathrm{GL}(V)$, we may form the induced representation $\mathrm{Ind}_H^G(\rho)$ given by

$$\mathrm{Ind}_H^G(\rho) := \{f\colon G \to V : f(hg) = \rho(h)f(g) \text{ for all } g \in G, h \in H\}.$$

The action of G on $\mathrm{Ind}_H^G(\rho)$ is given by

$$(\gamma \cdot f)(x) = f(x\gamma) \quad \text{for all } \gamma, x \in G.$$

Remark. Let π be any representation of G. Then

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G(\rho)) \cong \mathrm{Hom}_H(\mathrm{Res}_H^G(\pi), \rho).$$

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3 14 September 2012: The General Theory of Smooth Representations

Recall: G is an ℓ-group means that G is a topological group satisfying:

(i) G is Hausdorff, locally compact, totally disconnected

(ii) $1 \in G$ has a basis of neighborhoods consisting of open locally compact Hausdorff subgroups.

(iii) G is an open pro finite subgroup.

All of the examples that we will see in this course will be where G is a closed subgroup of $\mathrm{GL}_n(F)$, where F is a local non-Archimedean field, with topology induced from the topology on F.

3.1 Compact ℓ-groups

Example 3.1. Let $K = \mathrm{GL}_n(\mathscr{O}_F) \subseteq \mathrm{GL}_n(F)$.

Proposition 3.1. *Let K be a compact ℓ-group. Then*

(a) Every smooth irreducible representation of K is finite dimensional.

(b) Every short exact sequence in $\mathscr{R}(K)$ splits

(c) Every smooth representation is a direct sum of irreducible representations.

Remark. It follows from the above proposition that every compact ℓ-group is a profinite group. ◇

Proof. We leave (a) and (c) as exercises. We need to check that if $\pi\colon M \twoheadrightarrow N$ is a surjective morphism in $\mathscr{R}(K)$, then π splits. That is, we have a K-invariant section. First take any linear section $s : N \to M$ of π. We would like to change s so that it is K-equivariant. Let μ_K be the unique Haar measure on K with $\mu_K(K) = 1$. Then define

$$\sigma := \int_L g^{-1} \circ s \circ g \, d\mu_K(g).$$

This is a K-equivariant section of π, which gives a splitting for the surjection π. □

Alternative Proof of (b). There is an alternative interpretation for σ that is more elementary (as in, there is no measure theory involved). Here is a more concrete description. Take any finite dimensional K-subrepresentation $N_0 \subset N$. Then $M_0 := \operatorname{span}(K \cdot s(N_0))$ is a finite dimensional subspace of M. So this is a subrepresentation and π restricts to a surjection $M_0 \twoheadrightarrow N_0$. The K-action on M_0 and N_0 factors through the quotient $K \twoheadrightarrow K/H =: \Gamma$. Then

$$\sigma = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \gamma^{-1} \circ s|_{N_0} \circ \gamma \colon N_0 \to M_0$$

is a K-equivariant section of $\pi|_{M_0}$. From here, the last thing to check is that all the different sections that we get from considering different finite dimensional K-subrepresentations of N_0 are compatible with each other. The reason we have this compatibility is because the original linear section $s\colon N \to M$ was globally defined. $\square$

3.2 Schur's Lemma

Let G be an ℓ-group and let $M \in \mathscr{R}(G)$ be an irreducible representation.

Lemma. *Let K be an algebraically closed field of characteristic 0. Then*

$$\operatorname{End}_G(M) = K,$$

assuming that K is uncountable and G is second countable (as a topological space).

Remark. G is second countable if and only if $(G : K)$ is at most countable for all compact open subgroup $K \subset G$. $\diamond$

Example 3.2. Take $G = \overline{\mathbb{Q}}(t)^\times$. This is a discrete group and acts on $\overline{\mathbb{Q}}(t)$. This is an irreducible representation over $\overline{\mathbb{Q}}$.

Proof of Lemma. This is very similar to the proof of the Nullstellensatz in algebraic geometry (assuming uncountability of the field).

Consider $D = \operatorname{End}_G(M)$. This is a division algebra over $\mathbb{C}$. We will show that the dimension of D is at most countable. This will imply that $D = \mathbb{C}$.

Pick $a \in D \setminus \mathbb{C}$. This means that $a = \lambda \neq 0$ for all $\lambda \in \mathbb{C}$. Look at the inverses of these elements. We know $\left\{\frac{1}{a-\lambda}\right\}_{\lambda \in \mathbb{C}}$ is uncountable and hence they are linearly dependent. But then this means that a has to satisfy a polynomial condition over $\mathbb{C}$, and this contradicts the algebraic closedness of $\mathbb{C}$.

Choose $v \in M$, $v \neq 0$. Then by irreducibility, this vector v generates M as a representation. So any endomorphism of M is determined uniquely by v; that is, the map $D \to M$, $f \mapsto f(v)$ is injective. But $\dim_{\mathbb{C}}(M)$ is at most countable because $G \cdot v$ is at most countable (by the assumption that G is second countable). This completes the proof. □

3.3 Induction Functors

Let G be an ℓ-group and let $H \subset G$ be a closed subgroup. Then we have a restriction functor

$$\mathrm{Res}_H^G \colon \mathscr{R}(G) \to \mathscr{R}(H).$$

This functor has a right adjoint

$$\mathrm{Ind}_H^G \colon \mathscr{R}(H) \to \mathscr{R}(G).$$

This means that for any representation M of G and any representation N of H, we have

$$\mathrm{Hom}_H(\mathrm{Res}_H^G(M), N) \cong \mathrm{Hom}_G(M, \mathrm{Ind}_H^G(N)).$$

Consider $\mathscr{R}(G) \hookrightarrow \mathrm{Rep}(G)$, where $\mathrm{Rep}(G)$ denotes all representations of G. This inclusion has a right adjoint

$$M \mapsto M^{\mathrm{sm}} := \{v \in M \colon \mathrm{stab}_G(v) \text{ is open}\}.$$

Explicitly, this means that for $N \in \mathscr{R}(G)$ and $M \in \mathrm{Rep}(G)$, then we have

$$\mathrm{Hom}_G(N, M) = \mathrm{Hom}_G(N, M^{\mathrm{sm}}).$$

On the other hand, the restriction functor $\mathrm{Rep}(G) \to \mathrm{Rep}(H)$ has a right adjoint, namely, old-Ind_H^G. It follows formally that the *smooth induction* is defined to be

$$\mathrm{Ind}_H^G(N) = (\text{old-}\,\mathrm{Ind}_H^G(N))^{\mathrm{sm}}.$$

This is right adjoint to $\mathrm{Res}_H^G \colon \mathscr{R}(G) \to \mathscr{R}(H)$.

Explicitly, $\mathrm{Ind}_H^G(N)$ is the set of all $f \colon G \to N$ such that $f(hx) = h \cdot f(x)$ for all $x \in G, h \in H$ and such that the stabilizer of f in G under the action $(g \circ f)(x) = f(xg)$ is open. The second condition means that f locally constant. Note that f locally constant does *not* imply the second condition.

3.4 Compact Induction

This is easy, drawing from our discussion of smooth induction. Define

$$c\text{-}\operatorname{Ind}_H^G(N) = \left\{ f \in \operatorname{Ind}_H^G(N) \colon \text{“support of } f \text{ modulo } H\text{” is a compact subset of } H\backslash G \right\}.$$

Here, we can check that the second property in our definition of smooth representations is *equivalent* to saying that f is locally constant.

Remark. 1. If $H\backslash G$ is compact, then $c\text{-}\operatorname{Ind}_H^G(N) = \operatorname{Ind}_H^G(N)$. Here, a typical example is when $G = \mathrm{GL}_n(F)$ and $H = B$ (or a subgroup containing B). (Even if $H\backslash G$ is not compact, sometimes the above happens for a representation N. These representations are important.)

2. If H is open in G, (for instance, take $G = \mathrm{GL}_n(F)$ and $H = \mathrm{GL}_n(\mathscr{O}_F)$), then $c\text{-}\operatorname{Ind}_H^G$ is left adjoint to Res_H^G. Otherwise, Res_H^G has no left adjoint (EXERCISE). (Can check (think about infinite products in this category) that $\mathscr{R}(G) \to \mathscr{R}(H)$ does not preserve infinite products. (Need to work out how infinite products work in this category.) (Right adjoints preserve products, and left adjoints preserve coproducts.) $\diamond$

4 17 September 2012: Smooth Irreducible Representations of $\mathrm{GL}_2(F)$

We will discuss this topic when F is a non-Archimedean local field. There are two categories:

- Principal Series representations
- Supercuspidal representations (in the literature, these are also called cuspidal, or absolutely cuspidal, etc.)

We will describe the principal series representations completely and give some examples of the supercuspidal representations.

4.1 Construction of the Principal Series

We will basically do what we did in the finite-field case.

Let $G = \mathrm{GL}_2(F)$ and let $B = \left\{ \begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix} \right\} \subseteq G$. Consider the surjection

$$B \twoheadrightarrow T = F^\times \times F^\times.$$

Choose a smooth character $\chi : T \to \mathbb{C}^\times$ (that is, $\ker(\chi)$ is open). Note that this is the same as choosing two characters of $F^\times$, say $\chi_1, \chi_2 : F^\times \to \mathbb{C}^\times$. Let $\widetilde{\chi} : B \to \mathbb{C}^\times$ be the inflation of χ and form $\rho_\chi = \mathrm{Ind}_B^G(\widetilde{\chi}) = c\text{-}\mathrm{Ind}_B^G(\widetilde{\chi})$. As a topological space, $G/B \cong \mathbb{P}^1(F)$ and hence is compact.

4.1.1 First Observations

We get the following for free: If $\chi_1 = \chi_2$, then ρ_χ contains $\chi_1 \circ \det : \mathrm{GL}_2(F) \to F^\times \to \mathbb{C}^\times$ as a subrepresentation. Indeed,

$$\mathrm{Hom}_G(\chi_1 \circ \det, \rho_\chi) = \mathrm{Hom}_B(\chi_1 \circ \det|_B, \widetilde{\chi})$$

is one-dimensional. This is the only thing we get for free.

Claim 3. $\mathrm{End}_G(\rho_\chi) = \mathbb{C}$.

We will find that we have three cases:

(1) If $\chi_1 = \chi_2$, then ρ_χ has a 1-dimensional subrepresentation with irreducible quotient.

(2) ρ_χ has a 1-dimensional quotient with irreducible kernel.

(3) ρ_χ is irreducible.

Our technique will be to analyze the restriction of the representation ρ_χ to $\left\{\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right\} \subset B$.

Notice the duality between the first two cases. We now turn our attention to this.

4.1.2 Duality and Induction for Finite Groups

For finite groups, given a finite $\Gamma \supset H$, then given $\rho : H \to \mathrm{GL}(V)$, we obtain $\rho^* : H \to \mathrm{GL}(V^*)$ given by $h \mapsto \rho(h^{-1})^*$. Also,

$$\mathrm{Ind}_H^\Gamma(\rho^*) \cong (\mathrm{Ind}_H^\Gamma \rho)^*.$$

More generally, if $H \subset \Gamma$ are abstract groups, then

$$\mathrm{Ind}_H^\Gamma(\rho^*) \cong (\mathrm{ind}_H^\Gamma(\rho))^*.$$

Reminder: we will construct a pairing

$$\langle \cdot, \cdot \rangle : \mathrm{Ind}_H^\Gamma(\rho^*) \times \mathrm{ind}_H^\Gamma *(\rho) \to \mathbb{C}$$

given by the following. For $\varphi : \Gamma \to V^*$ and $\psi : \Gamma \to V$ satisfying some properties, we obtain a function $\gamma \mapsto \langle \varphi(\gamma) \mid \psi(\gamma) \rangle \in \mathbb{C}$. This function is invariant under left translations. So it descends to a function $H \backslash \Gamma \to \mathbb{C}$ with finite support. Then set

$$\langle \varphi, \psi \rangle := \sum_{x \in H \backslash \Gamma} \langle \varphi(x) \mid \psi(x) \rangle.$$

So we have obtained a Γ-equivariant linear isomorphism $\mathrm{Ind}_H^\Gamma(\rho^*) \to (\mathrm{ind}_H^\Gamma(\rho))^\times$. (Note: $\langle \cdot \mid \cdot \rangle$ is the usual evaluation $V^* \times V \to \mathbb{C}$.)

4.1.3 Duality and Induction for ℓ-groups

Let G be an ℓ-group and let $H \subset G$ be a closed subgroup. Then given $\rho : H \to \mathrm{GL}(V)$ a smooth representation, we have $\rho^\vee : H \to \mathrm{GL}(V^\vee)$, the smooth dual of ρ and $V^\vee := (V^*)^{\mathrm{sm}}$. (In the Zelevinsky paper, they write $\widetilde{\rho} = \rho^\vee$.)

The naïve expectation is the that $\mathrm{Ind}_H^G(\rho^\vee) \cong (c\text{-}\,\mathrm{Ind}_H^G(\rho))^\vee$. This is FALSE. There is a statement similar to this, but we need to replace ρ with something else.

We would like to imitate the previous argument. Take $\varphi \in \mathrm{Ind}_H^G(\rho^\vee)$ and $\psi \in c\text{-}\mathrm{Ind}_H^G(\rho)$. This function is invariant under left multiplication by H, and hence descends to an element of $C_c^\infty(H\backslash G)$. Note that C^∞ means locally constant and c means compact support. We give $H\backslash G$ the quotient topology.

Now we would like to define $\langle\varphi,\psi\rangle$. Then we get

$$\langle\varphi,\psi\rangle = \int_{H\backslash G}\langle\varphi \mid \psi\rangle\, dx.$$

Question: What is dx? It is a right-translation invariant measure on $H\backslash G$ which is invariant under the right G-action. BUT: dx doesn't always exist. For instance, it doesn't exist when $G = \mathrm{GL}_2(F)$ and $H = B$ as in our case.

We would like to figure out under what assumptions this measure does exist and if it doesn't exist we want to know how to salvage the situation so that it works out.

Sketch. Suppose dx is a right-invariant measure on $H\backslash G$. Then we get

$$\int dx\colon C_c^\infty(H\backslash G) \to \mathbb{C},$$

a linear functional invariant under right translation.

Introduce

$$av_H\colon C_c^\infty(G) \to C_c^\infty(H\backslash G), \quad f \mapsto \left(g \mapsto \int_H f(hg)\, d\mu_H^r(h)\right),$$

where μ_H^r is a chosen right Haar measure on H. It turns out that this map av_H is surjective (EXERCISE). Also, av_H commutes with the right translation action of G.

Remark. The function $av_H(f)$ is invariant under LEFT translation by H, so it descends to the quotient on the left by H. ◇

So $C_c^\infty(G) \to \mathbb{C}$, $f \mapsto \int_{H\backslash G} av_H(f)\, dx$ is a right-invariant linear functional, so there exists $c \in \mathbb{C}$ such that

$$\int_{H\backslash G} av_H(f)\, dx = c\int_G f\, d\mu_G^r$$

for all $f \in C_c^\infty(G)$ (where μ_G^r is a right Haar measure on G).

Remark. Space of right-invariant linear functions on $C_c^\infty(G)$ is one-dimensional. ◇

Let $f \in C_c^\infty(G)$ be arbitrary. If $\gamma \in H$, consider

$$(\lambda_\gamma f)(g) := f(\gamma^{-1}g).$$

Then

$$\begin{aligned} av_H(\lambda_\gamma f)(g) &= \int_H (\lambda_\gamma f)(hg)\, d\mu_H^r(h) \\ &= \int_H f(\gamma^{-1}hg)\, d\mu_H^r(h) \\ &= \int_H f(hg)\, d\mu_H^r(\gamma h) \\ &= \delta_H(\gamma)\cdot av_H(f). \end{aligned}$$

The character $\delta_H(\gamma) : H \to \mathbb{Q}_{>0}^\times$ is called the *modulus character* or *moduli character*.

The upshot is that

$$(\lambda_\gamma f) - \delta_H(\gamma)\cdot f \in \ker(av_H).$$

Hence we must have

$$\int_G (\lambda_\gamma f - \delta_H(\gamma)\cdot f)\, d\mu_G^r = 0.$$

But

$$\int_G (\lambda_\gamma f)\, d\mu_G^r = \delta_G(\gamma)\cdot \int_G f\, d\mu_B^r.$$

So the upshot here is that if dx exists, then $\delta_G(\gamma) = \delta_H(\gamma)$ for all $\gamma \in H$. So the restriction of the modulus character of G to H is the modulus character of H.

Warning: But this does not work in general. In our situation when $G = \mathrm{GL}_2(F)$ and B is the Borel subgroup , the modulus character for G is trivial but the modulus character for B is *not* trivial.

Remark. It turns out that the reverse implication is true. That is, if dx exists, then $\delta_G|_H = \delta_H$. ◇

Why is the modulus character rational valued? Take $K \subseteq G$ a compact open subgroup. Then the modulus character $\delta_G(\gamma)$ can be computed as follows: We have

$$\int \mathbb{1}_K\, d\mu_G^r = \mu_G^r(K)$$

and so

$$\int \mathbb{1}_{\gamma K}\, d\mu_G^r = \delta_G(\gamma)\mu_G^r(K).$$

Now,

$$\int \mathbb{1}_\gamma K\gamma^{-1}\, d\mu_G^r = \mu_G^r(\gamma K\gamma^{-1}),$$

so finally,

$$\delta_G(\gamma) = \frac{(\gamma K\gamma^{-1} : K\cap jKj^{-1})}{(K : K\cap jKj^{-1})}\mu_H^r(h).$$

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(Optional exercises for this class can be found at www.umich.edu/~mityab/teaching/m711f12.)

Let $\chi : T \to \mathbb{C}^\times$ be a smooth character. We may inflate this to $\widetilde{\chi} : B \to \mathbb{C}^\times$. Then set

$$\rho_\chi := \operatorname{Ind}_B^G(\widetilde{\chi}) = c\text{-}\operatorname{Ind}_B^G(\widetilde{\chi}).$$

Definition 5.1. A *principal series representation* of G is a subquotient of ρ_χ for some χ.

5.1 Classification Theorem for Principal Series Representations

Theorem 1.

(a) *There are three mutually exclusive possibilities:*

  (i) ρ_χ *is irreducible.*

  (ii) ρ_χ *has a 1-dimensional subrepresentation with irreducible quotient. (This happens if and only if* $\chi_1 = \chi_2$*. EXERCISE.)*

  (iii) ρ_χ *has a 1-dimensional quotient with irreducible kernel. (This is obtained by applying smooth duality to the previous.) (This happens if and only if* $\frac{\chi_1(a)}{\chi_2(a)} = ||a||^2$*, where* $||\cdot||$ *is the normalized absolute value.)*

(b) *Classification of irreducible principal series representations*

(c) *If* ρ *is a smooth irreducible representation of* G*, then* ρ *is a principal series representation if and only if* $\operatorname{Hom}_U(\rho|_U, 1_U) \neq 0$.

5.1.1 Strategy for (a)

We outline the proof of (a). The statement follows from the following steps:

1. Prove that $\rho_\chi^\vee \cong \rho_{\chi^{-1}\cdot\nu}$, where $\nu \cdot \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right) = \frac{||a||}{||b||}$. (Think about Haar measure and also the quotient $B\backslash G$.)

2. $\operatorname{Res}_B^G(\rho_\chi)$ has a filtration by B-subrepresentations $W_1 \subseteq W_2 \subseteq W_3 = \rho_\chi|_B$ such that W_1 is irreducible (and infinite-dimensional) and W_2/W_1, W_3/W_2 are 1-dimensional.

3. Every smooth finite-dimensional irreducible representation of $\mathrm{GL}_2(F)$ is 1-dimensional.

   Idea. If $\rho : G \to \mathrm{GL}(V)$ is a smooth finite-dimensional representation, then $\ker(\rho)$ is open. Furthermore, $\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \in \ker(\rho)$ when $x \in F$ is "small enough." This means that $\ker(\rho)$ contains $\mathrm{SL}_2(F)$. It follows then that if ρ is irreducible, then ρ must be one-dimensional. □

4. ρ_χ has a 1-dimensional sub representation if and only if $\rho_\chi^\vee$ has a 1-dimensional quotient. Also, ρ_χ is a 1-dimensional quotient if and only if $\rho_\chi^\vee$ has a 1-dimensional quotient. (These two statements are the same if we restrict our attention to admissible representations. In general, the square of the duality is not the identity, and so these two statements may not be the same.)

   Proof. EXERCISE: Duality is an exact functor on $\mathscr{R}(G)$ (where G is any ℓ-group). Let $V \hookrightarrow W$ be an injection of smooth representations. This induces a surjection $W^* \twoheadrightarrow V^*$. It requires proof to show that $(W^*)^{\mathrm{sm}} \twoheadrightarrow (V^*)^{\mathrm{sm}}$.

   Using the above, then we get both implications $(\Rightarrow)$ in the statement of (4). Then $(\Leftarrow)$ follows because $\rho_\chi^\vee = \rho_{\chi^{-1}\cdot\nu}$. □

Step 2 of the above outline gives that ρ_χ cannot have an infinite-dimensional subrepresentation with an infinite-dimensional quotient. Hence if ρ_χ is irreducible, then either ρ_χ has a finite-dimensional subrepresentation or a finite-dimensional quotient. And finally this means that it either has a one-dimensional subrepresentation or a one-dimensional quotient. This completes the proof.

5.2 Haar Measures and Examples

Let G be any ℓ-group. There are two viewpoints on a left Haar measure:

- Analytic. There exists a nonzero Borel measure μ_G^ℓ on G which is left-invariant. It is finite on compact sets, and hence we may integrate on functions with compact support.
- Algebraic. There exists a nonzero linear functional $I_G^\ell : C_c^\infty(G) \to \mathbb{C}$ which is invariant under left translations. This functional is unique up to scaling.

Theorem 2. *There exists a nonzero linear functional $I_G^\ell : C_c^\infty(G) \to \mathbb{C}$ which is invariant under left translations. This functional is unique up to scaling.*

Proof. Choose a compact open subgroup (c.o.s.) $K \subset G$, prescribe $I_G^\ell(\mathbb{1}_K)$ arbitrarily, where $\mathbb{1}_K$ is the indicator function of K. Then for any open subgroup $H \subset K$, we must have $I_G^\ell(\mathbb{1}_H) = I_G^\ell(\mathbb{1}_K)/(K : H)$. Therefore for any c.o.s. $H \subseteq G$, then

$$I_G^\ell(\mathbb{1}_H) = \frac{(H : K \cap H)}{(K : K \cap H)} \cdot I_G^\ell(\mathbb{1}_K).$$

This determines I_G^ℓ on all of $C_c^\infty(G)$. □

Example 5.1. Let $G = (F, +)$. Then there is a unique Haar measure dx such that $dx(\mathscr{O}_F) = 1$.

Example 5.2. What about $G = F^\times$? Observation: if $a \in F$, $dx(a \cdot \mathscr{O}_F) = ||a||$. Notation: if $a \in F^\times$, we may write $a = \pi^n \cdot u$, where $\pi \in \mathscr{O}_F$ is a generator of the maximal ideal, $n \in \mathbb{Z}$, and $u \in \mathscr{O}_F^\times$. Then define $||a|| = q^{-n}$, where $q = |\mathscr{O}_F/(\pi)|$. Indeed, $a\mathscr{O}_F = \pi^n \cdot \mathscr{O}_F$. If $n >=\geq 0$, then $\pi^n \mathscr{O}_F \subseteq \mathscr{O}_F$ with index q^n, and if $n < 0$, then $\mathscr{O} \subseteq \pi^n \mathscr{O}_F$ with index q^{-n}. Then $\frac{dx}{||x||}$ is a Haar measure on $F^\times$.

Example 5.3. $G = \mathrm{GL}_n(F)$. If $A \subseteq F^n$ is measurable, then for all $g \in \mathrm{GL}_n(F)$, then

$$(dx^n)(gA) = ||\det(g)|| \cdot (dx^n)(A).$$

Let dg be the "Lebesgue" measure on $\mathrm{Mat}_n(F) = F^{n^2}$. Then $\frac{dg}{||\det(g)||^n}$ is both a left and a right Haar measure on $\mathrm{GL}_n(F)$.

Remark. For some groups, the left Haar measure and the right Haar measure are different. For instance, for the Borel subgroup, we need different formulas for the left- and right-invariant Haar measures. ◇

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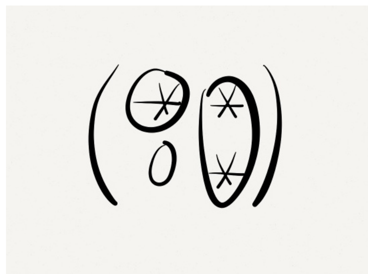
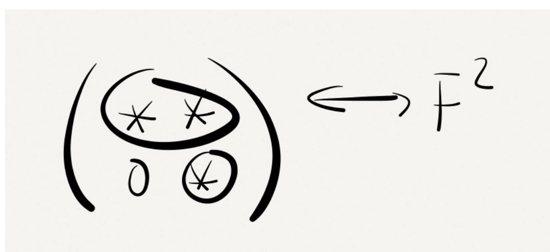
Recall: On $\mathrm{GL}_n(F)$ we have a left and right Haar measure: $\frac{dg}{\|\det(g)\|^n}$, where dg is the restriction of the Lebesgue measure on $\mathrm{Mat}_n(F) = F^{n^2}$.

Take G to be the Borel subgroup $\{\left(\begin{smallmatrix} x & y \\ 0 & z \end{smallmatrix}\right)\}$ inside $\mathrm{GL}_2(F)$. Then under the *left* multiplication action, we have a left Haar measure $\frac{dx\,dy\,dx}{\|x\|^2\|z\|}$.



Under the right multiplication action, we have the right Haar measure $\frac{dx\,dy\,dz}{\|x\|\cdot\|z\|^2}$, so B is not unimodular.



6.1 The Module/Modulus/Modular Character

Let G be any ℓ-group. Let $I : C_c^\infty(G) \to \mathbb{C}$ be a nonzero linear functional invariant under the left G-action. This is unique up to scaling.

Given $\gamma \in G$ and $f \in C_c^\infty(G)$, define a right representation

$$(\rho_\gamma f)(g) = f(g\gamma).$$

Then $f \mapsto I(\rho_\gamma f)$ is also left-invariant and so by uniqueness, there exists a unique $\delta_G(\gamma) \in \mathbb{C}^\times$ such that $I(\rho_\gamma f) = \delta_G(\gamma)^{-1} \cdot I_G(f)$. We therefore get a map $\delta_G \colon G \to \mathbb{C}^\times$, which is called the *modulus* of G.

In terms of μ_G^ℓ, we have that

$$\int_G f(g)\, d\mu_G^\ell(g\gamma^{-1}) = \int_G f(g\gamma)\, d\mu_G^\ell(g) = \delta_G^{-1}(\gamma) \int_G f(g)\, d\mu_G(g).$$

So

$$d\mu_G^\ell(g\gamma) = \delta_G(\gamma)\, d\mu_G^\ell(g) \quad \text{for all } \gamma \in G.$$

Example 6.1. Let $G = B \subseteq \mathrm{GL}_2(F)$. Write $g = \left(\begin{smallmatrix} x & y \\ 0 & z \end{smallmatrix}\right)$, $\gamma = \left(\begin{smallmatrix} a & ? \\ 0 & b \end{smallmatrix}\right)$. Recall that we have $d\mu_G^\ell(g) = \frac{dx\, dy\, dz}{\|x\|^2 \cdot \|z\|}$, and so

$$d\mu_G^\ell(g\gamma) = \frac{\|\det(\gamma)\| \cdot \|b\|}{\|a\|^2 \cdot \|b\|} \frac{dx\, dy\, dz}{\|x\|^2 \cdot \|z\|}.$$

Therefore the modulus character of B is

$$\delta_B(\gamma) = \frac{\|b\|}{\|a\|}.$$

Claim 4. Let G be any ℓ group and fix $K \subseteq G$ to be any compact open subgroup. Then

$$\delta_G(\gamma) = \frac{(K : K \cap \gamma K \gamma^{-1})}{(\gamma K \gamma^{-1} : K \cap \gamma K \gamma^{-1})}.$$

Proof. Without loss of generality, we may assume that $I(\mathbb{1}_K) = 1$. Then

$$\delta_G(\gamma)^{-1} = I(\rho_\gamma \mathbb{1}_K) = I(\mathbb{1}_{K\gamma^{-1}}) = I(\mathbb{1}_{\gamma K \gamma^{-1}}).$$

The result follows. DETAIL □

Consequences.

1. $\delta_G|_K \equiv 1$, and in particular, $\ker(\delta_G)$ is open.
2. $\delta_G : G \to \mathbb{Q}_{>0}$
3. If G has a normal compact open subgroup, then $\delta_G \equiv 1$. In particular, if G is compact, then $\delta_G \equiv 1$.

Claim 5. $\delta_G^{-1} \cdot \mu_G^\ell$ is a right Haar measure.

Proof. If $f \in C_c^\infty(G)$, then for all $\gamma \in G$,

$$\begin{aligned}\int_G f(g\gamma)\delta_G^{-1}(g)\, d\mu_G^\ell(g) &= \int f(g)\delta_G^{-1}(g\gamma^{-1})\, d\mu_G^\ell(g\gamma^{-1}) \\ &= \delta_G(\gamma) \cdot \delta_G(\gamma^{-1}) \int f(g)\delta_G(g)^{-1}\, d\mu_G^\ell(g).\end{aligned}$$ □

By a similar computation, we have that if μ_G^r is a right Haar measure on G, then for all $f \in C_c^\infty(G)$.

$$\int_G f(\gamma^{-1}g)\, d\mu_G^r(g) = \delta_G(\gamma)^{-1} \int_G f(g)\, d\mu_G^r(g)$$

Now back to the question about what happens to induction under duality...

6.2 Induction and Duality

Let G be any ℓ group and let $H \subseteq G$ be a closed subgroup. Choose $\rho \in \mathscr{R}(H)$, a smooth representation.

Claim 6.

$$\operatorname{Ind}_H^G(\rho^\vee \otimes \delta_H^{-1}\delta_G) \cong (c\text{-}\operatorname{Ind}_H^G(\rho))^\vee. \tag{1}$$

Corollary 6.1. *If* $G = \mathrm{GL}_2(F)$ *and* $B \subseteq G$ *is the Borel subgroup, then*

$$(\operatorname{Ind}_B^G(\widetilde{\chi}))^\vee \cong \operatorname{Ind}_B^G(\widetilde{\chi}^{-1} \otimes \delta_B^{-1}).$$

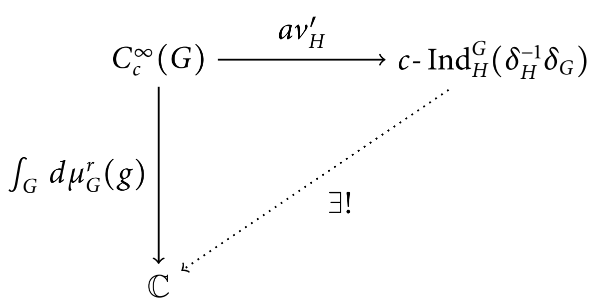
6.2.1 Construction of the Isomorphism in 1

We have a map

$$\operatorname{Ind}_H^G(\rho^\vee \otimes \delta_H^{-1}\delta_G) \otimes_{\mathbb{C}} c\text{-}\operatorname{Ind}_H^G(\rho) \to c\text{-}\operatorname{Ind}_H^G(\rho^\vee \otimes \rho \otimes \delta_H^{-1}\delta_G) \to c\text{-}\operatorname{Ind}_H^G(\delta_H^{-1}\delta_G),$$

and we'd like to construct a map from the rightmost object to $\mathbb{C}$.

The idea of this construction is the following.



The key point is that $\ker(av'_H)$ is contained in the kernel of $\int_G d\mu_G^r(g)$.

av'_H is equivariant with respect to the right action of G and surjective.

Formula for av'_H: For $f \in C_c^\infty(G)$,

$$av'_H(f)(g) = \int_H \frac{\delta_H(h)}{\delta_G(h)} f(hg)\, d\mu_H^r(h).$$

It is easy to check:

- $av'_H \colon C_c^\infty(G) \to c\text{-}\operatorname{Ind}_H^G(\delta_H^{-1}\delta_G)$

- av'_H commutes with the right G-action
- Given any $\gamma \in G$, define $(\lambda_\gamma f)(g) = f(\gamma^{-1} \cdot g)$. Then for all $\gamma \in H$,

  $$av'_H(\lambda_\gamma f) = \delta_G^{-1}(\gamma) \cdot av'_H(f).$$

  Then

  $$(\lambda_\gamma f) - \delta_G^{-1}(\gamma) \cdot f \in \ker(av'_H).$$

- $\lambda_\gamma f - \delta_G^{-1}(\gamma) \cdot f$ is also in the kernel of $\int_G d\mu_G^r$.

Medium difficulty to check:

- av'_H is surjective
- $\ker(av'_H)$ is spanned by $\{(\lambda_\gamma f) - \delta_G^{-1}(\gamma) \cdot f : f \in C_c^\infty(G)\}$.
- we actually get an isomorphism between $\mathrm{Ind}_H^G(\rho^\vee \times \delta_H^{-1}\delta_G$ and $c\text{-}\mathrm{Ind}_H^G(\rho))^\vee$.

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7.1 Set-Up

Given a character χ of $T = F^\times \times F^\times$, we obtain its inflation $\widetilde{\chi}$ to the standard Borel subgroup $B \subseteq G := \mathrm{GL}_2(F)$.

Proposition 7.1. *The representation* $\mathrm{Ind}_B^G(\widetilde{\chi})$ *has a filtration by B-subrepresentations* $W_1 \subseteq W_2 \subseteq W_3$, *where* W_1 *is irreducible and* W_3/W_2, W_2/W_1 *are 1-dimensional* W_3 *is the whole space.*

From this, we obtain a complete description of what it means for this induced representation is.

Proof. Let us try to analyze the space $\mathrm{Ind}_B^G(\widetilde{\chi}) = \{f : G \to \mathbb{C} : \text{properties}\}$. Recall that we have a Bruhat decomposition

$$G = B \cup BwB, \quad \text{where } w = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right).$$

Also, if $U \subseteq B$ is the unipotent subgroup, then the map

$$B \times U \to BwB, \quad (b, u) \mapsto bwu$$

is a homeomorphism. This means that for any function $f \in \mathrm{Ind}_B^G(\widetilde{\chi})$, then f is determined by $f(1)$ and $\{f(wu) : u \in U\}$.

Let us analyze the first condition—the image of f at the identity. We have $\mathrm{Ind}_B^G(\widetilde{\chi}) \twoheadrightarrow \mathbb{C}$, and this map is B-linear if we let B act on $\mathbb{C}$ via $\widetilde{\chi}$. Let $W_2 = \ker(\mathrm{Ind}_B^G(\widetilde{\chi}) \twoheadrightarrow \widetilde{\chi})$.

Let us analyze W_2. (Here is a question: Look at the space $C^\infty(\mathbb{P}^1(F))$ (locally constant functions on $\mathbb{P}^1(F)$). Consider the surjection $C^\infty(\mathbb{P}^1(F)) \twoheadrightarrow \mathbb{C}$. What is the kernel? It is $C_c^\infty(F)$, i.e. the space of locally constant, compactly supported functions on F.)

Exercise 1. We have a linear isomorphism $W_2 \to C_c^\infty(U)$ given by $f \mapsto f(w \cdots ?)$. The inverse of this is $h \mapsto \widetilde{h}$, where $\widetilde{h} : bwu \mapsto \widetilde{\chi}(b) \cdot h(u)$ and $b \mapsto 0$. These two conditions define $\widetilde{h}$ on all of G. ◇

Now we need to analyze how B acts on $C_c^\infty(U)$. We have $B = T \ltimes U$, where $T = F^\times \times F^\times$ is the maximal torus consisting of diagonal matrices. So we can analyze the action of U and T to obtain the action of B. Now, U acts on $C_c^\infty(U)$ by (right) translations. If $t \in T$, then for all $h \in C_c^\infty(U)$, then

$$\begin{aligned}(t\cdot h)(u) &= (t\cdot \widetilde{h})(wu)\\ &= \widetilde{h}(wut)\\ &= \widetilde{h}((wtw^{-1})w(t^{-1}ut))\\ &= \widetilde{\chi}(wtw^{-1})h(t^{-1}ut)\\ &= \chi^w(t)\cdot h(t^{-1}ut).\end{aligned}$$

This gives us the action of T, and hence we have now described the action of B on $C_c^\infty(U)$.

Consider the linear functional $\int_U du\colon C_c^\infty(U) \twoheadrightarrow \mathbb{C}$. Let us determine the action of B on $\mathbb{C}$ for which $\int_U du$ becomes B-linear. U acts trivially and we may identify

$$U = \{\left(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right)\} \cong (F,+), \quad du = dx = \text{Lebesgue measure on } F.$$

We have

$$\begin{pmatrix}a^{-1} & 0\\ 0 & b^{-1}\end{pmatrix}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix} = \begin{pmatrix}1 & ba^{-1}x\\ 0 & 1\end{pmatrix}.$$

So if $h \in C_c^\infty(U) = C_c^\infty(F)$, then for $t = \left(\begin{smallmatrix}a & 0\\ 0 & b\end{smallmatrix}\right)$, then we have

$$\int_F (t\cdot h)(x)\,dx = \chi^w(t)\int_F h(ba^{-1}x)\,dx = \chi^w(t)\cdot\frac{\|a\|}{\|b\|}\int_F h(x)\,dx.$$

The upshot is then that

$$\int_U du\colon W_2 \twoheadrightarrow \widetilde{\chi}^w\cdot\delta_B^{-1}$$

is B-linear and we will take W_1 to be the kernel of this map.

It remains to show that W_1 is irreducible as a representation of G. It is enough to look at the *mirabolic subgroup* $M := \{\left(\begin{smallmatrix}* & *\\ 0 & 1\end{smallmatrix}\right)\} \subseteq B$. Notice that $M = F^\times \ltimes F$, where $F^\times$ acts on F by scaling. We have that

$$W_1 = \{f \in C_c^\infty(F) : \int_F f(x)\,dx = 0\},$$

and on this space, F acts by translations and $F^\times$ acts by scaling. (Jake: does this imply that it's irreducible? You need more argument than we've given, I think?) □

Remark. This is the only infinite-dimensional representation of M! All others are one-dimensional. ◇

Remark. There is a different way of thinking about this statement so that this is obvious. This is, in some sense, the actual reason why this representation is irreducible.

The category $\mathscr{R}(F,+)$ is equivalent to the category of sheaves of $\mathbb{C}$-vector spaces on F. This equivalence is given by the Fourier transform. In the same line of thought, $\mathscr{R}(M)$ is equivalent, as a category, to the category of $F^\times$-equivariant sheaves on F (where $F^\times$ acts on F by scaling. Then $F = \{0\} \cup F^\times$, where $F^\times$ acts trivially on $\{0\}$ and acts freely on $F^\times$. Then the only equivariant sheaf is just the constant, which is exactly the sheaf corresponding to W_1 in our proof. ◇

7.2 Co-Invariants

Let G be any ℓ-group. Consider the *Jacquet functor* $J_G : \mathscr{R}(G) \to \mathbb{C} - \text{vect}$ given by $V \mapsto V / \operatorname{span}_{\mathbb{C}}\{g \cdot v - v : g \in G, v \in V\}$.

Example 7.1. $J_G(C_c^\infty(G))$ (with $C_c^\infty(G)$ a representation of G via right translation action of G) is 1-dimensional, namely: $\int_G d\mu_G^r(g)\colon C_c^\infty(G) \to \mathbb{C}$.

Remark. IMPORTANT: If G is a compact ℓ-group, then $J_G\colon \mathscr{R}(G) \to \mathbb{C}$-vect is exact. ◇

We can in fact say more.

Lemma. *If G is a filtered union of its compact open subgroups, then J_G is exact.*

Example 7.2. F has this property and $F^\times$ does not. Indeed, $F = \cup_{n\geq 1} \pi^{-1}\mathscr{O}_F$.

Remark. The word "filtered" is important. For instance, take any counterexample to Burnside's problem—that is, a finitely generated group wherein every element is torsion but the group is not finite. This would be a counterexample to the lemma if we had not required G to be a *filtered* union. ◇

Proof of the Lemma. If $K \subseteq G$ is a compact open subgroup, then $J_K\colon \mathscr{R}(G) \to \mathbb{C}$-vect is exact. (This is the composition with the restriction to $\mathscr{R}(K)$ and then taking coin variants to $\mathbb{C}$-vect.) This is because any smooth representation of K is a direct sum of irreducibles. In fact,

$$\varinjlim J_K \cong J_G,$$

and the fact that taking filtered colimits preserves exactness gives the result (and exactness is *not* preserved if we do not have the word "filtered"). The isomorphism above is given by the following. Let $K \subseteq K' \subseteq G$ be compact open subgroups. Then $J_K(V) \twoheadrightarrow J_{K'}(V)$. these are the maps via which we define the above direct limit. To define a map in the reverse, we have a canonical map $J_K(V) \twoheadrightarrow J_G(V)$. WORK THIS OUT IN DETAIL. $\square$

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Continuation from last time:

Lemma. *Consider $C_c^\infty(F)$ as a representation of the mirabolic group $M = F^\times \ltimes F$. Then $W = \ker(\int_F dx \colon C_c^\infty(F) \to \mathbb{C})$ is irreducible as a representation of M.*

Proof. Recall that the Jacquet functor $J_F \colon \mathscr{R}(F,+) \to \mathbb{C}$-vect is exact. More generally, if $\theta : F \to \mathbb{C}^\times$ is any smooth character, then $J_F^\theta \colon \mathscr{R}(F,+) \to \mathbb{C}$-vect given by $V \mapsto J_F(V \otimes \theta)$ is also exact. If $0 \neq V \in \mathscr{R}(F,+)$, then V has an irreducible subquotient, which must be 1-dimensional. From the exactness of J_F^θ, we may conclude that if $V \in \mathscr{R}(F,+)$, then V is nonzero if and only if $J_F^\theta(V)$ is nonzero for some smooth $\theta \colon F \to \mathbb{C}^\times$.

Now we are ready to prove that W is irreducible. Suppose, for a contradiction, that W is not irreducible and let W' be a nonzero subspace of W that is an M-subrepresentation. Then there exist smooth $\theta_1, \theta_2 \colon F \to \mathbb{C}^\times$ such that $J_F^{\theta_1}(W')$ and $J_F^{\theta_2}(W/W')$ are both nonzero. Then, again by exactness, $J_F(W) = 0$. Hence θ_1 and θ_2 are nontrivial.

The key point here is that $F^\times$ acts (simply) transitively on the set of nontrivial smooth characters $F \to CC^\times$. F "self-dual." Fix a nontrivial $\psi \colon F \to \mathbb{C}^\times$. Then every $\theta \colon F \to \mathbb{C}^\times$ has the form $x \mapsto \psi(ax)$ for a unique $a \in F$.

Hence $J_F^{\theta_1}(W/W')$ is nonzero. So $J_F^{\theta_1}(W)$ has dimension 2, which implies that $\dim J_F^{\theta_1}(C_c^\infty(F)) = 2$, which is a contradiction because $C_c^\infty(F) \otimes \theta_1 \cong C_c^\infty(F)$. □

Remark. Detail of the justification that F is self-dual. $F \supseteq \mathscr{O}_F \ni \pi$. Then we have a chain

$$\cdots \pi^2 \mathscr{O}_F \subseteq \pi \mathscr{O}_F \subseteq \mathscr{O}_F \subseteq \pi^{-1} \mathscr{O}_F \subseteq \cdots$$

The level of θ is the unique n such that $\theta|_{\pi^{n-1}\theta_F} \not\equiv 1$ and $\theta|_{\pi^n \mathscr{O}_F} \equiv 1$. Without loss of generality, we may assume $n = 1$. Then θ induces a character of $\mathscr{O}_F / \pi \mathscr{O}_F \cong \mathbb{F}_q$ and this is self-dual. Then there an $a \in \mathscr{O}_F$ such that $\theta(x) = \psi(ax)$ for all $x \in \mathscr{O}_F$. ◇

What is the upshot of all this work? (Work since last Friday.)

Recall we had $\chi = (\chi_1, \chi_2) \colon T = F^\times \times F^\times \to \mathbb{C}^\times$ and $\rho_\chi = \operatorname{Ind}_B^G(\widetilde{\chi})$. We had three possibilities:

(i) ρ_χ is irreducible

(ii) ρ_χ has a 1-dimensional subrepresentation (which happens if and only if $\chi_1 = \chi_2$)

(iii) ρ_χ has a 1-dimensional quotient (which happens if and only if $\chi_1(a)/\chi_2(a) = ||a||^2$ for all $a \in F^\times$)

8.1 Classification of the Irreducible Representations

Proposition 8.1. *If $\chi, \xi\colon T \to \mathbb{C}^\times$ are smooth characters, then*

$$\dim_{\mathbb{C}} \mathrm{Hom}_G(\rho_\chi, \rho_\xi) = \begin{cases} 1 & \text{if } \xi = \chi \text{ of } \xi = \chi^W \cdot \delta_B^{-1} \\ 0 & \text{otherwise.} \end{cases}$$

(Note that χ^W means the inflation of χ to W.)

Proof. Adjunction gives that

$$\mathrm{Hom}_G(\rho_\chi, \rho_\xi) \cong \mathrm{Hom}_B(\rho_\chi|_B, \widetilde{\xi}).$$

Also, we have a filtration $W_1 \subset W_2 \subset W_3$, where $W_3 = \rho_\chi|_B$, W_1 is irreducible over B, and $W_3/W_2 \cong \widetilde{\chi}$ and $W_2/W_1 \cong \widetilde{\chi}^W \cdot \delta_B^{-1}$.

Hence

$$\mathrm{Hom}_B(\rho_\chi|_B, \widetilde{\xi}) = \mathrm{Hom}_B(W_3/W_1, \widetilde{\xi}) = \mathrm{Hom}_T(W_3/W_1, \xi),$$

where U acts trivially on W_3/W_1 (and $J_U(W_3) = W_3/W_1$.

Now, W_3/W_1 is an extension of χ by $\chi^W \cdot \delta_B^{-1}$. Then we have two cases.

Case 1. If $\chi \neq \chi^W \cdot \delta_B^{-1}$. EXERCISE: the extension splits because T is commutative. (Cannot have a nontrivial extension of two different one-dimensional representations?) This implies the proposition.

Case 2. If $\chi = \chi^W \cdot \delta_B^{-1}$. Then ρ_χ is irreducible and the proposition follows from Schur's lemma.

This completes the proof. □

Definition 8.1. The Steinberg representation St_G is the irreducible quotient of $\mathrm{Ind}_B^G(1)$, which is $C^\infty(\mathbb{P}^1(F))/\{\text{constants}\}$.

Remark. For $G = \mathrm{GL}_2(\mathbb{F}_q)$, $\mathrm{St}_G = \mathrm{Fun}(\mathbb{P}^1(\mathbb{F}_q)/\{\text{constants}\})$. ◇

Claim 7. The Steinberg representation is self-dual.

Proof. We have showed that $(\mathrm{Ind}_B^G(1))^\vee \cong \mathrm{Ind}_B^G(\delta_B^{-1})$. By the previous proposition, there exists a nonzero G-linear map

$$\varphi\colon \mathrm{Ind}_B^G(1) \to \mathrm{Ind}_B^G(\delta_B^{-1}).$$

Note that $\mathrm{Ind}_B^G(1)$ is an extension of St_G by 1 and $\mathrm{Ind}_B^G(\delta_B^{-1})$ is an extension of 1 by $\mathrm{St}_G^\vee$. So φ has to induce an isomorphism $\mathrm{St}_G \cong \mathrm{St}_G^\vee$. □

More generally, consider $\chi = (\nu, \nu)\colon T \to \mathbb{C}^\times$ coming from $\nu\colon F^\times \to \mathbb{C}^\times$. Then ρ_χ is an extension of $\mathrm{St}_G \otimes (\nu \circ \det)$ by $(\nu \circ \det)$. Indeed,

$$\rho_\chi \cong (\nu \circ \det) \otimes \mathrm{Ind}_B^G(1).$$

This means that $\rho_\chi^\vee = \rho_{\chi^{-1}\cdot\delta_B^{-1}}$ is an extension of $(\nu^{-1} \circ \det)$ by $\mathrm{St}_G \otimes (\nu^{-1} \circ \det)$.

Putting everything we've done together, we get the following statements as corollaries.

Corollary 8.2. *The irreducible principal series representations of* $\mathrm{GL}_2(F)$ *over a local field F are*

- $\nu \circ \det$ *(where* $\nu\colon F^\times \to \mathbb{C}^\times$*),*
- $(\nu \circ \det) \otimes St_G$*, and*
- *and those* ρ_χ *that are irreducible.*

The only "overlaps" are $\rho_\chi \cong \rho_{\chi^w\cdot\delta_B^{-1}}$ *(when* ρ_χ *is irreducible).*

Corollary 8.3. *Let* ρ *is be any smooth irreducible representation of* $\mathrm{GL}_2(F)$*. Then* ρ *is principal series if and only if* ρ *is isomorphic to a subrepresentation of* $\mathrm{Ind}_B^G(\widetilde{\chi})$ *for some* χ*, which happens if and only if* $J_U(\rho) \neq 0$*. (Indeed,* ρ *is isomorphic to a subrepresentation of* $\mathrm{Ind}_B^G(\widetilde{\chi})$ *for some* χ *if and only if* $\rho|_B$ *has a* 1*-dimensional quotient, which happens if and only if* $\rho|_U$ *has a trivial quotient.)*

9 28 September 2012: Fairy Tales

Fix some notation. Let k be either a finite field $\mathbb{F}_q$ or a local field F. $G = \mathrm{GL}_n(k)$ and B the standard Borel in G. We have a projection to the diagonal $B \twoheadrightarrow T = (k^\times)^n$.

9.1 The Steinberg Representation St_G

For $n = 2$, the Steinberg representation has a nice presentation:

$$\mathrm{St}_G = C^\infty(\mathbb{P}^1(k))/\{\text{constants}\}.$$

For $n > 2$, the Steinberg representation is a subquotient of $\mathrm{Ind}_B^G(1)$. If $k = F$, St_G is the unique irreducible quotient of $\mathrm{Ind}_B^G(1)$. So the nonsemisimplicity of the local field case actually gives us enough rigidity to say this. If $k = \mathbb{F}_q$, then 1_G and St_G are exactly the two irreducible summands of $\mathrm{Ind}_B^G(1)$ that appear with multiplicity 1.

Remark. We have $\mathrm{End}_G(\mathrm{Ind}_B^G(1)) \cong S_n$. This is a subtle thing. This isomorphism is not natural and by no means canonical. This would make a nice research project to figure out where this comes from. The proof of this is quite unsatisfactory.

From this, we get a bijective correspondence

$$\{\text{irreducible summands of } \mathrm{Ind}_B^G(1)\} \leftrightarrow \{\text{irreducible representations of } S_n\}.$$

But the non-canonical-ness of the isomorphism of algebras above makes it very hard (impossible?) to pin down what this correspondence is.

9.2 More General Principal Series

Take any (smooth) $\chi\colon T \to \mathbb{C}^\times$ and form $\rho_\chi = \mathrm{Ind}_B^G(\widetilde{\chi})$. When $k = \mathbb{F}_q$, then we may compute $\mathrm{End}_G(\rho_\chi)$ using adjunction. The "size" depends on the stabilizer of χ in S_n. In particular, ρ_χ is irreducible if and only if $\chi^w \neq \chi$ for all nontrivial $w \in S_n$. When $k = F$, it is possible to use induction on n to study ρ_χ. The analogue of the mirabolic is: M_n = MATRIX $\cong \mathrm{GL}_{n-1}(F) \ltimes F^{n-1}$. We can describe all smooth representations of M_n in terms of $\mathrm{GL}_{n-1}(F)$ and M_{n-1}. (The best reference for this is Bernstein and Zelevinsky's paper.)

9.3 Other Irreducible Representations of G?

For $n = 2$, we only have two types: principle series representations and (super)cuspidal representations. In general, however, we have more. We first need to generalize the standard

Borel to the standard parabolic:

$$P = \{(\text{block upper triangular with a structure given by some partition of } n)\}.$$

Each such standard parabolic has a decomposition as a semidirect product

$$P = L_P \ltimes U_P.$$

We call L_P the "Levi factor of P" (it is the block-diagonal part of P and is a product of GL_m's) and U_P the unipotent radical of P (we look at the image of the block structure with 1's along the diagonal and control the rest). Note that the unipotent radical is the kernel of the natural projection $P \twoheadrightarrow L_P$.

If $\rho = a$ is a smooth irreducible representation of L_P, we can inflate it to a representation $\widetilde{\rho}$ of P via the aforementioned quotient map $P \to L_P$ and then form $\mathrm{Ind}_P^G(\widetilde{\rho})$.

This gives us a characterization of supercuspidals.

Definition 9.1. A (smooth) irreducible representation V of G is *(super)cuspidal* if it is not a subquotient of $\mathrm{Ind}_P^G(\widetilde{\rho})$ for some proper (standard) parabolic P.

Equivalently, V is a supercuspidal representation if and only if $J_{U_P}(V) = 0$ for all proper parabolic subgroups $P \subseteq G$.

Fact 9.1. *If π is any smooth irreducible representation of G, then π is a subquotient of* $\mathrm{Ind}_P^G(\widetilde{\rho})$ *for some parabolic $P \subseteq G$ and some supercuspidal irreducible representation ρ of L_P.*

9.4 Admissibility

This is a very important notion, though it is a bit technical.

Let G be any ℓ-group.

Definition 9.2. If $\rho\colon G \to \mathrm{GL}(V)$ is a smooth representation of G, it is called *admissible* if, for all compact open subgroups $K \subseteq G$,

$$\dim_{\mathbb{C}}(V^K) < \infty.$$

The upshot: Admissibility is equivalent to self-duality. We can form the smooth dual $\rho^\vee\colon G \to \mathrm{GL}(V^\vee)$ and then $\rho^{\vee\vee}\colon G \to \mathrm{GL}(V^{\vee\vee})$. We have a natural map $V \to V^{\vee\vee}$.

Exercise 2. V is admissible if and only if this is an isomorphism.

Why should you believe in this? V is the union of all $\dim_{\mathbb{C}}(V^K)$, ranging over all possible K. Then we may reduce the above to the statement that if W is an abstract vector space, then $\dim(W) < \infty$ if and only if $W \to (W^*)^*$ is an isomorphism.

Remark. Note that the principle series representations of $\mathrm{GL}_n(F)$ are “obviously” admissible. This follows from a more general fact:

Claim 8. Let G be any ℓ-group and let $H \subseteq G$ be a closed subgroup such that $H\backslash G$ is compact. Then for all smooth admissible representations ρ of H, $\mathrm{Ind}_H^G(\rho)$ is also admissible.

Idea. Let $K \subseteq G$ be any compact open subgroup. We may bound $\dim_{\mathbb{C}}(\mathrm{Ind}_H^G(\rho))^K$ in terms of two quantities:

- $\dim \rho^{(K\cap H)}$, and
- the number of K-orbits on $H\backslash G$.

This is basically the idea of the proof. □

Theorem 3. *Any smooth irreducible representation of* $\mathrm{GL}_n(F)$ *is admissible.*

The same is try for any reductive group over F, though it isn't true for groups in general. For instance, it fails for the mirabolic subgroup. The BLAH representation is not admissible. (FIGURE THIS OUT.)

10 1 October 2012: Hecke Algebras

Recall that for any abstract group Γ and we consider $\mathrm{Rep}(\Gamma)$, the category of all representations of Γ, then we have that $\mathrm{Rep}(\Gamma)$ is isomorphic to the category of $\mathbb{C}[\Gamma]$-modules (where $\mathbb{C}[\Gamma]$ denotes the group algebra).

From now on, we will fix an ℓ-group G. If G is not discrete, then $\mathscr{R}(G)$ is *not* equivalent to the category of A-modules for any (unital) ring A. The Hecke algebra is the next best thing you can get. With this perspective, the Hecke algebra is the analogue of the group algebra in this setting.

10.1 Definition and Construction

Definition 10.1. The *Hecke algebra of G* is

$$\mathscr{H}(G) = C_c^\infty(G)$$

together with the multiplication

$$(f_1 * f_2)(g) = \int_G f_1(\gamma) f_2(\gamma^{-1} g)\, d\mu_G^\ell(\gamma),$$

where μ_G^ℓ is a chosen left Haar measure on G. (We can define this canonically as well by taking locally constant measures instead of locally constant functions.)

Here is the construction of a $\mathscr{H}(G)$-module from a representation. Let $\rho\colon G \to \mathrm{GL}(V)$ be a smooth representation. Given $f \in \mathscr{H}(G)$ and $v \in V$, define

$$f * v = \int_G f(g)\rho(g)(v)\, d\mu_G^\ell(g).$$

This integral can be rewritten as a finite sum. This turns V into a left $\mathscr{H}(G)$-module which is *nondegenerate*, i.e. $\mathscr{H}(G) \cdot V = V$. Indeed, if $v \in V$, then there exists a compact open subgroup $K \subseteq G$ with $v \in V^K$. Then $e_K * v = v$, where $e_K = \frac{1}{\mu_G^\ell(K)} \cdot \mathbb{1}_K \in \mathscr{H}(G)$. Note that $e_K * e_K = e_K$, so that e_K is an idempotent with respect to convolution.

Fact 10.1. *This construction also gives an isomorphism of categories between $\mathscr{R}(G)$ and nondegenerate $\mathscr{H}(G)$-modules.*

10.2 Why are these algebras useful?

10.2.1 Reason 1

Suppose $\rho: G \to \mathrm{GL}(V)$ is a smooth representation. Then ρ is admissible (recall that this means that $\dim V^{\rho(K)} < \infty$ for all compact open subgroups $K \subseteq G$) if and only if for all $f \in \mathscr{H}(G)$,

$$\rho(f): V \to V, \quad v \mapsto f * v$$

has finite rank (is of "trace class").

So if ρ is admissible, we get a linear functional $\Theta_\rho: \mathscr{H}(G) \to \mathbb{C}$ given by $f \mapsto \mathrm{tr}(\rho(f))$, and this is called the *character of* ρ. (One should think of this as a sort of generalized function on G.)

Here is a deep nontrivial theorem (due to Harish-Chandra).

Theorem 4 (Harish-Chandra)**.** *If $G = \mathrm{GL}_n(F)$, where F is a non-archimedean local field, and ρ is a smooth irreducible representation of G (so ρ is admissible), then Θ_ρ is represented by a locally constant function on $G^{r.s.s} \subseteq G$ (where $G^{r.s.s.}$ means the regular semisimple elements of G, i.e. $G^{r.s.s.} = \{g \in \mathrm{GL}_n(F) : g$ has n distinct eigenvalues over $\overline{F}\}$.)*

This means that there exists a locally constant function $\theta_\rho: G^{\text{r.s.s.}} \to \mathbb{C}$ such that

$$\Theta_\rho(f) = \int_{G^{\text{r.s.s.}}} f(g)\theta_\rho(g)\, d\mu_G^\ell(g) \qquad \text{for all } f \in \mathscr{H}(G).$$

10.2.2 Reason 2: Structure theory of smooth representations

Definition 10.2. Let G be any ℓ-group. Fix a compact open subgroup $K \subseteq G$. Define

$$\begin{aligned} \mathscr{H}(G,K) &= \{f \in \mathscr{H}(G) : f(k_1 g k_2) = f(g) \text{ for all } g \in G, k_1, k_2 \in K\} \\ &= \{\text{functions } K\backslash G/K \to \mathbb{C} \text{ with finite support}\} \\ &= e_K * \mathscr{H}(G) * e_K. \end{aligned}$$

Recall that $e_K = \frac{1}{\mu_G^\ell(K)} \cdot \mathbb{1}_K$ (normalized indicator function). So $\mathscr{H}(G,K)$ is an algebra with unit e_K.

How are these used to study representations?

Remark. If $\rho: G \to \mathrm{GL}(V)$ is any smooth representation, then $V^K = e_K * V$ becomes a unital $\mathscr{H}(G,K)$-module. ◇

Remark. So $\mathscr{H}(G) = \varinjlim \mathscr{H}(G,K)$, where the limit ranges over compact open subgroups K as K gets smaller and smaller. ◇

Proposition 10.2. *We will state without proof some key facts about Hecke algebras.*

1. *If $V \in \mathscr{R}(G)$ then V is irreducible if and only if for any compact open subgroup $K \subseteq G$, either $V^K = 0$ or V^K is a simple module over $\mathscr{H}(G,K)$.*
2. *Fix a compact open subgroup $K \subseteq G$. Then $V \mapsto V^K$ is a bijection between isomorphism classes of irreducible representations V of G for which $V^K \neq 0$ and simple modules over $\mathscr{H}(G,K)$.*

Question: When is $V^K = 0$?

10.3 Examples

10.3.1 Satake Isomorphism

Take $G = \mathrm{GL}_n(F)$ and take $K = \mathrm{GL}_n(\mathscr{O}_F)$. The Cartan decomposition says that $K\backslash G/K$ has the following representatives: $\mathrm{diag}(\pi^{a_1}, \dots, \pi^{a_n})$ (where π is a chosen uniformizer and $a_1 \geq \cdots \geq a_n$ are integers).

(This next thing is very pretty!)

Corollary 10.3. *$\mathscr{H}(G,K)$ is commutative.*

The above is called the *spherical Hecke algebra*.

Proof. On $\mathscr{H}(G)$, the map

$$\varphi\colon \mathscr{H}(G) \to \mathscr{H}(G), \quad f(g) \mapsto f(g^{\mathrm{t}})$$

is an anti-involution:

$$\varphi(f_1 * f_2) = \varphi(f_2) * \varphi(f_1).$$

It is clear that φ preserves $\mathscr{H}(G,K)$ and $\varphi|_{\mathscr{H}(G,K)} = \mathrm{Id}$ since diagonal matrices are invariant under transposition. The result follows. □

Corollary 10.4. *Let $V \in \mathscr{R}(G)$ be irreducible. It follows from the above that* $\dim V^{\mathrm{GL}_n(\mathscr{O}_F)} \leq 1$.

Now we can talk about the Satake isomorphism. Consider the polynomial algebra $\mathbb{C}[x_1, \ldots, x_n][x_n^{-1}]$. The claim is that there is an isomorphism

$$\mathbb{C}[x_1, \ldots, x_n][x_n^{-1}] \mapsto \mathscr{H}(G, K),$$

where $G = \mathrm{GL}_n(F)$ and $K = \mathrm{GL}_n(\mathscr{O}_F)$. This map is defined by mapping x_i to the indicator function of the double coset of $\mathrm{diag}(\pi, \ldots, \pi, 1, \ldots, 1)$ where there are j π's and $(n - j)$ 1's.

In the case that $n = 1$, we have

$$\mathscr{H}(F^\times, \mathscr{O}_F^\times) = \mathbb{C}[F^\times / \mathscr{O}_F^\times] = \mathbb{C}[\mathbb{Z}] = \mathbb{C}[x, x^{-1}].$$

Remark. There are analogues of the Satake isomorphism for when you have a different reductive group in place of $\mathrm{GL}_n(F)$.

QUESTION FROM JOHN: Is there a natural grading on $\mathscr{H}(G, K)$ in the Satake isomorphism without knowing the isomorphism? Look at the differences in the power of the uniformizer on the diagonal. Note that f_a is the indicator function of $K \cdot \mathrm{diag}(\cdots) \cdot K$. $f_a * f_b = f_{a+b}$ + "lower order terms"

10.3.2 The Iwahori-Hecke Algebra

Consider

$$I = \{g \in \mathrm{GL}_n(\mathscr{O}_F) : g \text{ is upper-triangular modulo } \pi\}.$$

This is called the *standard Iwahori subgroup*. The Iwahori-Hecke algebra is $\mathscr{H}(G, I)$. It is very well understood and there is a very explicit presentation of $\mathscr{H}(G, I)$ and the simple representations over this algebra are classified. For a reference, see Chriss-Ginsburg.

11 3 October 2012: Supercuspidal Representations

Recall the definition. Let $G = \mathrm{GL}_n(F)$. A smooth irreducible representation ρ of G is called *supercuspidal* if $J_{U_P}(\rho) = 0$, where P is any (standard) parabolic subgroup $P \subseteq G$ and U_P is the unipotent radical of P.

We have a nontrivial theorem.

Theorem 5. *(For general G.) ρ is supercuspidal if and only if every matrix coefficient of ρ has compact support modulo $Z = \mathrm{Z}(G)$.*

Remark. Let $\rho\colon G \to \mathrm{GL}(V)$ be any smooth representation. Choose $v \in V$, $v^\vee \in V^\vee$ (where $V^\vee$ is the set of linear functionals on V whose stabilizers are...) and form

$$f_{v,v^\vee}\colon G \to \mathbb{C}, \quad g \mapsto \langle v^\vee \mid \rho(g)(v) \rangle.$$

These are the matrix coefficients of ρ. If ρ is irreducible and G is second-countable, then by Schur's lemma, the center has to act by scalars (i.e. $\rho(Z)$ consists of scalars). Thus it makes sense to consider $\mathrm{supp}(f \cdot v, v^*)/Z$.

We will not prove or justify this theorem and just take it as a black box. From now, we will take the above theorem's description of supercuspidals to be the definition.

Let G be any ℓ-group. Let $K \subseteq G$ be any compact open subgroup. Let ρ be any smooth irreducible representation of K. Obstruction: the center of G.

How to deal with $Z = \mathrm{Z}(G)$. By Schur's lemma, $Z \cap K$ acts on ρ by scalars. Extend this action to Z and call the resulting homomorphism $\theta\colon Z \to \mathbb{C}^\times$. (Note that any such extension is smooth.) We obtain a representation ρ_θ of $Z \cdot K$ where K acts via ρ and Z acts via θ.

Define

$$R(\rho, \theta) := c\text{-}\mathrm{Ind}_{ZK}^{G}(\rho_\theta).$$

We have a Mackey irreducibility criterion:

(M) If $g \in G \setminus ZK$, then

$$\mathrm{Hom}_{K \cap gKg^{-1}}(\rho|_{K \cap gKg^{-1}}, \rho^g|_{K \cap gKg^{-1}}) = 0,$$

where $\rho^g(\gamma) = \rho(g^{-1}\gamma g)$.

Now we state the main theorem for this week.

Theorem 6. *(a) Assume G is second-countable. Then $R(\rho, \theta)$ is irreducible if and only if (M) holds.*

(b) *Assume in addition that every smooth irreducible representation of G is admissible. If (M) is satisfied, then* $c\text{-}\operatorname{Ind}_{ZK}^{G}(\rho_\theta) = \operatorname{Ind}_{ZK}^{G}(\rho_\theta)$ *and* $R(\rho,\theta)$ *is supercuspidal.*

Proof of (a). We will prove the equivalence between the following conditions:

(i) $R(\rho,\theta)$ is irreducible

(ii) $\dim_{\mathbb{C}} \operatorname{End}_G(R(\rho,\theta)) = 1$

(iii) ρ satisfies (M)

We will show $(ii) \Rightarrow (iii) \Rightarrow (i)$.

Remark. Recall that this is quite different form the principle series representations. Indeed, in the principle series representations, (ii) is always true, but (i) is not. ◇

The idea of the proof is the following. Let $H \subseteq G$ be any open subgroup and choose $\pi \in \mathscr{R}(H)$. Suppose W is a nonzero G-subrepresentation of $c\text{-}\operatorname{Ind}_H^G(\pi)$. Then we have a nonzero map $W \to \operatorname{Ind}_H^G(\pi)$, i.e. $\operatorname{Hom}_G(W, \operatorname{Ind}_H^G(\pi)) \neq 0$. We can use injunction from here: That is,

$$\operatorname{Hom}_G(W, \operatorname{Ind}_H^G(\pi)) \cong \operatorname{Hom}_H(W|_H, \pi) \cong \operatorname{Hom}_H(\pi, W|_H),$$

where the last isomorphism holds since $W|_H$ and π are semisimple and finite-dimensional as representations of H. (What we actually care about is the one is nonzero if and only if the other is nonzero; the isomorphism is actually stronger than what we want.)

Remark. If $H = ZK$ and $\pi = \rho_\theta$ as above, then Z acts on $c\text{-}\operatorname{Ind}_{ZK}^G(\rho_\theta)$ by the scalar θ. So ZK-subrepresentations of $W|_{ZK}$ are the same as K-subrepresentations. But K is compact! ◇

We need to understand $c\text{-}\operatorname{Ind}_H^G(\pi)|_H$. Because H is open in G,

$$\begin{aligned} c\text{-}\operatorname{Ind}_H^G(\pi) &= \{f\colon G \to V : f(hg) = \pi(h)(f(g)),\ f \text{ is supported on a finite union of right cosets of } H\} \\ &= \bigoplus_{C \in H\backslash G/H} \{f\colon C \to V : f(hx) = \pi(h)(f(x)) \text{ for all } h \in H, x \in C, \text{ and support condition}\}. \end{aligned}$$

The last isomorphism is an isomorphism of H-representations. On the right-hand side, C ranges over double cosets HgH, H acts by right translation. We will call the summand on the right-hand side W_C.

Choose a double coset $C = HgH$. We get an injection $W_C \hookrightarrow \{\text{functions } H \to V\}$ via the map $f \mapsto (h \mapsto f(gh))$.

EXERCISE. This is an isomorphism of H-representations $W_C \to c\text{-}\operatorname{Ind}_{H\cap gHg^{-1}}^{H}(\pi^g)$, where $\pi^g\colon gHg^{-1} \to \operatorname{GL}(V)$ is given by $\gamma \mapsto \pi(g^{-1}\gamma g)$.

Proof. □

The upshot of all of this is the following:

$$\operatorname{Res}^G_H(c\text{-}\operatorname{Ind}^G_H(\pi)) \cong \bigoplus_{g\in S}(c\text{-}\operatorname{Ind}^H_{H\cap gHg^{-1}}(\pi^g)),$$

where S is a set of representatives of the double cosets of H in G.

We have essentially proved that the stated three conditions are equivalent. □

Corollary 11.1. *If π is irreducible (or finitely generated),*

$$\operatorname{Hom}_H(\pi, c\text{-}\operatorname{Ind}^G_H(\pi)) \cong \bigoplus_{g\in S}\operatorname{Hom}_H(\pi, c\text{-}\operatorname{Ind}^H_{H\cap gHg^{-1}}(\pi^g)).$$

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Recall:

(M) If $g \in G$, $g \notin ZK$, then $\mathrm{Hom}_{K\cap gKg^{-1}}(\rho|_{K\cap gKg^{-1}}, \rho^g|_{K\cap gKg^{-1}}) = 0$. The following are equivalent:

(i) $\mathscr{R}(\rho,\theta) = c\text{-}\mathrm{Ind}_{ZK}^G(\rho_\theta)$ is irreducible

(ii) $\dim_{\mathbb{C}} \mathrm{End}_G(\mathscr{R}(\rho,\theta)) = 1$

(iii) ρ has property (M)

Recall the set-up. Let G be any ℓ-group and let $H \subseteq G$ be an open subgroup. Consider $\pi \in \mathscr{R}(H)$ finitely generated. Then

$$c\text{-}\mathrm{Ind}_H^G(\pi)|_H = \bigoplus_{g\in S} c\text{-}\mathrm{Ind}_{H\cap gHg^{-1}}^H(\pi^g),$$

where S is a set of representatives of double cosets of H in G. Assume that $H \cap gHg^{-1}$ has finite index in H for all $g \in G$.

Remark. If $H = ZK$ where $K \subseteq G$ is a compact open subgroup, then this does hold. This assumption implies that

$$\mathrm{Hom}_H(\pi, c\text{-}\mathrm{Ind}_H^G(\pi)) \cong \bigoplus_{g\in S} \mathrm{Hom}_{H\cap gHg^{-1}}(\pi|_{H\cap gHg^{-1}}, \pi^g|_{H\cap gHg^{-1}}).$$

The right-hand side contains $\mathrm{End}_H(\pi)$.

Corollary 12.1. *Assume that π is irreducible and G is second countable. Then* $\mathrm{End}_G(c\text{-}\mathrm{Ind}_H^G(\pi))$ *is one-dimensional if and only if π satisfies the standard Mackey criterion: for all $g \in G \setminus H$,* $\mathrm{Hom}_{H\cap gHg^{-1}}(\pi, \pi^g) = 0$.

Now we specialize to the case when $H = ZK$, $\pi = \rho_\theta$, where $\rho \in \mathscr{R}(K)$ is irreducible, ρ_θ is an extension of ρ to ZK.

In this case, (M) for ρ is equivalent to the standard Mackey criterion for ρ_θ. So (ii) and (iii) in (M) are equivalent.

Proof of (ii) and (iii) implies (i). Let $W \subseteq \mathscr{R}(\rho,\theta)$ be a nonzero G-subrepresentation. Then $\mathrm{Hom}_G(W, \mathrm{Ind}_H^G(\rho_\theta)) \neq 0$ so $\mathrm{Hom}_{ZK}(W, \rho_\theta) \neq 0$ so $\mathrm{Hom}_{ZK}(\rho_\theta, W) \neq 0$.

On the other hand, (ii) implies that $\mathrm{Hom}_{ZK}(\rho_\theta, \mathscr{R}(\rho,\theta))$ is one-dimensional. So there exists a unique ZK-stable subspace $W_0 \subseteq \mathscr{R}(\rho,\theta)$ that affords the representation ρ_θ. So $W_0 \subseteq W$.

EXERCISE. W_0 generates $\mathscr{R}(\rho,\theta)$ as a G-representation.

Proof. □

Hence $W = \mathscr{R}(\rho,\theta)$.

Theorem 7 (b)**.** *Same assumptions and notation as in (a). Also, assume that every smooth irreducible representation of G is admissible and (M) holds for K and ρ. Then* $c\text{-}\mathrm{Ind}_K^G(\rho) = \mathrm{Ind}_K^G(\rho)$ *and is supercuspidal.*

Proof. Note that $\rho^\vee$ also satisfies (M). Hence $c\text{-}\mathrm{Ind}_Z K^G(\rho_\theta^\vee)$ is irreducible by (a) and hence admissible. (Remember that admissibility is exactly the condition we need to make duality work.) Therefore $c\text{-}\mathrm{Ind}_{ZK}^G(\rho_\theta^\vee)^\vee$ is irreducible.

Now, $c\text{-}\mathrm{Ind}_{ZK}^G(\rho_\theta^\vee)^\vee \cong c\text{-}\mathrm{Ind}_{ZK}^G(\rho_\theta)$ because $\delta_{ZK} = 1$ and $\delta_G|_{ZK} = 1$. (The latter follows from the properties of δ_G that we proved earlier.) So

$$c\text{-}\mathrm{Ind}_{ZK}^G(\rho_\theta) = \mathrm{Ind}_{ZK}^G(\rho_\theta).$$

(Note that $\rho_\theta^\vee$ means $(\rho_\theta)^\vee$.)

Now for the proof of supercuspidality.

STEP 1. $R(\rho,\theta)$ has one nonzero matrix coefficient whose support equals ZK. Take a nonzero $v \in V$ and nonzero $v^\vee \in V^*$. Define $f\colon G \to V$ to be

$$f(g) = \begin{cases} 0 & g \notin ZK \\ \rho_\theta(g)(v) & g \in ZK. \end{cases}$$

Then $f \in c\text{-}\mathrm{Ind}_{ZK}^G(\rho_\theta)$. Similarly, $v^\vee$ gives an element $f^\vee \in c\text{-}\mathrm{Ind}_{ZK}^G(\rho_\theta^\vee) = (c\text{-}\mathrm{Ind}_{ZK}^G(\rho_\theta))^\vee$. Then $g \mapsto f^\vee(g\cdot f)$ is a nonzero matrix coefficient of $c\text{-}\mathrm{Ind}_{ZK}^G(\rho_\theta)$ with support contained in ZK.

STEP 2. Let G be any ℓ-group and let ρ be any admissible irreducible smooth representation W of G.

CLAIM. If W has a nonzero matrix coefficient with compact support modulo $Z(G)$, then all matrix coefficients of W have compact support modulo $Z(G)$.

The proof of this is actually quite simple. We have a map

$$W \otimes_C W^\vee \to C^\infty(G) \supseteq X = \{\text{functions with compact support modulo } Z\}$$

defined by matrix coefficients. View all of these as representations of $G \times G$. It turns out that this is a map of $G \times G$ representations. But $W \otimes_{\mathbb{C}} W^\vee$ is irreducible as a $(G \times G)$-representation, and so this completes the proof. □

IN GENERAL: Let G_1 and G_2 be ℓ-groups and let W_i be an admissible smooth irreducible representation of G_i. Then $W_1 \otimes_{\mathbb{C}} W_2$ is irreducible as a representation of $G_1 \times G_2$.

Remark. If $K_i \subseteq G_i$ is a compact open subgroup, then $\mathscr{H}(G_1 \times G_2, K_1 \times K_2) = \mathscr{H}(G_1, K_1) \otimes \mathscr{H}(G_2, K_2)$.

Counterexample to the remark when.......

Consider the semi direct product $M = F^\times \ltimes F$ and let $W = \ker(\int : C_c^\infty(F) \to \mathbb{C})$. Then consider $W \otimes W$ as a representation of $M \times M$. There is a good chance that this is not irreducible.

13 8 October 2012: Supercuspidal Representations of $\mathrm{GL}_n(F)$

13.1 Invariant Approach

Let W be an n-dimensional vector space over F and let $G = \mathrm{Aut}_F(W) \cong \mathrm{GL}_n(F)$. Fix a lattice (i.e. an $\mathscr{O}_F$-submodule with $F \otimes_{\mathscr{O}_F} \Lambda \to W$ an isomorphism) $\Lambda \subseteq W$. Let $K := \mathrm{Aut}_{\mathscr{O}_F}(\Lambda) \cong \mathrm{GL}_n(\mathscr{O}_F)$ be a maximal compact subgroup of $\mathrm{GL}_n(F)$.

Fix a uniformiser $\pi \in \mathscr{O}_F$. Then $\mathscr{O}_F/(\pi) = \mathbb{F}_q$. Look at the quotient $\Lambda/\pi\Lambda$. This is an n-dimensional vector space over $\mathbb{F}_q$. Then $\mathrm{Aut}_{\mathbb{F}_q}(\Lambda/\pi\Lambda) = \mathrm{GL}_n(\mathbb{F}_q)$.

We have a natural surjection $K \to \mathrm{GL}_n(\mathbb{F}_q)$. Choose a cuspidal irreducible representation ρ of $\mathrm{GL}_n(\mathbb{F}_q)$, inflate it to a representation $\widetilde{\rho}$ to K and then extend $\widetilde{\rho}$ to a representation $\widetilde{\rho}_\theta$ or ZK. Note that $Z = F^\times \cdot \mathrm{Id}_W \subseteq G$ and $Z \cap K = \mathscr{O}_F^\times \cdots \mathrm{Id}_W$. So $Z = (Z \cap K) \times (\pi^{\mathbb{Z}} \cdot \mathrm{Id}_W)$. (So picking θ is the same as picking a choice of $\theta(\pi) \in \mathbb{C}^\times$ and this determines θ completely.)

Claim 9. If $g \in G$, $g \notin Z \cdot K$, then $\mathrm{Hom}_{K \cap gKg^{-1}}(\widetilde{\rho}|_{\dots}, \widetilde{\rho}^g|_{\dots}) = 0$. Hence

$$c\text{-}\mathrm{Ind}_{ZK}^G(\widetilde{\rho}_\theta) = \mathrm{Ind}_{ZK}^G(\widetilde{\rho})$$

is an irreducible supercuspidal representation of G.

Remark. This has depth 0, and every depth 0 supercuspidal for $\mathrm{GL}_n(F)$ arises in this way.

Recall that a representation $\rho\colon \mathrm{GL}_n(\mathbb{F}_q) \to \mathrm{GL}(V)$ is called *cuspidal* if, for any proper parabolic subgroup $P \subseteq \mathrm{GL}_n(\mathbb{F}_q)$, we have $V^{\rho(U_P)} = 0$, where U_P is the unipotent radical of P.

We identify $\mathrm{GL}_n(\mathbb{F}_q) \cong \mathrm{Aut}_{\mathbb{F}_q}(\overline{\Lambda})$ where $\overline{\Lambda} = \Lambda/\pi\Lambda$. Choose any flag

$$0 \neq A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq A_n \subseteq \overline{\Lambda}.$$

The corresponding parabolic subgroup is

$$P = \{g \in \mathrm{Aut}_{\mathbb{F}_q}(\overline{\Lambda}) : g(A_i) = A_i \text{ for all } i\}$$

and

$$U_P = \{g \in P : g \text{ acts trivially on } A_i/A_{i-1} \text{ for all } i\}.$$

Proof of Claim. To prove the claim, it suffices to look at representatives of ZK-double cosets in G.

QUESTION: What is G/K?

Remark. $G = \mathrm{Aut}_F(W)$ acts transitively on the space of all $\mathcal{O}_F$-lattices in W. We may identify the space of all $\mathcal{O}_F$-lattices with G/K. This is the *affine Grassmanian for* $\mathrm{GL}_n(F)$. Hence $K\backslash G/K$ can be identified with K-orbits on the affine Grassmanian G/K (this is the correct approach for proving the Cartan decomposition). Similarly, G/ZK can be identified with the space of homothety classes of $\mathcal{O}_F$-lattices in W. Then $ZK\backslash G/ZK$ is the set of K-orbits in G/ZK.

Now assume that $g \notin ZK$. Without loss of generality, $g(\Lambda) \subseteq \Lambda$ (can multiply) and also that $g(\Lambda) \nsubseteq \pi\Lambda$. Then $g(\Lambda) \neq \Lambda$, so by Nakayama's lemma, $g(\Lambda) + \pi\Lambda \neq \Lambda$. Hence the image of $g(\Lambda)$ in $\overline{\Lambda} = \Lambda/\pi\Lambda$ is a proper $\mathbb{F}_q$-subspace $0 \neq A \subsetneq \overline{\Lambda}$. Define P to be the stabilizer of A in $\mathrm{Aut}_{\mathbb{F}_q}(\overline{\Lambda})$.

First note that $K \cap gKg^{-1}$ is the stabilizer of $g(\Lambda)$ in K. The idea is that we would like to find a subgroup $H \subseteq K \cap gKg^{-1}$ with two properties:

(1) Under the surjection $r\colon K \to \mathrm{Aut}_{\mathbb{F}_q}(\overline{\Lambda})$, then H maps onto the space $U_P = \{\gamma \in \mathrm{Aut}_{\mathbb{F}_q}(\overline{\Lambda}) : \gamma$ acts trivially on A and on $\overline{\Lambda}/A\}$.

(2) $g^{-1}Hg \subseteq \ker(r) = \{\gamma \in K : \gamma$ acts trivially on $\Lambda/\pi\Lambda\}$.

Recall that $\widetilde{\rho}$ is the composition of $r\colon K \to \mathrm{Aut}_{\mathbb{F}_q}(\overline{\Lambda})$ with $\rho\colon \mathrm{Aut}_{\mathbb{F}_q}(\overline{\Lambda}) \to \mathrm{GL}(V)$. If H is a subgroup that satisfies (1) and (2), then

$$V^{\widetilde{\rho}(H)} = V^{\rho(U_P)} = 0$$

and on the other hand, $\widetilde{\rho}^g|_H$ is trivial. So finding a subgroup H that satisfies (1) and (2) is really the main idea of this proof.

Let us take H to be the largest possible subgroup satisfying (2). Introduce $U^1 = \ker(r) \cong 1 + \pi \cdot \mathrm{Mat}_n(\mathcal{O}_F) \subseteq \mathrm{GL}_n(\mathcal{O}_F)$. Take $H = gU^1g^{-1} \cap K \cap r^{-1}(U_P)$. Then $H \subseteq K \cap gKg^{-1}$ and (2) holds automatically. Hence we need only check that $r|_H\colon H \to U_P$ is surjective.

It is clear that $r(gU^1g^{-1} \cap K) \subseteq r(gKg^{-1} \cap K) = P$. So it suffices to show that the restriction $U_P \subseteq r|_{gU^1g^{-1}\cap K}$. Now,

$$gU^1g^{-1} \cap K = \{\gamma \in G : \gamma(\Lambda) = \Lambda,\ \gamma(g(\Lambda)) = g(\Lambda),\ \gamma \text{ acts trivially on } g(\Lambda)/\pi g(\Lambda)\}.$$

We postpone the check that $U_P \subseteq r(gU^1g^{-1} \cap K)$ for next time. □

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14.1 Leftovers

We first complete the proof from the proposition last time.

Recall from last time that we had Λ a free $\mathcal{O}_F$-module of rank n and $\mathcal{O}_F$-module of rank n and $\Lambda' \subsetneq \pi \cdot \Lambda$. $g(\Lambda) = \Lambda' \subsetneq \Lambda$. We have a reduction homomorphism $\mathrm{GL}_n(\mathcal{O}_F) \cong K = \mathrm{Aut}_{\mathcal{O}_F}(\Lambda) \to \mathrm{Aut}_{\mathbb{F}_q}(\overline{\Lambda})$. We have $\overline{\Lambda} = \Lambda/\pi\Lambda$ a n-dimensional vector space over $\mathbb{F}_q = \mathcal{O}_F/(\pi)$.

Let A be the image of λ' in $\overline{\Lambda}$. We have $0 \neq A \subsetneq \overline{\Lambda}$.

Claim 10. If $\varphi \in \mathrm{Aut}_{\mathbb{F}_q}(\overline{\Lambda})$, $\varphi(A) = A$, and φ acts trivially on A and on $\overline{\Lambda}/A$, then there exists a $\gamma \in K$ with $r(\gamma) = \varphi$, $\gamma(\Lambda') = \Lambda'$ and γ acts trivially on $\Lambda'/\pi\Lambda'$.

Proof. Find an $\mathcal{O}_F$-basis $e_1, \ldots, e_n$ of Λ such that $e_1, \ldots, e_k \in \Lambda'$ and $(\overline{e_1}, \ldots, \overline{e_k})$ is a basis for A over $\mathbb{F}_q$. Define $\gamma \in K = \mathrm{Aut}_{\mathcal{O}_F}(\Lambda)$ by $\gamma(e_i) = e_i$ for $1 \leq i \leq k$ and for $k+1 \leq j \leq n$, $\gamma(e_j) = e_j$+(suitable $\mathcal{O}_F$-linear combination of $e_1, \ldots, e_k$) to ensure that $\overline{\gamma(e_j) - e_j} = \varphi(\overline{e_j}) - \overline{e_j}$.

We need to check the following. If $w \in \Lambda'$, then $\gamma(w) - w \in \pi \cdot \Lambda'$. (If we can show, this, then γ has all the required properties. Indeed, this gives $\gamma(\Lambda') \subseteq \Lambda'$ and then the result on the quotient gives the desired result by Nakayama's lemma.) This is straightforward. For $w \in \Lambda'$, there exist $a_1, \ldots, a_k \in \mathcal{O}_F$ such that $w - a_1 e_1 - \cdots - a_k e_k \in \pi\Lambda$ since the reductions form a basis for A. Then there exist $b_1, \ldots, b_n \in \mathcal{O}_F$ such that $w = b_1 e_1 + \cdots + b_n e_n$ and $b_{k+1}, \ldots, b_n \in \pi\mathcal{O}_F$. Hence $\gamma(w) - w = \sum_{i=k+1}^n b_i(\gamma(e_i) - e_i)$. In this summand, $b_i \in \pi\mathcal{O}_F$ and $\gamma(e_i) - e_i \in \Lambda'$ and hence $\gamma(w) - w \in \pi\Lambda'$. □

14.2 Summary

From an irreducible cuspidal representation of $\mathrm{GL}_n(\mathbb{F}_q)$, we produced a family of supercuspidal irreducible representations of $\mathrm{GL}_n(F)$, parametrized by $\mathbb{C}^\times$. Let's return to the story of irreducible cuspidal representations of $\mathrm{GL}_n(\mathbb{F}_q)$.

14.3 Construction of Irreducible Cuspidal Representations of $\mathrm{GL}_n(\mathbb{F}_q)$

We begin by describing a generalization of the Drinfeld curve. Consider the affine subvariety $X \subseteq \mathbb{A}^n_{\mathbb{F}_q}$ defined by

$$\prod_{(a_1,\ldots,a_n) \in \mathbb{F}_q{}^n \setminus \{0\}} (a_1 x_1 + \cdots a_n x_n) = 1.$$

On X, we have commuting actions of $\mathrm{GL}_n(\mathbb{F}_q)$ and $\mathbb{F}_{q^n}^\times = \{\lambda \in \overline{\mathbb{F}_q} : \lambda^{q^n-1} = 1\}$ (acting by dilations).

Let ℓ be a prime different from $\mathrm{char}(\mathbb{F}_q)$.

Fact 14.1. *Given $\chi\colon \mathbb{F}_{q^n}^\times \to \overline{\mathbb{Q}}_\ell^\times$ with trivial stabilizer in $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$. Then $H_c^{n-1}(X, \overline{\mathbb{Q}}_\ell)[\chi]$ is an irreducible cuspidal representation of $\mathrm{GL}_n(\mathbb{F}_q)$. Moreover, every irreducible cuspidal representation arises in this way.*

Remark. $H_c^i \cong (H^{2d-i})^\times$.

There is a (more complicated) analogue of this for $\mathrm{GL}_n(F)$.

14.4 Lubin-Tate Tower

The Lubin-Tate Tower gives a geometric realization of supercuspidals for $\mathrm{GL}_n(F)$. (The Drinfeld curve was discovered as a corollary of this.) The plan for the rest of the semester is the following.

1. Vague comments about the Local Langlands Correspondence
2. Formal groups and modules
3. Construct the Lubin-Tate tower
    - non-Arthimedean analogues of analytic geometry
4. Relationship with the Drinfeld curve X
5. Complements

To motivate the local Langlands conjecture, we will repackage our discussion of depth 0 supercuspidal representations of $\mathrm{GL}_n(F)$ so that it looks more similar to the aforementioned construction of the cuspidal representations of $\mathrm{GL}_n(\mathbb{F}_q)$.

- $\chi\colon \mathbb{F}_{q^n}^\times \to \overline{\mathbb{Q}}_\ell^\times$ with trivial $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$-stabilizer.
- This gives rise to ρ, the corresponding cuspidal representation of $\mathrm{GL}_n(\mathbb{F}_q)$.
- Then we can get $\mathrm{Ind}_{Z\cdot \mathrm{GL}_n(\mathscr{O}_F)}^{\mathrm{GL}_n(F)}(\rho_\theta)$, where choosing θ is the same as choosing $\theta(\pi\cdot I_n) \in \overline{\mathbb{Q}}_\ell^\times$.

- Then we combine θ and χ in the following way.

- $F_n \supseteq F$ is the unique unramified degree n extension. Let $\mathcal{O}_{F_n}$ be the integral closure of $\mathcal{O}_F$ in F_n. Unramified means that π also generates the maximal ideal of $\mathcal{O}_{F_n}$. Then we get a picture [insert picture]. (Note that we can obtain F_n by adjoining $(q^n - 1)$th roots of 1. We have $F_n^\times = \mathcal{O}_{F_n}^\times \times \pi^{\mathbb{Z}}$. We have a surjection $\mathcal{O}_{F_n}^\times \twoheadrightarrow \mathbb{F}_{q^n}^\times \xrightarrow{\chi} \overline{\mathbb{Q}}_\ell^\times$. Then we get, combining χ with θ, $F_n^\times \to \overline{\mathbb{Q}}_\ell^\times$ and this has trivial stabilizer in the Galois group.

More generally, for any separable degree-n extension $E \supseteq F$ with smooth character $\nu \colon E^\times \to \mathbb{Q}_\ell^\times$ satisfying some property, then we may construct a supercuspidal irreducible representation of $\mathrm{GL}_n(F)$ from the pair (E, ν).

Some remarks:

- Nobody knows how to do this geometrically. Lubin-Tate doesn't directly answer this question.

- Also, this doesn't give all possible supercuspidal irreducible representations. Perhaps the most interesting supercuspidal irreducible representations are the ones that do not come from this construction!

Weil group and Local Langlands next time!

15 12 October 2012: What is the Local Langlands Correspondence About?

From our perspective, we can think of it as a "classification" of smooth irreducible representations of $\mathrm{GL}_n(F)$. Very roughly speaking, these things correspond to n-dimensional representations of the Galois group $\mathrm{Gal}(\overline{F}/F) =: G_F$.

15.1 The Weil Group of F

This is a subgroup $\mathscr{W}_F \subseteq G_F$. Consider the finite unramified extensions of F. For all $d \geq 1$, there exists a unique such extension $F_d \supseteq F$ of degree d and we have isomorphisms $\mathrm{Gal}(F_d/F) \to \mathrm{Gal}(\mathscr{O}_{F_d}/\mathscr{O}_F) \to \mathrm{Gal}(\mathbb{F}_{q^d}/\mathbb{F}_q)$. We have a surjection $G_F \to \mathrm{Gal}(F_d/F)$. Put these together and we get a surjection

$$G_F \twoheadrightarrow \varprojlim_d \mathrm{Gal}(\mathbb{F}_{q^d}/\mathbb{F}_q) \cong \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q) \cong \widehat{\mathbb{Z}}.$$

We have a short exact sequence

$$1 \to \mathscr{I}_F \to G_F \to \widehat{\mathbb{Z}} \to 0,$$

where $\mathscr{I}_F$ is the inertia group. We take the Weil group $\mathscr{W}_F$ to be the preimage of $\mathbb{Z} \subseteq \widehat{\mathbb{Z}}$. The topology on $\mathscr{W}_F$ is such that $\mathscr{I}_F$ is open in $\mathscr{W}_F$ and the induced topology on $\mathscr{I}_F$ is the usual one induced by the Galois group G_F. In particular, $\mathscr{W}_F$ is an ℓ-group.

15.2 Structure of the Inertia Group $\mathscr{I}_F$

This fits into an exact sequence

$$1 \to \mathscr{I}_F^{\mathrm{wild}} \to \mathscr{I}_F \to \prod_{\ell \neq p} \mathbb{Z}_\ell^\times \to 1.$$

$\mathscr{I}_F^{\mathrm{wild}}$ is a pro-p group.

We have $F \supseteq \mathscr{O}_F$ and we may choose a uniformiser $\pi \in \mathscr{O}_F$. Let $\ell \neq p$ be a prime. Let $m \geq 1$ be an integer. Then

$$f(x) = x^{\ell^m} - \pi$$

is an Eisenstein polynomial over $\mathscr{O}_F$. So it is irreducible and $E := F[x]/(f(x))$ is a field extension of F of degree ℓ^m and x gives a uniformizer in E. So $E \supseteq F$ is totally ramified.

BACK UP FOR A SECOND. SOMETHING DODGY HERE. E doesn't have to be Galois over F. So we'll amend our set-up as follows. $\mathscr{I}_F = \ker(G_F \twoheadrightarrow \widehat{\mathbb{Z}}) = \mathrm{Gal}(\overline{F}/F^{\mathrm{ur}})$ where $F^{\mathrm{ur}} = (\overline{F})^{\mathscr{I}_F} = \cup_{d \geq 1} F_d$. This is the extension of F generated by all roots of unity whose order is prime to p. Then

$$\mathscr{O}_{F^{\mathrm{ur}}} = \bigcup_{d \geq 1} \mathscr{O}_{F_d}$$

is still a DVR with uniformizer π. Then let us take $E := F^{\mathrm{ur}}[x]/(f(x))$. Then E is a degree-ℓ^m Galois extension of F^{ur}.

So we get

$$\mathscr{I}_F = \mathrm{Gal}(\overline{F}/F^{\mathrm{ur}}) \twoheadrightarrow \mathrm{Gal}(E/F^{\mathrm{ur}}) \cong \mu_{\ell^m}(F^{\mathrm{ur}}),$$

where $\mu_{\ell^m}(F^{\mathrm{ur}}) := \{\lambda \in F^{\mathrm{ur}} : \lambda^{\ell^m} = 1\}$. Varying ℓ and m, we obtain

$$\mathscr{I}_F \twoheadrightarrow \prod_{\ell \neq p} \varprojlim_m \mu_{\ell^m}(F^{\mathrm{ur}}) \cong \prod_{\ell \neq p} \mathbb{Z}_\ell^\times,$$

where the first surjection is canonical and $\varprojlim_m \mu_{\ell^m}(F^{\mathrm{ur}}) = \mathbb{Z}_\ell^\times(1)$, but the second isomorphism is non-canonical.

Fact 15.1. *We have* $\mathscr{I}_F^{wild} = \ker(\varphi \colon \mathscr{I}_F \twoheadrightarrow \prod_{\ell \neq p} \mathbb{Z}_\ell^\times)$ *is a pro-*p *group. This kernel is quite difficult to understand.*

15.3 Upshot of Local Class Field Theory

One way to formulate local class field theory is that it gives a "canonical" isomorphism (of topological groups) between the abelianization of the Weil group $\mathscr{W}^{\mathrm{ab}} = \mathscr{W}_F/\overline{[\mathscr{W}_F, \mathscr{W}_F]} \to F^\times$ and $F^\times$. This is known as *local Artin reciprocity*. Later, when we talk about formal groups, we will mention one way to construct this isomorphism explicitly. An alternative way to formulate this is to say that we have a bijection between the one-dimensional smooth irreducible representations of the two sides of the aforementioned isomorphism.

15.4 The Local Langlands Correspondence

We would like to generalize local class field theory. The LHS is replaced by the collection of isomorphism classes of smooth irreducible representations of $\mathrm{GL}_n(F)$ over $\overline{\mathbb{Q}_\ell}$, and the RHS is replaced with n-dimensional continuous representations of $\mathscr{W}_F$ over $\overline{\mathbb{Q}_\ell}$ that are *Frobenius-semisimple*. *Frobenius-semisimple* means that for all $g \in \mathscr{W}_F$ with $g \mapsto \pm 1$ under

$\mathscr{W}_F \to \mathbb{Z}$, the action of on the representation is diagonalizable. This is very important! This definition knows about the topology of $\overline{\mathbb{Q}_\ell}$.

A different way to look at this is the following. Consider the set of n-dimensional Frobenius-semisimple Weil-Deligne representations of $\mathscr{W}_F$. This is a more algebraic way of viewing the RHS.

15.5 The Meaning of Continuity

Let V be a finite-dimensional $\overline{\mathbb{Q}}_\ell$-vector space and consider an abstract representation $\rho\colon \mathscr{W}_F \to \mathrm{GL}(V)$. We say that ρ is *continuous* if and only if there exists a finite extension $E \supseteq \mathbb{Q}_\ell$ and a basis $(v_1, \ldots, v_n)$ of V such that $\rho(\mathscr{W}_F)$ preserves $\mathrm{span}_E(v_1, \ldots, v_n)$. The resulting homomorphism $\mathscr{W}_F \to \mathrm{GL}_n(E)$ is continuous.

15.6 Remarks and an "Example"

Remark. The irreducible representations of $\mathscr{W}_F$ correspond exactly to the supercuspidal representations of $\mathrm{GL}_n(F)$. ◇

Example 15.1. Say that for all $m \geq 1$ you have a smooth action of $\mathscr{W}_F$ on $(\mathbb{Z}/\ell^m\mathbb{Z})^n$ by group automorphisms. Suppose that they are compatible with the reduction maps between the m's. Then we get an action (that might not be smooth in general, but will be continuous) of $\mathscr{W}_F$ on the inverse limit, which is exactly $\mathbb{Z}_\ell^n$. So we get a continuous homomorphism $\mathscr{W} \to \mathrm{GL}_n(\mathbb{Z}_\ell) \hookrightarrow \mathrm{GL}_n(\mathbb{Q}_\ell) \hookrightarrow \mathrm{GL}_n(\overline{\mathbb{Q}_\ell})$.

This construction works if we replace $\mathbb{Q}_\ell$ with E and $\mathbb{Z}_\ell$ with $\mathscr{O}_E$. In this set-up, we replace the cyclic groups with $(\mathscr{O}_E/\mathfrak{m}_E^m)^n$.

(This is somehow a more natural way to view these representations. That is, we construct it form the action on finite things instead of an action on a vector space directly.)

16 17 October 2012: Weil-Deligne Representations

No class on Friday. No office hours tomorrow.

Set up:

- F a local field
- $p = \mathrm{char}(\mathscr{O}_F/(\pi))$
- $\ell \neq p$ is a prime
- $q = |\mathscr{O}_F/(\pi)|$
- $\mathscr{I}_F = \mathrm{Gal}(F^{\mathrm{sep}}/F^{\mathrm{nr}})$
- $F \subseteq F^{\mathrm{nr}} \subseteq F^{\mathrm{sep}}$, where F^{nr} is a maximal unramified extension and F^{sep} is the separable closure.

We have the following diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathscr{I}_F & \longrightarrow & \mathscr{W}_F & \xrightarrow{\nu} & \mathbb{Z} & \longrightarrow & 0 \\ & & \downarrow t_\ell & & & & & & \\ & & \mathbb{Z}_\ell & & & & & & \end{array}$$

Let $E \supseteq \mathbb{Q}_\ell$ be a finite extension and let $\rho\colon \mathscr{W}_F \to \mathrm{GL}_n(E)$ be a continuous homomorphism. We would like to understand how far this homomorphism is from being smooth; i.e. we want to understand how far the kernel is from being open. This naturally creates the notion of a Weil-Deligne representation.

Consider $K = 1 + \mathrm{Mat}_{n\times n}(P_E)$, where P_E is the maximal ideal of $\mathscr{O}_E$. This is an open subgroup of $\mathrm{GL}_n(\mathscr{O}_E)$ and hence an open subgroup of $\mathrm{GL}_n(E)$. From here, we have that

$$\rho^{-1}(K) \cap \mathscr{I}_F \subseteq \mathscr{I}_F$$

is an open subgroup.

Recall that $t_\ell\colon \mathscr{I}_F \twoheadrightarrow \mathbb{Z}_\ell$ is constructed using the canonical map $\mathscr{I}_F \twoheadrightarrow \mathrm{Gal}(F^{\mathrm{nr}}[\pi^{1/\ell^m}]/F^{\mathrm{nr}})$ with the non canonical isomorphism $\mathrm{Gal}(F^{\mathrm{nr}}[\pi^{1/\ell^m}]/F^{\mathrm{nr}}) \cong \mathbb{Z}/\ell^m\mathbb{Z}$. The key fact here is the following.

Fact 16.1. $\ker(t_\ell)$ *has no finite quotients whose order is a power of ℓ. (This is an exercise in Kummer theory.)*

Hence:

Main Observation. $\rho^{-1}(K) \cap \ker(t_\ell) \subseteq \ker(\rho)$.

This is the main observation that makes this whole thing work! This means that we have a commutative diagram

$$\begin{array}{ccc} \rho^{-1}(K) \cap \mathscr{I}_F & \xrightarrow{\quad\rho\quad} & \mathrm{GL}_n(E) \\ & {\scriptstyle t_\ell}\searrow \qquad \nearrow{\scriptstyle \overline{\rho}_\ell} & \\ & U_\ell = t_\ell(\rho^{-1}(K) \cap \mathscr{I}) \overset{\text{open}}{\subseteq} \mathbb{Z}_\ell & \end{array}$$

The map $\overline{\rho}_\ell$ is a continuous homomorphism.

Claim 11. There is a unique $N \in \mathrm{Mat}_n(E)$ such that $\overline{\rho}_\ell(x) = \exp(x \cdot N)$ for all $x \in U_\ell$ that are sufficiently close to 0.

Proof. If $m \gg 1$, then we get well-defined maps

$$\log\colon 1 + \mathrm{Mat}_n(P_E^m) \to \mathrm{Mat}_n(P_E^m),$$
$$\exp\colon \mathrm{Mat}_n(P_E^m) \to 1 + \mathrm{Mat}_n(P_E^m).$$

ACTUALLY: Revise:

$$\log\colon 1 + \mathrm{Mat}_n(P_E^m) \to \text{small open additive subgroup of } \mathrm{Mat}_n(\mathscr{O}_E),$$
$$\exp\colon \text{small open additive subgroup of } \mathrm{Mat}_n(\mathscr{O}_E) \to 1 + \mathrm{Mat}_n(P_E^m).$$

If $0 \neq x_0 \in U_\ell$, then $\overline{\rho}_\ell(x_0)^{\ell^d} \in 1 + \mathrm{Mat}_n(P_E^m)$ for a large enough d. Define $N = \frac{1}{\ell^d}\log(\overline{\rho}_\ell(x_0)^{\ell^d})$. By construction, if $k \in \mathbb{Z}$ and $\ell^d \mid k$, then $\overline{\rho}_\ell(k \cdot x_0) = \exp(kx_0N)$. By continuity, $\overline{\rho}_\ell(x) = \exp(x \cdot N)$ for all $x \in \overline{\mathbb{Z} \cdot \ell^d x_0}$ (the closure inside $\mathbb{Z}_\ell$). But this closure is open on U_ℓ! ☐

The upshot of all this is the following. We started with a continuous homomorphism $\rho\colon \mathscr{W}_F \to \mathrm{GL}_n(E)$. We showed that there is an open subgroup U (this will be $t_\ell^{-1}(\text{open subgroup in } U_\ell)$) inside $\rho^{-1}(K) \cap \mathscr{I}_F$ such that $\rho(h) = \exp(t_\ell(h) \cdot N)$ for all $h \in U$. So ρ is smooth if and only if $N = 0$.

Fact 16.2. 1. *$\rho(\gamma)N\rho(\gamma)^{-1} = \|\gamma\| \cdot N$ for all $\gamma \in \mathscr{W}_F$, where $\|\gamma\| = q^{-\nu(\gamma)}$. Call this $(*)$.*

Remark. This follows from the uniqueness of N and the fact that (EXERCISE!) $t_\ell(\gamma h \gamma^{-1}) = \|\gamma\| \cdot t_\ell(h)$ for all $\gamma \in \mathscr{W}_F$ and $h \in \mathscr{I}_F$. Call this $(**)$.

2. *N is nilpotent.*

Remark. This follows from $(*)$ by choosing $\gamma \in \mathscr{W}_F$ such that $\|\gamma\|$ is not a root of 1. (In this case, same thing as saying $\|\gamma\| \neq 1$.)

Corollary 16.3. *If ρ is irreducible, it is smooth.*

Proof. $\ker(N)$ is $\rho(\mathscr{W}_F)$-invariant by $(*)$. So it must be the whole space. □

In general, construct a smoooth homomorphism $\rho' : \mathscr{W}_F \to \mathrm{GL}_n(E)$ as follows. Fix $\varphi \in \mathscr{W}_F$ with $\nu(\varphi) = 1$. Then for all $\gamma \in \mathscr{W}_F$, $\gamma = \varphi^k \cdot h$ for unique $k \in \mathbb{Z}$, $h \in \mathscr{I}_F$. Set $\rho'(\gamma) = \rho(\gamma) \cdot \exp(-t_\ell(h)N)$. Then $(*)$ and $(**)$ imply that ρ' is also a homomorphism. Moreover, $\rho'(\gamma)N\rho'(\gamma)^{-1} = \|\gamma\| \cdot N$ for all $\gamma \in \mathscr{W}_F$ and $\ker(\rho')$ is open by construction. (Call this $(***)$.)

Definition 16.1. A *Weil-Deligne representation* is a pair (ρ', N), where $\rho' : \mathscr{W}_F \to \mathrm{GL}_n(E)$ is a smooth homomorphism and $N \in \mathrm{Mat}_n(E)$ is a nilpotent operator such that $(***)$ holds.

Remark. Note that this makes sense for any field E of characteristic $\neq p$.

This process can be reversed and does not depend on the choice of φ. (This should be clear from the construction.)

Fact 16.4. *We get a bijection between n-dimensional continuous representations ρ of $\mathscr{W}_F$ over $\overline{\mathbb{Q}}_\ell$ and n-dimensional Wel-Deligne representations of $\mathscr{W}_F$ over $\overline{\mathbb{Q}}_\ell$. (Of course up to isomorphism.)*

Remark. There is something called the Weil-Deligne group. It is a group scheme over $\mathbb{Q}$. Vaguely... there is a natural notion of tensor product in the category of finite-dimensional Weil-Deligne representations over $\mathbb{Q}$. This has an obvious forgetful functor. And then you continue....

17 22 October 2012: Motivation for Formal Groups

Talk tomorrow. It will be (sort of) a summary for the rest of the course.

Recall: We have a bijection between n-dimensional continuous representations of $\mathscr{W}_F$ over $\overline{\mathbb{Q}}_\ell$ (up to isomorphism) and n-dimensional Weil-Deligne representations of $\mathscr{W}_F$ over $\overline{\mathbb{Q}}_\ell$ (again up to isomorphism). The second notion is completely algebraic and does not use the topology on $\overline{\mathbb{Q}}_\ell$.

Remark. If $\rho: \mathscr{W}_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ is a continuous homomorphism, then $\rho(\mathscr{W}_F) \subseteq \mathrm{GL}_n(E)$ for some finite extension $E \supseteq \mathbb{Q}_\ell$.

- $\overline{\mathbb{Q}}_\ell$ with the standard absolute value is not complete. This is because

$$\overline{\mathbb{Q}}_\ell = \cup_{i=1}^\infty E_i,$$

  where $\mathbb{Q}_\ell \subseteq E_1 \subseteq \cdots$ are finite extensions. Each E_i is closed in $\overline{\mathbb{Q}}_\ell$ and has empty interior. Baire category theorem (for compact Hausdorff) gives that $\overline{\mathbb{Q}}_\ell$ is not complete.

- $\rho: \mathscr{W}_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ is a continuous homomorphism. It is enough to show that $\rho(\mathscr{I}_F) \subseteq \mathrm{GL}_n(E)$ for some finite extension E of $\mathbb{Q}_\ell$. Now, the inertia subgroup $\mathscr{I}_F$ is a profinite group, so it is complete. So

$$\mathscr{I}_F = \cup_{i=1}^\infty (\mathscr{I}_F \cap \rho^{-1}(\mathrm{GL}_n(E_i))).$$

  (Reason that this is enough: Pick an arbitrary lift of the Frobenius.) Now, each guy on the RHS is closed in $\mathscr{I}_F$, and either the interior is empty or it is open and has finite index. Applying Baire category, we get that at least one of the guys on the RHS must be open, so some $\mathscr{I}_F \cap \rho^{-1}(\mathrm{GL}_n(E_i))$ has finite index in $\mathscr{I}_F$ for some i. Pick a representative on each coset and apply the same argument. ◇

Corollary 17.1. *Let ℓ and ℓ' be prime numbers not equal to $p = \mathrm{char}(\mathscr{O}_F/(\pi))$. We have a field isomorphism $f: \overline{\mathbb{Q}}_\ell \to \overline{\mathbb{Q}}_{\ell'}$. We get a bijection "induced by f" between continuous n-dimensional representations of the Weil group $\mathscr{W}_F$ over $\overline{\mathbb{Q}}_\ell$ and n-dimensional representations of the Weil group $\mathscr{W}_F$ over $\overline{\mathbb{Q}}_{\ell'}$.*

Remark. One should not take the above for granted.

The local Langlands correspondence is a bijection between smooth irreducible representations of $\mathrm{GL}_n(F)$ over $\overline{\mathbb{Q}}_\ell$ and continuous n-dimensional Frobenius-semisimple representations of $\mathscr{W}_F$ over $\overline{\mathbb{Q}}_\ell$. The former also corresponds to n-dimensional Frobenius-semisimple

Weil-Deligne representations of $\mathscr{W}_F$ over $\overline{\mathbb{Q}}_\ell$. In the first set, we have the supercuspidal representations. Under the local Langlands correspondence, these correspond to the smooth irreducible representations of $\mathscr{W}_F$ over $\overline{\mathbb{Q}}_\ell$.

17.1 Examples

Let E be a separable degree-n extension of the local field F. Then $\mathscr{W}_E \subseteq \mathscr{W}_F$ is an open subgroup of index n. We can try to induce one-dimensional characters of $\mathscr{W}_F$ to $\mathscr{W}_E$. What are the one-dimensional characters of $\mathscr{W}_F$? This is what local class field theory is about. We have

$$\mathscr{W}_E \twoheadrightarrow \mathscr{W}_E^{\text{ab}} = \mathscr{W}_E/\overline{[\mathscr{W}_E, \mathscr{W}_E]} \xrightarrow{\cong} E^\times,$$

where the rightmost isomorphism is given by the local Artin map. We will call the composition rec. Given a smooth $\theta\colon E^\times \to \overline{\mathbb{Q}}_\ell^\times$, then $\operatorname{Ind}_{\mathscr{W}_E}^{\mathscr{W}_F}(\theta \circ \text{rec})$ is a smooth n-dimensional representation and it is sometimes irreducible. In general it is not so easy to check that this is irreducible. However, we can say something.

Proposition 17.2. *The induced representation is irreducible if $E \supseteq F$ is Galois and θ has trivial stabilizer under the action of* $\operatorname{Gal}(E/F)$.

Remark. In the above, this is basically the Mackey irreducibility criterion. ◇

Remark. In general, $\operatorname{Ind}_{\mathscr{W}_E}^{\mathscr{W}_F}(\theta \circ \text{rec})$, assuming its irreducibility, corresponds to some smooth irreducible supercuspidal representation of $\mathrm{GL}_n(F)$. The question is: Which one? ◇

Remark.

1. There are algebraic constructions, but these are very messy in general!
2. The underlying geometry is completely unknown. The Lubin-Tate tower is (very roughly) a parameter space for the space of formal groups. It gives a geometric realization of irreducible supercuspidal representations of $\mathrm{GL}_n(F)$.

Final comment: I gave an answer to this question in the following special case: If $E \supseteq F$ is unramified and θ has the property that $\theta|_{1+(\pi)}$ is trivial, then we gave a construction of the corresponding supercuspidal representation of $\mathrm{GL}_n(F)$ on the righthand side, from $\operatorname{Ind}_{\mathscr{W}_F}^{\mathscr{W}_F}(\theta \circ \text{rec})$.

Remark. From $\chi\colon \mathbb{F}_{q^n}^\times \to \overline{\mathbb{Q}}_\ell^\times$, we get a cuspidal irreducible representation of $\mathrm{GL}_n(\mathbb{F}_q)$. It is completely unknown how to generalize this for the case of local fields.

17.2 Some First Words about Formal Groups

Let G be a Lie group and let $\mu\colon G \times G \to G$ be the multiplication map. Pick a coordinate chart centered at the identity. That is, we have $(U, x_1, \ldots, x_n)$ where U is an open neighborhood of $1 \in G$. Pick a coordinate chart so that $1 \mapsto 0$ in $\mathbb{R}^n$. That is, we have $U \simeq V \subseteq \mathbb{R}^n$ open and $1 \mapsto 0$.

Expand μ into a Taylor series $\mu = (\mu_1, \ldots, \mu_n)$ where each $\mu_i \in \mathbb{R}[[x_1, \ldots, x_n, y_1, \ldots, y_n]]$. This collection of power series satisfies the following properties:

1. Associativity: $\mu(\mu(x, y), z) = \mu(x, \mu(y, z))$
2. $\mu(x, 0) = x = \mu(0, x)$

These are the axioms for an n-dimensional formal group law. On Wednesday, we will give the definition. We will focus our attention on the one-dimensional case.

This has as much information as the Lie algebra, thought the Lie algebra is a much more manageable object. In the positive characteristic case, this gives you the correct analogue of Lie theory in the characteristic 0 situation.

18 24 October 2012

18.1 Remarks for Last Time

1. $E \supseteq F$ is a degree n separable extension. Then $\theta\colon E^\times \to \overline{\mathbb{Q}}_\ell^\times$ smooth gives a smooth n-dimensional representation $\operatorname{Ind}_{\mathscr{W}_E}^{\mathscr{W}_F}(\theta \circ \mathrm{rec})$.

   Fact 18.1. *If $p \nmid n$, then every n-dimensional smooth irreducible representation of $\mathscr{W}_F$ arises in this way! (Proof on the HW exercises.)*

2. For $p = n = 2$, then this is not always the case. A reference of this is Weil's 1974 paper in Inventiones Math.

18.2 An Introduction to Formal Groups

Technically: Actually, we are going to study formal group laws, i.e. choose coordinates. The ones we discuss in this course will be one-dimensional. So here, when we say "formal group," we mean a one-dimensional commutative formal group law. Last spring, Mitya gave a lot of lectures about this.

Definition 18.1. A *formal group over a commutative ring R* is $F \in R[[x, y]]$ satisfying:

- $F(x, 0) = x = F(0, x)$
- $F(F(x, y), z) = F(x, F(y, z))$
- $F(x, y) = F(y, x)$

Remark. Commutativity follows from the other axioms unless there exists an element $r \in R$ such that r is nilpotent and is killed by some nonzero integer. Hence the third property (commutativity) is not really much of a restriction. ◇

Remark. Existence of inverses is automatic. Indeed, there exists a unique formal power series $\iota(x) \in R[[x]]$ that has constant term 0 and $\iota(x) = x + O(x^2)$ such that $F(x, \iota(x)) = 0$. ◇

18.3 Examples

1. $\widehat{\mathbb{G}}_a(x, y) = x + y$.
2. $\widehat{\mathbb{G}}_m(x, y) = (1 + x)(1 + y) - 1 = x + y + xy$.

Remark. Side comment: Consider a functor $\alpha_p\colon \{\mathbb{F}_p\text{-algebras}\} \to \{\text{abelian groups}\}$ taking $A \mapsto \{a \in A : a^p = 0\}$. Then $\operatorname{Spec} \mathbb{F}_p[x]/(x^p)$ is a non-reduced group scheme. ◇

18.3.1 Construction of a Formal Group: Formal Completion

Let E be any one-dimensional algebraic group over a field $k = R$. Let $\widehat{E}$ be the corresponding formal group. Let $\widehat{\mathscr{O}}_{E,1} \cong k[[x]]$. Let $\mu\colon E \times E \to E$ be the multiplication map. From this, we get an induced map $k[[t]] = \widehat{\mathscr{O}}_{E,1} \to \widehat{\mathscr{O}}_{E,1} \widehat{\otimes}_k \widehat{\mathscr{O}}_{E,1} \cong k[[x, y]]$ given by $t \mapsto F$.

18.4 Fairy Tales about Formal Groups

There is a connection to algebraic topology here. Assumption: singular cohomology, simplicial cohomology, etc. There are also *generalized cohomology theories* which is something that gives a functor from the category of topological spaces to the category of graded abelian groups. So $X \mapsto \bigoplus_n H^n_{\text{gen}}(X)$.

Some assumptions:

- multiplicativity (so that we can upgrade the graded abelian groups to graded super-commutative rings)
- complex-oriented (roughly, the practical consequences is that this allows us to define chern classes of line bundles, i.e. we have a notion of $c_1(L) \in H^2_{\text{gen}}(X)$, where L is a complex line bundle on X) (if this cohomology is the usual one, when this is the usual first chern class) (if L and M are complex line bundles on X, then $c_1(L \otimes_{\mathbb{C}} M) = F(c_1(L), c_1(M))$. to have a universal formula like F that does not depend on X, this means that F is going to be a formal power series in two variable with coefficients that come from the cohomology of the point. note that $F \in R[[x, y]]$ where $R = H^*_{\text{gen}}(\text{point})$.)

  (a) (:53) For safety, let us impose the condition that $H^n_{\text{gen}}(*) = 0$ if n is odd, so R is actually commutative and there is no problem.)

  (b) $F(c_1(L), c_1(M))$ makes sense in $\prod_n H^{2n}_{\text{gen}}(X)$.

Remark. Side comment. $X \to *$ induces a map $H^*_{\text{gen}}(*) \to H^*_{\text{gen}}(X)$. $c_1(L \otimes_{\mathbb{C}} M) \in H^2_{\text{gen}}(X)$. ◇

KEY POINT: F is a formal group law over R.

(Maybe there will be an informal seminar to go over this kind of thing in the Winter 2013 term.)

19 9 November 2012: The Lubin-Tate Construction of Local Class Field Theory

Let F be as usual. We have $F^{\text{sep}} \supset F^{\text{ab}}$

(missed 20 min here)

Fact 19.1. *There exists a collection of isomorphisms $rec_F: G_F^{ab} \to \widehat{F}^\times$ for all possible F such that the following diagrams commute: (see john's notes for (1) and (2)).*

19.1 Local Kronecker-Weber

$\mathbb{Q}_p^{\text{ab}} = \mathbb{Q}_p^{\text{nr}} \cdot \mathbb{Q}_p^{\text{tr}}$, where $\mathbb{Q}_p^{\text{nr}} = \mathbb{Q}_p(\mu_{p'})$, $\mu_{p'} = \{a \in \overline{\mathbb{Q}}_p : a^m = 1 \text{ for some } p \nmid m\}$, and $\mathbb{Q}_p^{\text{tr}} = \mathbb{Q}_p(\mu_{p^\infty})$ where $\mu_{p^\infty} = \{a \in \overline{\mathbb{Q}}_p : a^{p^m} = 1 \text{ for some } m \geq 1\}$.

$$\begin{aligned} &\text{Gal}(\mathbb{Q}_p^{\text{ab}}/\mathbb{Q}_p) \to \text{Gal}(\mathbb{Q}_p^{\text{nr}}/\mathbb{Q}_p) \times \text{Gal}(\mathbb{Q}_p^{\text{tr}}/\mathbb{Q}_p) \\ &\text{Gal}(\mathbb{Q}_p^{\text{nr}}/\mathbb{Q}_p) \to \widehat{\mathbb{Z}} \\ &\text{Gal}(\mathbb{Q}_p^{\text{tr}}/\mathbb{Q}_p) \to \mathbb{Z}_p^\times \end{aligned}$$

We have the obvious map $\text{Gal}(\mathbb{Q}_p^{\text{tr}}/\mathbb{Q}_p) \to \text{Aut}(\mu_{p^\infty})$.

ALso, $\mu_{p^\infty} \cong \mathbb{Q}_p/\mathbb{Z}_p$ (non-canonically) as abelian groups. This implies that for $\text{Aut}(\mathbb{Q}_p/\mathbb{Z}_p) \cong \mathbb{Z}_p^\times$, we have a canonical isomorphism $\text{Aut}(\mathbb{Q}_p/\mathbb{Z}_p) \cong \text{Aut}(\mu_{p^\infty})$.

19.2 The General Lubin-Tate Construction

Choose $f(x) \in \mathcal{O}_F[[x]]$ such that $f(x) = \pi \cdot x + \mathcal{O}(x^2)$ and $f(x) \cong x^q \pmod{\pi}$. Take $F^{\text{ab}} = F^{\text{nr}}$ to contain roots of $f \circ \cdots \circ f$ (composed m times) for all $m \geq 1$.

Example 19.1. Take $F = \mathbb{Q}_p$, $f(x) = (1+x)^p - 1$. Get local Kronecker-Weber.

Let Σ be the unique formal $\mathcal{O}_F$-module over $\mathcal{O}_F$ such that $[\pi]_\Sigma(x) = f(x)$.

Informally: Look at the π^m-torsion points of Σ. Consider $\Sigma(\mathcal{O}_F^{\text{sep}})$ as a set, $m_{\mathcal{O}_F^{\text{sep}}}$, equipped with operations $(x, y) \mapsto \Sigma(x, y)$ and $a: x \mapsto [a]_\Sigma(x)$ for $a \in \mathcal{O}_F$. So we get an honest $\mathcal{O}_F$-module.

So $\Sigma(\mathcal{O}_{F^{\text{sep}}})$ is an $\mathcal{O}_F$-module in the usual sense.

Consider $M = \{x \in \Sigma(\mathcal{O}_{F^{\text{sep}}} : [\pi^d]_\Sigma(x) = 0 \text{ for some } d \geq 1\}$. This is an $\mathcal{O}_F$-submodule. Key claim: $M \cong F/\mathcal{O}_F$ as an $\mathcal{O}_F$-module.

Exercise 3. It is enough to prove that for any $d \geq 1$, $\{x \in m_{F^{\text{sep}}} : [\pi^d]_\Sigma(x) = 0\}$ has size q^d.

Proof of Exercise. Without loss of generality, assume that $f(x) = \pi x + x^q$ (i.e. $f(x) = [\pi]_\Sigma(x)$). Then

$$f(f^{d-1}(x)) = \pi \cdot f^{d-1}(x) + f^{d-1}(x)^q = f^{d-1}(x)(\pi + f^{d-1}(x)^{q-1}).$$

The rightmost factor is an Eisenstein polynomial of degree $(q-1)q^{d-1}$. □

$G_F = \mathrm{Gal}(F^{\mathrm{sep}}/F)$ acts on M and we get a canonical homomorphism $G_F \to \mathrm{Aut}_{\mathcal{O}_F-\mathrm{mod}}(M) \cong \mathrm{Aut}_{\mathcal{O}_F-\mathrm{mod}}(F/\mathcal{O}_F) \cong \mathcal{O}_F^\times$.

FACTS. $G_F \twoheadrightarrow \mathcal{O}_F^\times$ and if K is the kernel, then the fixed field of K is a maximal abelian totally ramified extension. That is, $F^{\mathrm{ab}} = F^{\mathrm{nr}} \cdot (F^{\mathrm{sep}})^K$ and $(F^{\mathrm{sep}})^K$ is a maximal abelian totally ramified extension of F. Then we get an isomorphism $\mathrm{Gal}((F^{\mathrm{sep}})^K/F) \to \mathcal{O}_F^\times$.

(send frobenius to -1 in order to make it compatible with the above.)

20 12 November 2012: Deformation Spaces of Formal Modules

Our goal is to find geometric objects that carry a natural action of $\mathrm{GL}_n(F) \times \mathscr{W}_F$.

20.1 Analogy

Let E be an elliptic curve over a number field $K \subseteq \mathbb{C}$. If $N \geq 1$, then $E(\mathbb{C})_{N\text{-torsion}} \cong (\mathbb{Z}/N\mathbb{Z})^{\times}$ as groups. In fact, $G_K = \mathrm{Gal}(\overline{K}/K)$ acts naturally on $E(\mathbb{C})_{N\text{-torsion}}$.

$X_N := \mathrm{Isom}((\mathbb{Z}/N\mathbb{Z})^{\times}, E(\mathbb{C})_{N\text{-torsion}})$ carries an action of $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}) \times G_K$. If $M \mid N$, then we get a natural surjection $X_N \twoheadrightarrow X_M$ compatible with $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}) \times G_K \twoheadrightarrow \mathrm{GL}_2(\mathbb{Z}/M\mathbb{Z}) \times G_K$. Therefore $\varprojlim_N X_N$ carries a continuous action of $\mathrm{GL}_2(\widehat{\mathbb{Z}}) \times G_K$. We can do the same in families of elliptic curves.

This is an analogue to our situation in that we would like to replace K with F and E with a formal $\mathscr{O}_F$-module.

20.2 The Idea of Deformation Theory

Let Σ be a formal $\mathscr{O}_F$-module over $\overline{\mathbb{F}_q}$. This corresponds to a collection of "structure constants" (the coefficients of all the power series) satisfying certain polynomial relations. Deforming a constant $a_0 \in \overline{\mathbb{F}_q}$ means replacing it with a polynomial $a_0 + a_1\varepsilon + \cdots + a_n\varepsilon^n$ where ε is a formal symbol with $\varepsilon^{n-1} = 0$. This is a formal deformation of level n.

This naive approach would suggest replacing $\overline{\mathbb{F}_q}$ with $\overline{\mathbb{F}_q}[\varepsilon]/(\varepsilon^{n+1})$. The correct analogue is the following notion.

Definition 20.1. Let R be any local $\mathscr{O}_F$-algebra. Recall that this includes the assumption that the uniformiser $\pi \in \mathscr{O}_F$ map to something in m_R, the maximal ideal of R.

Suppose Σ is a formal $\mathscr{O}_F$-module over the residue field $k = R/m_R$. A *deformation of* Σ *with base* R is a pair $(\mathscr{F}, i)$, where $\mathscr{F}$ is a formal $\mathscr{O}_F$-module over R and i is an isomorphism of formal $\mathscr{O}_F$-modules between Σ and $\mathscr{F} \mod m_R$.

Remark.

1. Can also ignore the isomorphism i and instead require that $\Sigma = \mathscr{F} \mod m_R$.
2. Special case: $m_R^n = 0$ for some $n \geq 1$. This corresponds to deformations of order $n-1$.
3. The notion of a morphism of deformations is clear. If Σ has finite height, then all deformations of Σ are rigid. ◇

Lemma. *Let R be a local separated $\mathscr{O}_F$-algebra. (Separated means that $\cap_{n\geq 1} m_R^n = 0$.) Let $\mathscr{F}$ and $\mathscr{F}'$ be formal $\mathscr{O}_F$-modules over R such that $\mathscr{F} \mod m_R$ is a formal $\mathscr{O}_F$ module over the residue field $k = R/m_R \supset \mathbb{F}_q$ and has finite height (over the maximal ideal).*

Then $\mathrm{Hom}_{\mathscr{O}_F}(\mathscr{F}, \mathscr{F}') \hookrightarrow \mathrm{Hom}_{\mathscr{O}_F}(\mathscr{F} \mod m_R, \mathscr{F}' \mod m_R)$. *(The claim is the injectivity of this map.)*

Proof. By induction, we are reduced to showing that if $m_R^{n+1} = 0$ for some $n \geq 1$ and $\varphi\colon \mathscr{F} \to \mathscr{F}'$ is a morphism of formal $\mathscr{O}_F$-modules such that $\varphi \equiv 0 \mod m_R^n$, then $\varphi = 0$. We have:

$$\begin{aligned}\varphi(\mathscr{F}(x,y)) &= \mathscr{F}'(\varphi'(x), \varphi'(y)) = \varphi(x) + \varphi(y)\\ \varphi([a]_{\mathscr{F}}(x)) &= [a]_{\mathscr{F}'}(\varphi(x)) = a \cdot \varphi(x), \quad \text{for all } a \in \mathscr{O}_F.\end{aligned}$$

Now, $m_R^n = m_R^n / m_R^{n+1}$ is a vector space over the residue field $k = R/m_R$. Let $\lambda\colon m_R^n \to k$ be any linear functional. Then $\lambda(\varphi)$ is a morphism of formal $\mathscr{O}_F$-modules from $\mathscr{F} \mod m_R$ and $\widehat{\mathbb{G}}_a$ over k. So $\lambda(\varphi) = 0$. □

(There are no morphisms between two formal modules of different height.) (What was the definition of height again?)

Here is our main definition.

Definition 20.2. Let $\mathscr{C}$ be the category of complete local $\widehat{\mathscr{O}}_F^{\mathrm{nr}}$-algebras R such that $\widehat{\mathscr{O}}_F^{\mathrm{nr}}/(\pi) \to R/m_R$ is an isomorphism. The morphisms are local homomorphisms of $\mathscr{O}_F$-algebras. Fix Σ to be a formal $\mathscr{O}_F$-module over $\overline{\mathbb{F}_q}$ of height $h \geq 1$. We get a deformation functor $\mathscr{C} \to \mathrm{Sets}$, $R \mapsto \{\text{deformations of } \Sigma \text{ with base } R\}$.

Remark. Recall that F^{nr} is the maximal unramified extension of F and $\mathscr{O}_F^{\mathrm{nr}}$ is its ring of integers (a DVR with the same uniformizer as $\mathscr{O}_F$). The residue field is $\widehat{\mathscr{O}}_F^{\mathrm{nr}}/(\pi) = \overline{\mathbb{F}_q}$. $\widehat{\mathscr{O}}_F^{\mathrm{nr}}$ is the completion of $\mathscr{O}_F^{\mathrm{nr}}$. ◇

Remark. Note that complete means that $R \to \varprojlim_n R/m_R^n$ is an isomorphism. ◇

Remark. This functor is representable. This is essentially the level 0 of the Lubin-Tate tower.

21 16 November 2012

Recall that we have $\mathscr{C} = \{$local, complete $\widehat{\mathscr{O}}_F^{\mathrm{nr}}$-algebras R such that $\widehat{\mathscr{O}}_F^{\mathrm{nr}}/(\pi) = \overline{\mathbb{F}}_q \cong R/m_R\}$. Let Σ be a formal $\mathscr{O}_F$-module over $\overline{\mathbb{F}}_q$ of height $h \geq 1$.

We have a deformation functor

$$\mathrm{Def}^n_\Sigma \colon \mathscr{C} \to \mathrm{Sets}, \quad R \mapsto \{(\mathscr{F}, \varphi)\}/\cong,$$

(level $n \leftrightarrow$ level π^n) where $\mathscr{F}$ is a formal $\mathscr{O}_F$-module over R such that $(\mathscr{F} \mod m_R) = \Sigma$ and $\varphi\colon (\pi^{-n}\mathscr{O}_F/\mathscr{O}_F)^h \to \mathscr{F}(R)$[1] is a morphism of $\mathscr{O}_F$-modules such that:

$$\prod_{a \in (\pi^{-1}\mathscr{O}_F/\mathscr{O}_F)^h} (x - \varphi(a)) \mid [\pi]_{\mathscr{F}}(x) \quad \text{in } R[[x]]. \tag{2}$$

Lemma. *If R is an integral domain and $\pi \neq 0$ in R, then (2) holds if and only if φ is an isomorphism onto $\mathscr{F}(R)_{\pi^n\text{-torsion}}$.*

Proof. Assume that (2) holds. Note that $\mathscr{F}(R)_{\pi^n\text{-tors}}$ consists of the roots of $[\pi^n]_{\mathscr{F}}(x)$ in m_R. Recall that Weierstrass preparation says that $[\pi^n]_{\mathscr{F}}(x) = w_n(x) \cdot v_n(x)$, where $\deg(w_n) = q^{nh}$, $v_n \in R[[x]], v_n(0) = 1$.[2] So $|\mathscr{F}(R)_{\pi^n\text{-tors}}| \leq q^{nh} = |(\pi^{-n}\mathscr{O}_F/\mathscr{O}_F)^h|$.[3] Hence it is suffices to show that φ is injective.

It is enough to prove that $\varphi|_{(\pi^{-1}\mathscr{O}_F/\mathscr{O}_F)^h}$ is injective. So it is enough to prove that if $\beta \in m_R$ and $[\pi]_{\mathscr{F}}(\beta) = 0$, then $[\pi]'_\beta(\beta) \neq 0$. We have

$$[\pi]_{\mathscr{F}}(\mathscr{F}(x,y)) = \mathscr{F}([\pi]_{\mathscr{F}}(x), [\pi]_{\mathscr{F}}(y)) \quad \text{in } R[[x,y]].$$

Now,

$$\frac{\partial}{\partial y}\colon [\pi]'_{\mathscr{F}}(\mathscr{F}(x,y)) \cdot \frac{\partial \mathscr{F}}{\partial y}(x,y) = [\pi]'_{\mathscr{F}}(y) \cdot \frac{\partial \mathscr{F}}{\partial y}([\pi]_{\mathscr{F}}(x), [\pi]_{\mathscr{F}}(y)).$$

Set $x = \beta$, $y = 0$:

$$[\pi]'_{\mathscr{F}}(\beta) \cdot ? = \pi \cdot 1 \neq 0.$$ □

[1] $\mathscr{F}(R)$ is m_R as a set, but the addition is the usual addition coming from $\mathscr{F}$

[2] $[\pi^n]_{\mathscr{F}}(x) = \pi^n \cdot x + \cdots + ? \cdot x^{q^{nh}} + O(x^{q^{nh}+1})$, where the ? is not in m_R but all coefficients of preceding terms are in m_R. So $[\pi]_\Sigma(x) = ? \cdot x^{q^h} + O(x^{q^h+1})$.

[3] $\varphi\colon (\pi^{-n}\mathscr{O}_F/\mathscr{O}_F)^h \to \mathscr{F}(R)$. $a \in (\pi^{-1}\mathscr{O}_F/\mathscr{O}_F)^h$ implies that $\varphi(a)$ is a root of $[\pi]_{\mathscr{F}}(x)$. This shows the right to left implication...?!

Theorem 8 (Drinfeld). *$Def^n_\Sigma\colon \mathscr{C} \to$ Sets is representable by $A_n \in \mathscr{C}$. Moreover, A_n is a regular (Noetherian) local ring, and if $\varphi^{univ}\colon (\pi^{-n}\mathscr{O}_F/\mathscr{O}_F)^h \to \mathscr{F}^{univ}(A_n)$ is the universal thing, then $(\varphi_n^{univ}(e_1), \varphi_n^{univ}(e_2), \dots, \varphi_n^{univ}(e_h))$ is a regular sequence generating m_{A_n} where $e_i = (0, \dots, \pi^{-n}, \dots, 0)$, where the only nonzero slot is in the ith place.*

$A_0 = \mathscr{O}_F^{\mathrm{nr}}[[u_1, \dots, u_{n-1}]]$

In general, we get a surjection $\alpha_n\colon \widehat{\mathscr{O}}_F^{\mathrm{nr}}[[x_1, \dots, x_h]] \twoheadrightarrow A_n$. A_n has dimension h and the first guy has dimension $h+1$ and they are both regular. So the kernel has to be a principal ideal that is generated by some element in $m \smallsetminus m^2$, where m is the maximal idea of the domain.

Example 21.1. Let $F = \mathbb{F}_q((\pi))$ and $n = 1$. We have: $\varphi_1^{\mathrm{univ}}\colon (\pi^{-1}\mathscr{O}_F/\mathscr{O}_F)^h = \mathbb{F}_q^n \to \mathscr{F}^{\mathrm{univ}}(A_1) = m_{A_1}$. (This last equality as a set.) Assume that $\Sigma(x, y) = x + y$ and $[a]_\Sigma(x) = ax$ for all $a \in \mathbb{F}_q$ and $[\pi]_\Sigma(x) = x^{q^h}$. Also assume that $\mathscr{F}^{\mathrm{univ}}(x, y) = x + y$ and $[a]_{\mathscr{F}^{\mathrm{univ}}}(x) = ax$ for all $a \in \mathbb{F}_q$ and $[\pi]_{\mathscr{F}^{\mathrm{univ}}}(x) = \pi x + u_1 x^q + u_2 x^q + \cdots + u_{h-1} x^{q^{h-1}} + x^{q^h}$ (over $A_0 = \widehat{\mathscr{O}}_F^{\mathrm{nr}}[[u_1, \dots, u_{h-1}]]$).

We know that $\prod_{\alpha \in \mathbb{F}_q^h}(x - \varphi(\alpha))$ divides $[\pi]_{\mathscr{F}^{\mathrm{univ}}}(x)$ and $\alpha = a_1 e_1 + \cdots + a_h e_h$. This implies that $\prod_{(a_1, \dots, a_h) \in \mathbb{F}_q^h}(x - a_1 \cdot \varphi(e_1) - \cdots - a_n \cdot \varphi(e_n))$ divides $[\pi]_{\mathscr{F}^{\mathrm{univ}}}(x)$. So they are equal because they have the same degree.

Compare the linear coefficients and get $\pi = \prod_{(a_1, \dots, a_n) \in \mathbb{F}_q^n}(a_1\varphi(e_1) + \cdots + a_n\varphi(e_n))$. ◇

On Monday, we will begin rigid analytic spaces.

22 19 November 2012: Rigid Analytic Spaces

22.1 Tate's Original Motivation

Let E be an elliptic curve over $\mathbb{C}$. We may consider its group of complex points $E(\mathbb{C})$, which has the structure of a complex Lie group and is isomorphic to $\mathbb{C}/\Lambda$ for a lattice $\Lambda \subseteq \mathbb{C}$. Without loss of generality, we may assume that Λ has a basis consisting of $1, \tau$, where the imaginary part of τ is positive ($\Im(\tau) > 0$). So $\mathbb{C}/\Lambda \cong (\mathbb{C}/\mathbb{Z})/\langle\text{image of } \tau\rangle$. The map $\mathbb{C} \to \mathbb{C}^\times$, defined by $z \mapsto \exp(2\pi i z)$, identifies $\mathbb{C}/\mathbb{Z} \cong \mathbb{C}^\times$, so we get $E(\mathbb{C}) \cong \mathbb{C}^\times/q^{\mathbb{Z}}$, where $q = \exp(e\pi i\tau)$. So $0 < |q| < 1$. ($E(\mathbb{C})$ is algebraic and $\mathbb{C}^\times$ is $\mathbb{G}_m(\mathbb{C})$.)

A simplified version of Tate's result is the following. Let $\lambda \in \mathbb{C}_p = \widehat{\overline{\mathbb{Q}}}_p$ and $\lambda \neq 0, 1$. Let E be the projectivization of $\{y^2 = x(x-1)(x-\lambda)\} \subseteq \mathbb{A}^2_{\mathbb{C}_p}$. Then $E(\mathbb{C}_p) \cong \mathbb{C}_p^\times/q^{\mathbb{Z}}$ for some $q \in \mathbb{C}_p$ with $0 < |q| < 1$, provided that $|\lambda| < 1$.

Remark. This is not a very good way to state this result because we want the non-algebraically-closed case. In this case, we want the absolute value of the J-invariant to be bigger than one.

22.2 What makes non-Archimedean analysis more difficult?

Let K be a field that is a complete with respect to a nontrivial non-Archimedean absolute value. (Example: Let $K = F$ as before, or $K = \widehat{F}^{\mathrm{nr}}, \widehat{F}^{\mathrm{sep}}, \widehat{\overline{F}}$.)

The notion of an analytic function on an open set $U \subseteq K$ makes sense. The problem is the following: An analytic function on $\mathring{D} = \{x \in K : |x| < 1\}$, in general can *not* be represented by a power series converging on $\mathring{D}$. The obstruction? There are too many open sets. For example, $U = \{|x| < 1/2\}$ and $V = \mathring{D} \setminus U = \{1/2 \leq |x| < 1\}$ are both open, and so we can take $f\colon \mathring{D} \to K$ such that $f \equiv 1$ on U and $f \equiv 0$ on V.

The moral of this discussion is that we need to restrict the collection of open sets that we allow in this theory. But even this is not enough. Namely, we need to restrict the notion of an open covering that we allow.

Example 22.1. Consider the case $K = \mathbb{Q}_p$. Then $\mathbb{Z}_p = \{x \in K : |x| \leq 1\}$ can be covered by p disjoint closed discs of radius $1/p$. We will *not* allow these.

22.3 Analogy

Rigid analytic spaces over K behave a bit like schemes that are locally of finite type over K.

(PICTURE)

Definition 22.1. The set

$$K\langle z_1, \ldots, z_n\rangle = \{\sum_{I=(i_1,\ldots,i_n)} a_I z^I : a_I \in K, a_I \to 0 \text{ as } \ell(I) \to \infty\}$$

is the K-algebra of *restricted power series.*

Definition 22.2. An *affinoid K-algebra* is a K-algebra isomorphic to a quotient of $K\langle z_1, \ldots, z_n\rangle$ for some $n \geq 1$.

Properties 22.1.

1. Affinoid K-algebras are Noetherian.
2. On $K\langle z_1, \ldots, z_n\rangle$, we have the Gauss norm:

   $$\|\sum_I a_I z^I\| := \max_I |a_I|.$$

   This makes $K\langle z_1, \ldots, z_n\rangle$ into a Banach K-algebra, and $\|\cdot\|$ is multiplicative.

   Remark. For $f \in K\langle z_1, \ldots, z_n\rangle$, $\|f\| = \max |f(x)|$ where $x = (x_1, \ldots, x_n) \in \overline{K}^n$ and $|x_i| \leq 1$ for all i.
3. All ideals in $K\langle z_1, \ldots, z_n\rangle$ are closed with respect to $\|\cdot\|$.
4. Every affinoid K-algebra has a norm that makes it a Banach K-algebra. The resulting topology on B is independent of the choice of the presentation $V = K\langle z_1, \ldots, z_n\rangle/J$.
5. Every homomorphism of affinoid K-algebras is automatically continuous.
6. (Nullstellensatz) If L is an affinoid K-algebra and L is also a field, then L is a finite extension of K.

Now we need to discuss the corresponding analytic objects.

Notation. Fix an algebraic closure $\overline{K}$ of K. If B is an affinoid K-algebra, the set of maximal ideals $\operatorname{Max}(B)$ can be identified with

$$\bigcup_{K\subseteq L\subseteq\overline{K}, [L:K]<\infty} \operatorname{Hom}_{K\text{-alg}}(B, L)/\operatorname{Aut}_K(\overline{K}).$$

Example 22.2. Consider $B = K\langle z_1, \ldots, z_n\rangle$. Then any homomorphism $B \to L$ is determined by the image of the (topological) generators z_i and hence

$$\mathrm{Hom}_{K\text{-alg}}(B, L) = \overline{D}^n(L) = \{x \in L^n : |x_i| \le 1 \text{ for all } i\}.$$

The consequence of this (i.e. the upshot) is that we can now say concretely what the set of all maximal ideals looks like. That is,

$$\mathrm{Max}(K\langle z_1, \ldots, z_n\rangle) = \overline{D}^n(\overline{K}) / \mathrm{Aut}_K(\overline{K}).$$

More generally, if $B = K\langle z_1, \ldots, z_n\rangle/(f_1, \ldots, f_m)$, then

$$\mathrm{Max}(B) = \{\text{common zeros of } f_1, \ldots, f_m \text{ in } \overline{D}^n(\overline{K}) / \mathrm{Aut}_K(\overline{K})\}.$$

Remark. If $x \in \overline{D}^n(L)$, we get $\varphi\colon K\langle z_1, \ldots, z_n\rangle \to L$ where $\sum_I a_I z^I \mapsto \sum a_I x^I$. Conversely, if $\varphi \in \mathrm{Hom}_{K\text{-alg}}(K\langle z_1, \ldots, z_n\rangle, L)$, then $\{z_i^m\}_{m\ge1}$ is a bounded sequence and therefore the image $\{\varphi(z_i)^m\}_{m\ge1}$ is a bounded sequence in L. Hence $|\varphi(z_i)| \le 1$.

(If V is a topological vector space, then we say that $X \subseteq V$ is bounded if for any open neighborhood U around 0, there exists $\lambda \in K^\times$ such that $\lambda \cdot X \subseteq U$.)

23 21 November 2012: Affinoid Spaces

23.1 Definitions

Let K be a non-Archimedean field with an absolute value $|\cdot|$. Recall that

$$K\langle z_1, \ldots, z_n\rangle = \{\sum_I a_I z^I : a_I \to 0 \text{ as } |I| \to +\infty\},$$
$$\mathrm{Aff}_K = \{K\text{-algebras that are quotients of } K\langle z_1, \ldots, z_n\rangle\}.$$

Definition 23.1. The category of *affinoid spaces* over K is the opposite of $\mathrm{Aff}_K \ni B \leftrightarrow \mathrm{Sp}(B)$. Let $\mathrm{Max}(B)$ denote the "underlying set" of $\mathrm{Sp}(B)$.

Remark. If $f\colon B \to C$ is a morphism in Aff_K, we get an induced map $f^*\colon \mathrm{Max}(C) \to \mathrm{Max}(B)$ given by $m \mapsto f^{-1}(m)$.

The difficulty is the following: Let A be a finitely generated K-algebra and endow $\mathrm{Max}(A)$ with the Zariski topology. If $B \in \mathrm{Aff}_K$, there is no direct analogue of this (topology) for $\mathrm{Max}(B)$.

So suppose $f\colon B \to C$ is a morphism in Aff_K. Then we get $\mathrm{Sp}(f)\colon \mathrm{Sp}(C) \to \mathrm{Sp}(B)$. QUESTION: When should we call $\mathrm{Sp}(f)$ an open immersion?

The following are equivalent:

(i) Given any morphism $\varphi\colon B \to R$ in Aff_K, there exists $\psi\colon C \to R$ such that $\varphi = \psi \circ f$ if and only if $f^*(\mathrm{Max}(C)) \supset \varphi^*(\mathrm{Max}(R))$ inside $\mathrm{Max}(B)$. Moreover, ψ is unique if it exists.

Remark. (analogue) Recall that if $U \to X$ is a morphism of K-schemes of finite type,

PICTURE

(open immersion)

(ii) $f^*\colon \mathrm{Max}(C) \to \mathrm{Max}(B)$ is injective and for any $m \in \mathrm{Max}(C)$, the induced map $\widehat{B}_{f^{-1}(m)} \to \widehat{C}_m$ is an isomorphism.

Proof that (i) ⇒ (ii). We analyze some cases.

1. **B has a unique maximal ideal**

   In this case, (i) says that $f\colon B \to C$ is an isomorphism. Hence (ii) also holds. (Yoneda's lemma)

2. **General case**

   If $m \in \mathrm{Max}(B)$ such that $f(m)C \neq C$, then property (i) also holds for $B/m \to C/f(m)C$, and more generally, for $B/m^n \to C/f(m)^n C$ for all $n \geq 1$. So each of these is an isomorphism. This is equivalent to (ii).

□

Remark. The proof that (ii) implies (i) is quite difficult. It is a theorem of Gerritzen and Grauert.

23.2 Examples and Non-Examples

Consider $B \in \mathrm{Aff}_K$. $f \in B$ is a "function" in the following sense: If $m \in \mathrm{Max}(B)$, then $[B/m : K] < \infty$ and so $B/m \hookrightarrow \overline{K}$. Then for $x = m$ (think about m as a point), we will set $f(x)$ to be the image of fg in $B/m \hookrightarrow \overline{K}$. Because we need to pick an embedding, the "function" $f(x)$ is only well-defined up to the action of the Galois group $\mathrm{Aut}_K(\overline{K})$. In particular, $x \mapsto |f(x)|$ is a well-defined function $\mathrm{Max}(B) \to \mathbb{R}_{\geq 0}$.

Definition 23.2. A subset $U \subset \mathrm{Max}(B)$ is called an *open affinoid subdomain* if there exists an open immersion (i.e. satisfying (i) and/or (ii)) $\mathrm{Sp}(C) \to \mathrm{Sp}(B)$ such that U is the image of $\mathrm{Max}(C)$ in $\mathrm{Max}(B)$.

Caution. The set $\{x \in \mathrm{Max}(B) : f(x) \neq 0\}$ is *not* an open affinoid subdomain. ($f(x) \neq 0$ if and only if $f \notin m$)

Example 23.1.

Claim. Let $U = \{x \in \mathrm{Max}(B) : |f(x)| \leq 1\}$.

Proof of Claim. Let $C = B\langle z\rangle/(z - f)$. We claim that $\mathrm{Sp}(C) \to \mathrm{Sp}(B)$ is an open immersion whose set-theoretic image is $\{x \in \mathrm{Max}(B) : |f(x)| \leq 1\}$.

The universal property of $B\langle z\rangle$ is the following: Given a morphism $B\langle z\rangle \to R$ is the same as specifying a morphism $B \to R$ together with specifying that $z \mapsto a \in R$ where $\{a^n\}$ is bounded in R. Hence the universal property of the quotient $B\langle z\rangle/(z-f)$ is that specifying $B\langle z\rangle/(z-f) \to R$ is the same as specifying $B \to R$ with $f \mapsto$ a power-bounded element of R.

Here is a technical ingredient that we need for this proof: For $R \in \mathrm{Aff}_K$ and $a \in R$, $\{a^n\}_{n\geq 1}$ is bounded if and only if $|a(x)| \leq 1$ for all $x \in \mathrm{Max}(R)$.

Given this fact, the rest of the proof is clear. □

Similarly, $\{x \in \mathrm{Max}(B) : |f| \geq 1\} \leftrightarrow B\langle z\rangle/(zf-1)$ is also an open affinoid subdomain.

Example 23.2.

Claim. Take $B = K\langle z\rangle$, so $\mathrm{Sp}(B)$ is the closed unit disc. The open unit disc $U = \{x \in \mathrm{Max}(B) : |z(x)| < 1\}$ is *not* an affinoid subdomain. In fact, there does not exist a morphism $\mathrm{Sp}(C) \to \mathrm{Sp}(B)$ whose image equals U.

Remark. In fact, there does not exist a morphism $\mathrm{Sp}(C) \to \mathrm{Sp}(B)$ whose image equals U. Indeed, if it exists, then we have $B = K\langle z\rangle \to C$, $z \mapsto f$. This means that for any $x \in \mathrm{Max}(C)$, $|f(x)| < 1$. But there is a maximum principle which says that the function $x \mapsto |f(x)|$ gives a maximum on $\mathrm{Max}(C)$ (uses Noether normalization). But this is a contradiction because $z(x)$ does not achieve a maximum on the open unit disc. (We are secretly using the fact that the absolute value of K is nontrivial.)

We may also show that the punctured disc is not an affinoid subdomain.

24 26 November 2012: Rigid Analytic Spaces

We fix some notation, as usual.

- $(K, |\cdot|)$ a non-Archimedean field
- $\mathrm{Aff}_K = \{K\text{-algebras that arise as quotients of } K\langle z_1, \ldots, z_n\rangle\}$
- $\mathrm{Aff}_K^{\mathrm{op}}$, the opposite category of Aff_K
- We have a map $\mathrm{Aff}_K \to \mathrm{Aff}_K^{\mathrm{op}}$ given by $B \mapsto \mathrm{Sp}(B)$.

24.1 Reminders

1. If $B \in \mathrm{Aff}_K$, then $\mathrm{Max}(B)$ is the "underlying set" of $\mathrm{Sp}(B)$
2. For a morphism $\varphi: B \to C$ of affinoid algebras (i.e. a morphism in Aff_K), then we get an induced morphism $\varphi^*: \mathrm{Max}(C) \to \mathrm{Max}(B)$, given by $m \mapsto \varphi^{-1}(m)$.
3. If φ is as above, then $j: \mathrm{Sp}(C) \to \mathrm{Sp}(B)$ is an *open immersion* if it satisfies one of the following:
   (i) Given any morphism $f: Y \to \mathrm{Sp}(B)$ in $\mathrm{Aff}_K^{\mathrm{op}}$ whose image is (set-theoretically) contained in $j(\mathrm{Sp}(C))$, there exists a unique factorization $Y \to \mathrm{Sp}(C)$ such that the composition with j gives $f: Y \to \mathrm{Sp}(C) \to \mathrm{Sp}(B)$.
   (ii) φ^* is injective and for all $m \in \mathrm{Max}(C)$, φ induces an isomorphism $\widetilde{B}_{\varphi^{-1}(m)} \to \widetilde{C}_m$.

Definition 24.1. If $X \in \mathrm{Aff}_K^{\mathrm{op}}$, a subset $U \subseteq X$ is an (open) *affinoid subdomain* if U is the image of an open immersion $Y \to X$ in $\mathrm{Aff}_K^{\mathrm{op}}$. (The map $Y \to X$ is unique up to unique isomorphism.)

Notation. We will write $Y = \mathrm{Sp}(\mathscr{O}_X(U))$.

Theorem 9 (Tate's Acyclicity Theorem)**.** *The assignment $U \mapsto \mathscr{O}_X(U)$ satisfies the sheaf axioms for finite covers of open affinoid subdomains by open affinoid subdomains. (The Cech complex is acyclic.)*

Why is the class of open affinoids stable under finite intersections? That is, why is the intersection of two open affinoids again an open affinoid?

Proposition 24.1. *Given morphisms $A \to B$, $A \to C$ in Aff_K, the pushout exists and*

$$\mathrm{Max}(B\widehat{\otimes}_A C) \to \mathrm{Max}(B) \times_{\mathrm{Max}(A)} \mathrm{Max}(C).$$

(This is an isomorphism if $K = \overline{K}$. In general, it is an isomorphism if we assume that one of $\mathrm{Sp}(B) \to \mathrm{Sp}(A)$ *or* $\mathrm{Sp}(A) \to \mathrm{Sp}(B)$ *is an open immersion.)*

Sketch of Proof.

Step 1. $A = K$, $B = K\langle z_1, \dots, z_n\rangle$, $C = K\langle w_1, \dots, w_m\rangle$ implies that

$$B\widehat{\otimes}_K C = K\langle z_1, \dots, z_n, w_1, \dots, w_m\rangle.$$

Step 2. If $A = K$ and B, C arbitrary, then we may write B and C as quotients of $K\langle z_1, \dots, z_n\rangle$ and $K\langle w_1, \dots, w_m\rangle$.

Step 3. In this general case, construct $B\widehat{\otimes}_A C$ as a suitable quotient of $B\widehat{\otimes}_K C$. □

Remark. If $\mathrm{Sp}(C) \to \mathrm{Sp}(A)$ is an open immersion, then so is $\mathrm{Sp}(B\widehat{\otimes}_A C) \to \mathrm{Sp}(B)$.

Corollary 24.2.
1. *By dualizing, we get that Aff_K^{op} has fiber products.*
2. *Open immersions in Aff_K^{op} are stable under pullbacks.*
3. *If $X \in \mathit{Aff}_K^{op}$, then the class of open affinoid subdomains of X is closed under finite intersections.*

Remark. The finiteness condition in the acyclicity theorem is very important! Here is a counterexample. Assume $K = \overline{K}$ and take $X = \mathrm{Sp}\, K\langle z\rangle$. Then the underlying set of X is $\mathrm{Max}\, K\langle z\rangle = \{x \in K : |x| \le 1\}$, the closed unit disc. If $x_0 \in K$, $|x_0| \le 1$, and $c \in K^\times$, $|c| < 1$, then $U_{x_0,c} = \{x \in K : |x - x_0| \le |c|\}$ is an affinoid subdomain. In fact, these form a covering over X. That is, $X = \cup_{x_0 \in X} U_{x_0,c}$. But these sets are either equal or do not intersect, and hence this covering does not have the acyclicity property. Indeed, $\mathscr{O}_X(X) = K\langle z\rangle$ is a domain, and hence it cannot be written as a product of rings.

24.2 Main Definitions

Let $X \in \mathrm{Aff}_K^{\mathrm{op}}$.

1. A subset $U \subseteq X$ is *admissible* if given any morphism $f\colon Y \to X$ in $\mathrm{Aff}_K^{\mathrm{op}}$ with $f(Y) \subseteq U$, there exist affinoid subdomains $V_1, \dots, V_n \subseteq X$ such that $f(Y) \subseteq V_1 \cup \cdots \cup V_n \subseteq U$.

2. If $U, U_i \subseteq X$ $(i \in I)$ are admissible subsets such that $U = \cup_{i \in I} U_i$, the cover $\{U_i\}_{i \in I}$ of U is called *admissible* if given a morphism $f\colon Y \to X$ in $\mathrm{Aff}_K^{\mathrm{op}}$ with $f(Y) \subseteq U$, there exists an open affinoid cover $Y = W_1 \cup \cdots \cup W_m$ such that for all j, there exists an i such that $f(W_j) \subseteq U_i$.

3. We may now define connectedness for admissible subsets. The precise formulation is left as an exercise.

Example 24.1. 1. If $B \in \mathrm{Aff}_K$, $f_1, \dots, f_n \in B$, $U_i := \{x \in \mathrm{Max}(B) : f_i(x) \neq 0\}$, and $U := \cup_{i=1}^n U_i$, then U and each U_i is admissible in $X = \mathrm{Sp}(B)$, and the cover $\{U_i\}_{i=1}^n$ of U is also admissible.

Proof. Choose a sequence $c_1, c_2, c_3, \dots \in K^\times$ such that $M_j = |c_j| \to \infty$ as $j \to \infty$. Write $U_i^j = \{x \in U_i : |f_i(x)| \geq M_j^{-1}\}$. It is clear that U_i is the union of all of these and that $U_i^j \cong \mathrm{Sp}\, B\langle z\rangle/(z \cdot c_i \cdot f_i - 1)$. (Reason: The inequality is the same as $|c_j \cdot f_i(x)| \geq 1$. Bounded inverse.) Each U_i^j is an affinoid subdomain of X. □

EXERCISE: If $\alpha\colon Y \to X = \mathrm{Sp}(B)$ is a morphism in $\mathrm{Aff}_K^{\mathrm{op}}$ such that $\alpha(Y) \subseteq U$, then there exists $j >> 0$ such that $\alpha(Y) \subseteq U_1^j \cup \cdots \cup U_m^j$.

2. Take $X = \mathrm{Sp}\, K\langle z\rangle$. Then $U = \{x \in X : |z(x)| < 1\}$ is also admissible. Use a similar argument, except choose $c_1, c_2, \dots \in K^\times$ such that $|c_1| < |c_2| < |c_3| < \cdots$ and $|c_j| \to 1$. A small problem is that this might not be possible (e.g. in $\mathbb{Q}_p$). The solutions is to instead choose $c \in K$ with $0 < |c| < 1$ and use $U^n = \{x \in X : |z(x)^n| \leq |c|\}$. Then we do have the increasing union $U = U^1 \cup U^2 \cup \cdots$. Then use the maximum principle.

25 28 November 2012

Definition 25.1. A G-topology on a set X consists of the following data:

I. A collection $\mathscr{T}$ of subsets of X, called *admissible subsets*, such that $\varnothing, X \in \mathscr{T}$, and $\mathscr{T}$ is closed under finite intersections.

II. For each $U \in \mathscr{T}$, a subcollection of the set of all possible covers $U = \cup_{i\in I} U_i$ (where each $U_i \in \mathscr{T}$), called *admissible covers*.

This data satisfies the following axioms:

1. If $U \in \mathscr{T}$, then $\{U\}$ is an admissible cover of U.

2. If $V \in \mathscr{T}$, $V = \cup_{i\in I} V_i$ is an admissible cover, and $U \subseteq V$ is a subset such that $U \cap V_i \in \mathscr{T}$ for all i, then $U \in \mathscr{T}$.

3. If $U, V \in \mathscr{T}$ and $V = \cup_{i\in I} V_i$ is an admissible cover, then $U \cap V = \cup_{i\in I}(U \cap V_i)$ is also an admissible cover.

4. If $U = \cup_{i\in I} U_i$ is an admissible cover and $U_i = \cup_{j\in J_i} U_{ij}$ is an admissible cover for all $i \in I$, then $U = \cup_{i,j} U_{ij}$ is an admissible cover.

5. If $V \in \mathscr{T}$, $V_j \in \mathscr{T}$, and $V = \cup V_j$ has an admissible refinement, then the cover itself is admissible.

This allows us to have a good theory of sheaves on these spaces.

25.1 Main Example

Let $B \in \mathrm{Aff}_K$ and $X = \mathrm{Max}(B)$ "=" $\mathrm{Sp}(B)$. A subset $U \subseteq X$ is *admissible* if given a morphism $f\colon Y \to X$ in $\mathrm{Aff}_K^{\mathrm{op}}$ with $f(Y) \subseteq U$, there exist affinoid subdomains $V_1, \dots, V_n \subseteq X$ such that $f(Y) \subseteq V_1 \cup \cdots \cup V_n \subseteq U$.

If $U, U_i \subseteq X$ are admissible and $U = \cup_{i\in I} U_i$, this cover is *admissible* if given $f\colon Y \to X$ with $f(Y) \subseteq U$, the cover $Y = \cup_{i\in I} f^{-1}(U_i)$ has a finite refinement consisting of affinoid subdomains of Y.

Remark. 1. All G-topology axioms are satisfied.

2. Suppose $U \subseteq X$ is admissible and let $\{U_i\}_{i\in I}$ be all the affinoid subdomains of X contained in U. Then $U = \cup_{i\in I} U_i$, and this cover is admissible.

3. Given affinoid subdomains $V_1, V_2, \ldots, V_n \subseteq X$, then their union $V = V_1 \cup \cdots \cup V_n$ is admissible and this cover is admissible.

4. A finite cover of an admissible subset by admissible subsets may not be admissible. (Read on for a counterexample!) $\diamond$

Here is a counter example to 4. Indeed, we may take $X = \operatorname{Sp} K\langle z\rangle$, $U = \{x \in X : |z(x)| < 1\}$, $V = X \smallsetminus U = \operatorname{Sp} K\langle z, z^{-1}\rangle$. Note that V is already an affinoid subdomain of X. U is admissible. (Choose $c \in K^\times$ with $|c| < 1$. Then $U = \cup_{n=1}^{\infty} U_n$ where $U_n = \{x \in X : |c^{-1} \cdot z^n(x)| \leq 1\}$. This is the closed disc of radius $|c|^{1/n}$. Then we use a theorem about the image of...?) However, U, V do not form an admissible cover of X.

Recall that for $X \in \mathrm{Aff}_K^{\mathrm{op}}$, for any affinoid subdomain $U \subseteq X$, we have $\mathscr{O}_X(U) \in \mathrm{Aff}_K$. By Tate's acyclicity theorem, this assignment $U \mapsto \mathscr{O}_X(U)$ satisfies the sheaf axioms with respect to finite covers (of affinoid subdomains by affinoid subdomains).

Corollary 25.1. *If $B \in \mathrm{Aff}_K$, then $X = \operatorname{Sp}(B)$ is connected if and only if the algebra B has exactly two idempotents.*

Proof. X is connected if and only if $X \neq \varnothing$ and there does not exist an admissible cover $X = U \cup V$ such that $U, V \neq \varnothing$ and $U \cap V = \varnothing$.

Suppose $f \in B$ is a nontrivial idempotent. Then $U = \{x \in X : f(x) \neq 0\}$ and $V = \{x \in X : f(x) \neq 1\}$ are both affinoid subdomains of X.

Conversely, suppose that $X = U \cup V$ is an admissible cover by admissible sets U, V with $U, V \neq \varnothing$ and $U \cap V = \varnothing$. Apply the definition of an admissible cover to the identity map $X \to X$. Then this cover $\{U, V\}$ has a finite refinement consisting of affinoid subdomains: $U = U_1 \cup \cdots \cup U_m$ and $V = V_1 \cup \cdots \cup V_n$. Apply Tate's theorem to the cover $X = U_1 \cup \cdots \cup U_m \cup V_1 \cup \cdots \cup V_n$. Hence there exists $f \in \mathscr{O}_X(X) = B$ such that $f|_{U_i} = 1$ for all i and $f|_{V_j} = 0$ for al j. Thus f is a nontrivial idempotent. $\square$

Remark. Beware of stalks of sheaves! In general, stalks do not determine everything. On the other hand, $\mathscr{O}_X(X) \hookrightarrow \prod_{x \in X} \mathscr{O}_{X,x}$. The reason: if $x \in X = \operatorname{Max}(B)$, then $\mathscr{O}_{X,x} \to \widehat{B}_x$ and it follows that $B \hookrightarrow \prod_{m \in \operatorname{Max}(B)} \widehat{B}_m$. $\diamond$

26 30 November 2012: Definition of Rigid Analytic Spaces

Definition 26.1. 1. A *G-space* is a set equipped with a G-topology.

2. A *sheaf on a G-space* is a presheaf on the collection of all admissible subsets satisfying the sheaf axioms with respect to admissible covers.

Example 26.1. For $X \in \mathrm{Aff}_K^{\mathrm{op}}$, the assignment

$$\{\text{affinoid subdomains of } X\} \to K\text{-algebras}, \quad U \mapsto \mathscr{O}_X(U)$$

extends uniquely to a sheaf on X. (The stalks are local rings (hopefully artinean), so we get a "locally ringed G-space" over K.)

Definition 26.2. A morphism $X \to Y$ of G-spaces is a map $f\colon X \to Y$ such that for all admissible $U \subseteq Y$, $f^{-1}(U) \subseteq X$ is admissible, and for all admissible covers $U = \cup_{i\in I} U_i$ in Y, the corresponding cover $f^{-1}(U) = \cup_{i\in I} f^{-1}(U_i)$ is admissible in X.

Remark. Here are some obvious things to notice:

- The functor $f_*\colon \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ has a left adjoint $f^{-1}\colon \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$.
- Also have $R^n f_*\colon \mathrm{Ab}(X) \to \mathrm{Ab}(Y)$, and hence in particular, we have cohomology for sheaves of abelian groups.
- Also have the notion of a morphism of locally ringed G-spaces.

The above allows us to make a definition of a rigid analytic space. ◇

Definition 26.3. A *rigid analytic space* over K is a locally ringed G-space $(X, \mathscr{O}_X)$ over K such that there exists an admissible (!) cover $X = \cup_{i\in I} U_i$ such that each $(U_i, \mathscr{O}_X|_{U_i}) \cong \mathrm{Sp}(B_i)$ for some $B_i \in \mathrm{Aff}_K$.

26.1 Connection to Usual Notions of Algebraic Geometry

We will next discuss schemes that are locally of finite type over K. We would like to understand a functor

$$(\text{schemes locally of finite type over } K) \to \mathrm{Rig}_K, \quad Y \to Y^{\mathrm{an}}.$$

This is very similar to the functor (schemes over $\mathbb{C}$) → (complex-analytic spaces).

Caution. Y^{an} is almost never an affinoid space over K. In particular, can only get Y^{an} to be an affinoid space over K if Y is spec of a finite K-algebra.

Remark. We have an embedding

$$\mathrm{Rig}_K \hookrightarrow \text{Functors}(\mathrm{Aff}_K, \text{Sets}), \quad X \mapsto \text{the functor } B \mapsto \mathrm{Hom}_{\mathrm{Rig}_K}(\mathrm{Sp}(B), X).$$

Proposition 26.1. *If Y is a locally finite type K-scheme, the functor*

$$\mathit{Aff}_K \to \mathit{Sets}, \quad B \mapsto \mathrm{Hom}(\mathrm{Spec}(B), Y),$$

with Hom meaning as K-schemes, is representable by $Y^{an} \in \mathit{Rig}_K$.

Proof. Everything reduces to $Y = \mathbb{A}^1_K$ (so get $\mathbb{A}^n_K$) and then we get all affine schemes of finite type over K and then finally, we glue to get everything else.

$(\mathbb{A}^1_K)^{\mathrm{an}}$ has to represent $\mathrm{Aff}_K \to \text{Sets}$, $B \mapsto B$.

(For comparison, $\mathrm{Sp}\, K\langle z\rangle$ represents the functor $B \mapsto B^\circ$, where $B^\circ = \{b \in B : \{b^n\}_{n=1}^\infty \text{ is bounded in } B\}$.)

Remark. For any $b \in B$, there exists $c \in K^\times$ such that $cb \in B^\circ$.

Choose a sequence $c_1, c_2, \ldots \in K$ such that $1 < |c_1| < |c_2| < \cdots$ and $|c_i| \to \infty$ as $i \to \infty$. Consider $A_i = K\langle z/c_i\rangle \subseteq K[[z]]$. Each $\mathrm{Sp}(A_i)$ is an affinoid space over K, and $A_{i+1} \subseteq A_i$ induces an open immersion $\mathrm{Sp}(A_i) \hookrightarrow \mathrm{Sp}(A_{i+1})$.

Now take $X = \varinjlim_i \mathrm{Sp}(A_i)$. Then $\mathrm{Hom}_{\mathrm{Rig}_K}(\mathrm{Sp}(B), X) = B$ for all $B \in \mathrm{Aff}_K$. □

Remark. We have a canonical morphism of G-spaces $Y^{\mathrm{an}} \to Y$ which set-theoretically is a bijection onto the set of closed points of Y.

Pick an admissible affinoid subspace $U \subseteq Y^{\mathrm{an}}$. So $U = \mathrm{Sp}(B)$. We have $j\colon \mathrm{Sp}(B) \hookrightarrow Y^{\mathrm{an}}$ and a corresponding K-scheme morphism $\mathrm{Spec}(B) \to Y$. Then $\mathrm{Sp}(B) \to \mathrm{Spec}(B)$ and we can glue. How do we construct $\mathrm{Sp}(B) \to \mathrm{Spec}(B)$? There is an obvious inclusion $f\colon \mathrm{Max}(B) \hookrightarrow \mathrm{Spec}(B)$ and this is a morphism of G-spaces. Reason: Recall that we proved (on Monday?) that f^{-1} takes finite covers by basic affine opens in $\mathrm{Spec}(B)$ to admissible covers in $\mathrm{Max}(B)$. The rest of this justification is left as an exercise. (Use the maximum principle!)

26.2 Things That Also Work

1. Coherent sheaves on rigid spaces
2. Cartan's Theorems A and B on affinoid spaces. (Theorem A says BLAH and Theorem B says BLAH.)
3. Theorems of GAGA

27 7 December

Recall from last time: Let $(K, |\cdot|)$ be a complete non-Archimedean field (discretely valued). Have a uniformiser $\pi \in \mathscr{O}_K \subseteq K$. Nice formal schemes over $\mathscr{O}_K$: Start with X a scheme that is locally finite type over $\mathscr{O}_K$, $f\colon X \to \operatorname{Spec}(\mathscr{O}_K) = \{(0), (\pi)\}$ a structure morphism. Complete X along the special fiber. Take $Y = f^{-1}((\pi)) \subseteq X$ (a closed subspace). The formal completion $\widehat{X}_Y$ is the space Y with the structure sheaf given by the obvious inverse sheaf, i.e. with structure sheaf $\varprojlim_n \mathscr{O}_X(\pi^n \cdot \mathscr{O}_X)$. This is a *nice* formal $\mathscr{O}_K$-scheme.[4] (This is the same as Hartshorne's completion of a noetherian scheme (sheaf?).)

We need to handle things like $\widehat{X}_Y$ where $Y \subseteq f^{-1}((\pi))$ is a closed subset (but not necessarily the whole fiber).

Example 27.1. Take $X = \operatorname{Spec} \mathscr{O}_K[t]$. Then $\widehat{X}_{f^{-1}}((\pi)) = \operatorname{Spf}(\mathscr{O}_K\langle t\rangle)$ (this is nice). Also, $\widehat{X}_{\text{ideal}(\pi,t)} = \operatorname{Spf}(\mathscr{O}_K[[t]])$ (this is not nice).

Generalize (the construction $Y \mapsto Y \otimes_{\mathscr{O}_K} K$ from last time) to *special*[5] formal schemes over $\mathscr{O}_K$.[6] An *affine* special formal schemes correspond to $\mathscr{O}_K$-algebras of the form $A = \mathscr{O}_K\langle z_1, \dots, z_n\rangle[[w_1, \dots, w_m]]/I$, where I is a closed ideal and the topology comes from the ideal $(\pi, w_1, \dots, w_m)$.

Given such an A, we can form the rigid generic fiber $\operatorname{Spf}(A) \otimes_{\mathscr{O}_K} K$ as a rigid analytic space over K. (can never get an open unit disc from this construction.)

27.1 Very ad hoc construction

Define $\overline{D} = \operatorname{Sp}(K\langle x\rangle)$ (closed unit disc) and let $\mathring{D}$ be the open unit disc inside $\overline{D}$. Then $\operatorname{Spf}(A) \otimes_{\mathscr{O}_K} K$ is the closed subspace of $\overline{D}^n \times \mathring{D}^n$ cut out by the equations coming from I.

27.1.1 Motivation

Consider $\mathscr{O}_K\langle z\rangle \otimes_{\mathscr{O}_K} K = K\langle z\rangle$ (because coefficients of higher powers of z are already in $\mathscr{O}_K$ by definition of restricted formal power series). That is, $K\langle z\rangle = \{f \in K[[z]] : f(\alpha) \text{ converges for all } \alpha \in \overline{K} \text{ with } |\alpha| \le 1\}$. Also, $\mathscr{O}_K[[w]] \otimes_{\mathscr{O}_K} K = \{f = \sum_{i=0}^\infty a_i w^i \in K[[w]] : \{|a_i|\}_{i=1}^\infty \text{ is bounded}\} = \{f \in K[[w]] : f(\alpha) \text{ converges for all } \alpha \in \overline{K} \text{ with } |\alpha| < 1\}$.

[4] This terminology—"nice"—is not used in the literature.

[5] Berkovich

[6] References: Berthelot (not published, but available online), and de Jong (Crystalline Dieudonne theory...).

27.2 Lubin-Tate Tower

Let F be a local field. In this story, we will take $K = \widehat{F}^{\mathrm{nr}}$.

Earlier, we constructed "moduli spaces" $\mathrm{Spf}(\mathscr{A}_n)$ where $n = 0, 1, 2, \ldots$ as follows.

1. Fix Σ to be a formal $\mathscr{O}_F$-module over $\overline{\mathbb{F}}_q = \mathscr{O}_K/(\pi)$ of height h (unique up to $\cong$).

2. Deformation functor $\mathrm{Def}^n_\Sigma \colon \mathscr{C} \to \mathrm{Sets}$ taking $R \mapsto \{(\mathscr{F}, \varphi)\}/\cong$, where:

   (a) $\mathscr{F}$ is a formal $\mathscr{O}_F$-module over R such that $(\mathscr{F} \mod m_R) = \Sigma$.

   (b) If $n = 0$, there is no φ.

   (c) If $n \geq 1$, then φ is a level n structure on $\mathscr{F}$, i.e., an $\mathscr{O}_F$-module homomorphism $(\pi^{-n}\mathscr{O}_F/\mathscr{O}_F)^h \to \mathscr{F}(R)$ such that $\prod_{\alpha(\pi^{-1}\mathscr{O}_F/\mathscr{O}_F)^h}(x - \varphi(\alpha))$ divides $[\pi]_{\mathscr{F}}(x)$ in $R[[x]]$.

   Here, $\mathscr{C}$ is the category of complete local $\mathscr{O}_K$-algebras R such that $\overline{\mathbb{F}}_q \xrightarrow{\cong} R/m_R$.

   Then Def^n_Σ is representable by $\mathscr{A}_n \in \mathscr{C}$.

   Remark. Recall that $\mathscr{A}_0 \cong \mathscr{O}_K[[u_1, \ldots, u_{h-1}]]$ (noncanonically) and for $n \geq 1$, we have $\mathscr{A}_n \cong \mathscr{O}_K[[x_1, \ldots, x_n]]/(\text{principal ideal})$ (canonically).

   We have $\mathscr{A}_0 \to \mathscr{A}_1 \to \ \to \mathscr{A}_n$ and each of these maps is finite and flat.

Definition 27.1. Define $\mathfrak{X}_n := \mathrm{Spf}(\mathscr{A}_n) \otimes_{\mathscr{O}_K} K$. These are rigid analytic spaces over $K = \widehat{F}^{\mathrm{nr}}$. We get $\mathfrak{X}_0 \leftarrow \mathfrak{X}_1 \leftarrow \cdots$. This is called the *Lubin-Tate tower*. $\mathfrak{X}_0$ is an open polydisc of dimension $n - 1$. For $n \geq 1$, $\mathfrak{X}_n$ is a closed subspace of $\overset{\circ}{D}{}^h$ cut out by one analytic equation.

Relation between $\mathfrak{X}_1$ and the Drinfeld curve! (Discuss on Monday.)

27.3 Group actions on the tower

1. $\mathrm{GL}_n(\mathscr{O}_F)$ acts[7] on each $\mathfrak{X}_n$ through the quotient $\mathrm{GL}_h(\mathscr{O}_F/(\pi^n))$. In fact, $\mathrm{GL}_h(\mathscr{O}_F/(\pi^n))$ already acts on $\mathscr{A}_n$ by changing the level structures: $\varphi \colon (\pi^{-1}\mathscr{O}_F/\mathscr{O}_F)^h \to \mathscr{F}(R)$.

   In fact, $\mathfrak{X}_n \to \mathfrak{X}_0$ is an étale cover and a torsor for $\mathrm{GL}_h(\mathscr{O}_F/(\pi^n))$ acting on $\mathfrak{X}_n$.

[7] We ultimately want $\mathrm{GL}_h(F)$, but $\mathrm{GL}_h(\mathscr{O}_F)$ is a pretty big open subgroup of $\mathrm{GL}_h(F)$, so this is getting there).

2. $\mathrm{Aut}(\Sigma)$ acts on each Def_Σ^n and hence also on $\mathscr{A}_n$ and on $\mathfrak{X}_n$. In fact, $\mathrm{Aut}(\Sigma) \cong \mathcal{O}_D^\times$. What is $\mathcal{O}_D^\times$?

   Notation. Take $E \supseteq F$ to be the unique unramified degree-h extension of F. Then $\mathrm{Gal}(E/F) = \langle \varphi \rangle$, where φ is the Frobenius. Then take $\mathcal{O}_D = \mathcal{O}_E\{\Pi\}/(\Pi^h - \pi)$. The ring $\mathcal{O}_E\{\Pi\}$ is the twisted polynomial ring with relation $\Pi \cdot a = \varphi(a) \cdot \Pi$ for all $a \in \mathcal{O}_E$. So $D = \mathcal{O}_D[\pi^{-1}]$, and this is a central division algebra over F of dimension h^2.

Upshot: Not only do we get the local Langlands correspondence, but we also get the Jacquet-Langlands correspondence (relating representations of $\mathrm{GL}_h(F)$ and representations of $D^\times$).

28 10 December 2012: Integral Models

Let $(K, |\cdot|)$ be discretely valued, complete, non-Archimedean field with uniformiser π. Let Y be a rigid analytic space over K. Does there exist a (nice or special) formal scheme of X over $\mathscr{O}_K$ such that $X \otimes_{\mathscr{O}_K} K \cong Y$? In general, this is *hard* and there does not exist any good unique statements.

28.1

Suppose $Y = \mathrm{Sp}(B)$ for some $B \in \mathrm{Aff}_K$ and we ask for affine admissible integral models, i.e., an $\mathscr{O}_K$-subalgebra $A \subseteq B$, which is π-adically complete and topologically finitely generated over $\mathscr{O}_K$ with $B = K \cdot A$.

Remark. Such an A always exists! Indeed, we may choose a surjection $f\colon K\langle z_1, \ldots, z_n\rangle = T_n \twoheadrightarrow B$ (this is the Tate algebra in n variables) and take $A = f(\mathscr{O}_K\langle z_1, \ldots, z_n\rangle)$. ◇

In general, if $A \subseteq B$ satisfies the above properties, then $A \subseteq B^\circ$ (where B° denotes the set of power-bounded elements of B, i.e. $B^\circ := \{b \in B : \{b^k\}_{k=1}^{\infty} \text{ is bounded}\}$).

It is easy to see that if $\mathrm{Sp}(B)$ has a "maximal" integral model, then that model has to be $\mathrm{Spf}(B^\circ)$. The problem is that B° is not always topologically finitely generated over $\mathscr{O}_K$.

Example 28.1. Take $B = K\langle z\rangle/(z^2)$. Let $B^\circ = \mathscr{O}_K + K \cdot z$.

Lemma. *If B is reduced, then B° is π-adically complete and topologically finitely generated over $\mathscr{O}_K$.*

Proof. Noether normalization implies that there exists an injective K-algebra homomotphism $f\colon T_n \hookrightarrow B$ such that B is finite over T_n.

Claim 12. B° is finite over $T_n^\circ = \mathscr{O}_K\langle z_1, \ldots, z_n\rangle$.

Proof of Claim. First reduce to the case where B is a domain. We have an embedding $B \hookrightarrow \prod_{\mathfrak{p} \subseteq B}(B/\mathfrak{p})$, where the product varies over minimal primes, and this is a finite product of domains.

We have a standard fact: B° is integral over T_n°. Also, the field of fractions of B° is a finite extension of the field of fractions of T_n°. So "clearly," T_n° is Japanese. □

The claim completes the proof of the lemma. □

28.2 How do you compute B°?

Lemma. *Let $B \in \mathit{Aff}_K$ be reduced. Suppose $f\colon T_n \twoheadrightarrow B$ is a surjection in Aff_K and put $C = f(T_n^\circ) \subseteq B^\circ$. (Remember that $T_n^\circ \cong \mathscr{O}_K\langle z_1, \ldots, z_n\rangle/(\ker f \cap T_n^\circ)$.) If $C/\pi C$ is smooth over $\mathscr{O}_K/(\pi)$, then $C = B^\circ$.*

Sketch. First reduce to the case where B is an integral domain. Then C is a domain and C is formally smooth over $\mathscr{O}_K$. This implies that C is normal.

But B° is integral over C and they have the same field of fractions. □

28.3 Application to the Drinfeld hypersurface

(The computation for the curve is exactly the same.)

Let $F = \mathbb{F}_q((\pi))$ and pick $n \geq 2$. Recall the Lubin-Tate tower of complete $\widehat{\mathscr{O}}_F^{\mathrm{nr}}$-algebras

$$\mathscr{A}_0 \hookrightarrow \mathscr{A}_1 \hookrightarrow \mathscr{A}_2 \hookrightarrow \cdots,$$

where $\mathscr{A}_0 \cong \widehat{\mathscr{O}}_F^{\mathrm{nr}}[[u_1, \ldots, u_{h-1}]]$ (non-canonically) and $\mathscr{A}_1 \cong \widehat{\mathscr{O}}_F^{\mathrm{nr}}[[x_1, \ldots, x_h]]/(\pi - \prod_{a \in \mathbb{F}_q^h \smallsetminus \{0\}}(a_1x_1 + \cdots + a_hx_h))$ (canonically).

We have a Lubin-Tate tower of rigid analytic spaces $\mathfrak{X}_n = (\operatorname{Spf} \mathscr{A}_n) \otimes_{\widehat{\mathscr{O}}_F^{\mathrm{nr}}} F^{\mathrm{nr}}$:

$$\mathfrak{X}_0 \leftarrow \mathfrak{X}_1 \leftarrow \mathfrak{X}_2 \leftarrow \cdots.$$

Recall that $\mathfrak{X}_0$ is an open unit polydisc of dimension $h-1$ and $\mathfrak{X}_1$ is the closed subspace of $\mathring{D}^h$ cut out by the equation $\prod_{a \in \mathbb{F}_q^h \smallsetminus \{0\}}(a_1x_1 + \cdots + a_hx_h) = \pi$.

We would like to interpret the above geometrically. We do this in two steps:

1. Let $Y_1 = \mathfrak{X}_1 \cap D'^h$ where $D' = \mathring{D}$ is the closed disc of radius $|\pi|^{1/(q^h-1)}$.

   Explicitly: The closed unit disc is $\operatorname{Sp} K\langle z\rangle$ and $D' = \operatorname{Sp}\langle z, w\rangle/(w - \pi^{-1} \cdot z^{q-1})$.

   Motivation: If we were allowed to make the substitution $x_i = \pi^{1/(q^h-1)} \cdot y_i$, then we would get the equation for the Drinfeld curve. But this doesn't really make sense. Step 2 is to make sense of this.

2. Take $K = \widehat{F}^{\mathrm{nr}}[\omega]/(\omega^{q^h-1} - \pi)$. Here, $\omega \in K$ is a uniformiser. We now base change to K.

   Now,
$$Y_1 \otimes_{\widehat{F}^{\mathrm{nr}}} K \cong \operatorname{Sp}(K\langle y_1, \ldots, y_h\rangle/(\prod_{a \in \mathbb{F}_q^h \smallsetminus \{0\}}(a_1x_1 + \cdots + a_hx_h) - 1)).$$

Let the inside of the Sp be B. And let C be the image of $\mathscr{O}_K\langle y_1, \ldots, y_h\rangle$ in B.

Then $C/\omega C \cong \overline{\mathbb{F}}_q[y_1, \ldots, y_n]/(\text{same relation})$. So $\mathrm{Spec}(C/\omega C)$ is the Drinfeld hypersurface X and it is clear that X is smooth. The upshot is that the formal scheme $\mathrm{Spf}(C)$ is the canonical integral model of the affinoid $Y_1 \otimes_{\widehat{F}^{\mathrm{nr}}} K$ and the special fiber of this model is X.

Remark. $\mathrm{GL}_h(\mathscr{O}_F/(\pi)) = \mathrm{GL}_h(\mathbb{F}_q)$ acts on $\mathfrak{X}_1$. It preserves the affinoid subspace Y_1 and so $\mathrm{GL}_h(\mathbb{F}_q)$ acts on Y_1 and therefore also acts on $Y_1 \otimes_{\widehat{F}^{\mathrm{nr}}} K$ and hence on C and the hypersurface X. Moreover, the action on X given by this construction is the obvious one.

But this is not the end of the story. We also have an action of $\mathrm{Gal}(K/\widehat{F}^{\mathrm{nr}})$ on X and this action commutes with the action of $\mathrm{GL}_h(\mathbb{F}_q)$. Now, $\mathrm{Gal}(K/\widehat{F}^{\mathrm{nr}}) \cong \mathbb{F}_{q^h}^\times$. This is how you see the full picture for the Drinfeld hypersurface.

$H_c^{h-1}(X, \overline{\mathbb{Q}}_\ell)$ is a representation of the product of these two groups, and this is how one realizes, geometrically, all the cuspidal representations of $\mathrm{GL}_h(\mathbb{F}_q)$. ◇

29 12 December 2012

29.1 Computation of $\mathrm{End}_{\mathscr{O}_F}(\Sigma)$

Let $E \supseteq F$ be an unramified extension of degree h (so the residue field of E is $\mathbb{F}_{q^h}$) and let $f(x) = \pi \cdot x + x^{q^h}$. There is a unique formal $\mathscr{O}_E$-module $\widetilde{\Sigma}$ such that (got erased).

By construction, Σ is already defined over $\mathbb{F}_q \subseteq \overline{\mathbb{F}}_q$ and $[\pi]_\Sigma(x) = x^{q^h}$. In particular, $\sigma(x) = x^q \in \mathrm{End}_{\mathscr{O}_F}(\Sigma)$. You also have $\mathscr{O}_E$ acting on Σ by $\mathscr{O}_F$-endomorphisms. Also, σ^h acts in the same way as π.

Exercise 4. This is an easy exercise. If $a \in \mathscr{O}_E$, then

$$\sigma \circ [a]_\Sigma = [\varphi(a)]_\Sigma \circ \sigma,$$

where $\varphi \in \mathrm{Gal}(E/F)$ is the Frobenius generator.

So we get an $\mathscr{O}_F$-algebra homomorphism $\mathscr{O}_{D_{1/h}} = \mathscr{O}_E\{\sigma\}/(\sigma^h - \pi) \to \mathrm{End}_{\mathscr{O}_F}(\Sigma)$. An idea for checking that it is an isomorphism: check that you get an isomorphism modulo π (nontrivial) and check π-adic completion of both sides (easy). (Checking that you get an isomorphism modulo π involves the following: For $a_0 x + a_1 x^q + a_2 x^{q^2} + \cdots + a_{h-1} x^{q^{h-1}}$, one can add higher x^{q^n} terms to get an endomorphism.)

Theorem 10. *If ρ is a smooth irreducible representation of $G := \mathrm{GL}_n(F)$ over $\mathbb{C}$, then there is a locally constant function Θ_ρ on $G^{rss} \subseteq G$ such that for all $f \in C_c^\infty(G^{rss}) \subseteq C_c^\infty(G)$,*

$$\mathrm{tr}(\rho(f)) = \int_G f(g)\Theta_\rho(g)\, d\mu(g).$$

Remark. Recall that

$$\rho(f) = \int_G f(g)\rho(g)\, d\mu(g),$$

where μ is some Haar measure. This operator has finite rank and hence has a well-defined trace.

On the division algebra side... Let D be any central division algebra of dimension n^2 over F. Then we have $D^\times$ (this is an ℓ-group), which has center $F^\times$ and furthermore, $D^\times/F^\times$ is *compact*. (Hence the representation theory of D is much simpler than the representation theory of $\mathrm{GL}_n(F)$!)

We have an embedding of the set of conjugacy classes of $D^\times$ to the set of (semisimple) conjugacy classes of $\mathrm{GL}_n(F)$. (This map is very far from being surjective.) This is given by the following:

1. (Completely general) If $a \in D$, then $F[a] \subseteq D$ is a finite field extension of F. Moreover, the degree $[F[a] : F]$ divides n.

2. (Partial converse) If $K \supseteq F$ is a finite field extension such that the degree $[K : F]$ divides n, then there exists an F-embedding $K \hookrightarrow D$. (This is only true for local fields F.)

   (Completely general) Any two such embeddings are conjugate under $D^\times$. (Special case of Skoelem-Noether theorem)

3. The upshot is that we now get a bijection

   $$\{\text{conjugacy classes of } D^\times\} \leftrightarrow \{\text{reduced characteristic polynomial}\} \smallsetminus \{x^n\},$$

   where by "reduced characteristic polynomial," we mean a monic degree-n element of $F[x]$ which is a power of an irreducible polynomial.

Theorem 11 (Local Jacquet-Langlands Correspondence)**.** *Let ρ' be a smooth irreducible representation of $D^\times$ over $\mathbb{C}$. (This representation has to be finite-dimensional because the quotient of $D^\times$ by its center is compact!) Then there exists a smooth irreducible representation ρ of $\mathrm{GL}_n(F)$ over $\mathbb{C}$ such that $\mathrm{tr}(\rho') \leftrightarrow (-1)^{n-1} \cdot \Theta_\rho$.*

More precisely, $\mathrm{tr}(\rho') \leftrightarrow (-1)^{n-1} \cdot \Theta_\rho$ means that the values agree under the bijection

$$(\textit{regular elliptic } D^\times\textit{-conjugacy classes}) \longleftrightarrow (\textit{regular elliptic } \mathrm{GL}_n(F)\textit{-conjugacy classes}),$$

where regular elliptic means that the minimal polynomial of your element over F is separable of degree n.

($\mathscr{L}^1$ functions are not well-defined. So we look at the locus where the function is locally constant and then we have an honest function. This locus might actually bigger than G^{rss}, but it certainly contains this.)

(The regular elliptic is not dense because there is no way to perturb a bit to get an irreducible char poly to become reducible (i.e. have multiplicity > 1).)

(If you have a matrix whose eigenvalues are in F, it is not in the closure of the set of regular elliptic $\mathrm{GL}_n(F)$-conjugacy classes.)

29.2 Some Remarks

1. The Local Langlands philosophy is the following one. Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be reductive groups over F that become isomorphic over some extension of F. Then the representation theories of $\mathbb{G}_1(F)$ and $\mathbb{G}_2(F)$ are very closely related.

2. Let D_1, D_2 be central division algebras of dimension n^2 over F. Then we get a canonical bijection between conjugacy classes in $D_1^\times$ and $D_2^\times$. That is, we have a bijection

$$D_1^\times/\text{conjugacy} \longleftrightarrow D_2^\times/\text{conjugacy}.$$

Then Jacquet-Langlands gives a bijection between

$$\operatorname{Irr}(D_1^\times) \longleftrightarrow \operatorname{Irr}(D_2^\times),$$

where Irr denotes the set of isomorphism classes of irreducible representations. (Characters match!)

Look at $\mathscr{H}_1 = C_c^\infty(D_1^\times)$. This contains $\mathscr{H}_1^{D_1^\times}$, which is where the characters live (uh oh—this is not true). Take three conjugacy classes $C_1, C_2, C_3 \subseteq D_1^\times$. The "number of times" that C_3 appears in $C_1 \cdot C_2$ is $\alpha_{C_1,C_2}^{C_3} \in \mathbb{Z}_{\geq 0}$.

We get a vector space isomorphism

$$\mathscr{H}_1^{D_1^\times} \xrightarrow{\cong} \mathscr{H}_2^{D_2^\times}.$$

Reformulation: This is an algebra isomorphism with respect to convolution. (One way to check this: If you figure out the right way to define the structure constants, one way to check the isomorphism is to check that the structure constants agree. (This is not easy.))

If D_1 has dimension k^2 and D_2 has dimension n^2, $k \mid n$, then we have an injection $\operatorname{Irr}(D_2^\times) \hookrightarrow \operatorname{Irr}(\mathrm{GL}_{n/k}(D_1))$. (No good way to have an increasing system of division algebras and make sense of the resulting structure.)

3. Going back to the Jacquet-Langlands correspondence. We have

$$\operatorname{Irr}(D^\times) \hookrightarrow \operatorname{Irr}(\mathrm{GL}_n(F)).$$

What is the image? The answer is: "essentially $\mathscr{L}^2$ (mod center) representations". This class includes all supercuspidals.

"$\mathscr{L}^2$ mod center" means

$$\rho\colon \mathrm{GL}_n(F) \to \mathrm{GL}(V)$$

is a smooth irreducible representation. By Schur's lemma, it has a central character.

(a) χ_ρ is a central character of $\mathrm{GL}_n(F)$ ($F^\times \xrightarrow{x_\rho} \mathbb{C}^\times$ is a smooth homomorphism) and takes values in $S^1 \subseteq \mathbb{C}^\times$.

(b) every matrix coefficient of ρ is $\mathscr{L}^2$ modulo the center of $\mathrm{GL}_n(F)$.
(Recall that this means $v \in V$, $f \in V^\vee :=$ smooth dual, get a matrix coefficient (i.e. function) $g \mapsto f(\rho(g)(v))$.)

Essentially $\mathscr{L}^2$ mod center means that there exists a smooth $\eta\colon \mathrm{GL}_n(F) \to \mathbb{C}^\times$ such that $\eta \cdot \rho$ is $\mathscr{L}^2$ modulo the center.

4. If n is a prime, then it is easy to say what all of these representations are. That is, the essentially $\mathscr{L}^2$ mod center smooth irreducible representations of $G = \mathrm{GL}_n(F)$ are the supercuspidals (matrix coefficients have compact support modulo the center) and $(\chi \cdot \det) \cdot \mathrm{St}_G$, where χ is a character of $F^\times$. (Recall that St_G is the unique irrep inside the induced trivial representation from the Borel.)

("structure constant"?)

$\mathrm{GL}_n(F)$-side	$D^\times$-side
$(\chi \circ \det) \cdot \mathrm{St}_G$	$\chi \circ Nrd_D$
supercuspidals	smooth irreps of $D^\times$ of dimension > 1

(Nrd comes from the reduced determinant) (???)
Note that the picture is only this simple when n is prime!

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Correction of last time. Let D be a central division algebra over F of dimension n^2. Choose $F \subseteq E \subseteq D$, where E is an unramified degree n field extension of F and let $\varphi \in \mathrm{Gal}(E/F)$ be the Frobenius generator. Then there exists $\omega \in D^\times$ such that ω normalizes E and $\omega a \omega^{-1} = \varphi(a)$ for all $a \in E$ (*). Then $\omega^n \in C_D(E) = E$, so $\omega^n = u \cdot \pi^k$ for some $u \in \mathscr{O}_E^\times$, $k \in \mathbb{Z}$.

Assume $\omega^n \in F$. Without loss of generality, $0 \leq k \leq n-1$. If ω with $v \cdot \omega$ for some $v \in \mathscr{O}_E^\times$, then (*) does not change, and ω^n becomes $v \cdot \varphi(v) \cdot \varphi^2(v) \cdots \varphi^{n-1}(v)\omega^n = \mathrm{N}_{E/F}(v) \cdot \omega^n$. But $\mathrm{N}_{E/F}\colon \mathscr{O}_E^\times \to \mathscr{O}_F^\times$ is surjective and hence we may assume that $\omega^n = \pi^k$.

Claim 13. In the above, $\gcd(k, n) = 1$.

Proof of Claim. Let A be the F-subalgebra of D generated by E and ω. Since $\omega^n \in F$, then showing $\gcd(k, n) = 1$ is equivalent to showing that $A = D$. (ω is a root of the polynomial $x^n - \pi^k$ and $k \mid n$, then this polynomial would be reducible.) Clearly, A is a finite-dimensional semisimple algebra over F and A is a product of simple F-algebras. Also, $C_D(A) \subseteq E^\varphi = F$. In particular, A has center F. Therefore A is simple and $A = C_D(C_D(A)) = D$. □

30.1 Fairy Tales about the Local Langlands Correspondence

(This will be extremely imprecise.)

The statement gives a certain construction of a map that is a bijection between the Galois side and the GL_n side:

$$\begin{pmatrix} n\text{-dimensional Frobenius-semisimple} \\ \text{Weil-Deligne representations of } \mathscr{W}_F \end{pmatrix} \longleftrightarrow \begin{pmatrix} \text{smooth irreducible} \\ \text{representations of } \mathrm{GL}_n(F) \end{pmatrix}.$$

The correspondence takes a pair (ρ, N) to a representation $\rho(\sigma, N)$. Our representations will be taken over $\mathbb{C}$.

Let $\sigma\colon \mathscr{W}_F \to \mathrm{GL}(V)$ be a smooth representation over $\mathbb{C}$. We have a linear map $N\colon V \to V$ such that $\sigma(\gamma) \cdot N \cdot \sigma(\gamma)^{-1} = \|\gamma\| \cdot N$ for all $\gamma \in \mathscr{W}_F$.

Recall:

$$1 \to \mathscr{I}_F \to \mathscr{W}_F \to \mathbb{Z} \to 0.$$

The geometric Frobenius $(1/q)$ in $\mathscr{W}_F$ maps to 1 via $v\colon \mathscr{W}_F \to \mathbb{Z}$. Recall that $\|\gamma\| = q^{-v(\gamma)}$.

We have a map $\mathrm{rec}_F\colon \mathscr{W}_F \to F^\times$ (and hence $\mathscr{I}_F \to \mathscr{O}_F^\times$) that makes

$$1 \to \mathscr{O}_F^\times \to F^\times \to \mathbb{Z} \to 0,$$

where $F^\times \to \mathbb{Z}$ is the usual valuation. $\|\gamma\| = \|\operatorname{rec}_F(\gamma)\|$ where $\|\cdot\|$ is the standard normalized absolute value on F.

Frobenius-semisimple means that σ is semisimple.

Question: How to characterize this bijection uniquely? That is, how to define $\rho(\sigma, N)$.

For $n = 2$, this is uniquely characterized by meromorphic functions (actually rational functions in q^{-s}):

1. L-factor:
$$L((\sigma, N) \otimes \chi, s) = L(\rho(\sigma, N) \otimes \chi, s)$$
for all smooth $\chi\colon F^\times \to \mathbb{C}^\times$.

2. ϵ-factor:
$$\varepsilon((\sigma, N) \otimes \chi, s, \psi) = \varepsilon(\rho(\sigma, N) \otimes \chi, s, \psi)$$
for all χ and for all nontrivial smooth $\psi\colon F \to \mathbb{C}^\times$.

Properties 30.1.

1. (σ, N) is irreducible ($N = 0$ in this case) if and only if $\rho(\sigma, N)$ is supercuspidal.

2. $(\sigma, N) \mapsto \rho(\sigma, N)$ is compatible with twists by one-dimensional characters.

3. Compatibility with (smooth) duals

4. $\det(\sigma)$ is a one-dimensional representation and corresponds to some character of $F^\times$, and this character is the central character of $\rho(\sigma, N)$ via local class field theory.

5. (σ, N) is indecomposable if and only if $\rho(\sigma, N)$ is essentially $\mathscr{L}^2$ modulo the center.

Remark. Take $n = 2$. Then (σ, N) is indecomposable if and only if either σ is irreducible or $N \neq 0$.

If $N \neq 0$, then without loss of generality, we may assume that $N = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $\sigma\colon \mathscr{W}_F \to \mathrm{GL}_2(\mathbb{C})$ is a smooth homomorphism. The image of σ has to normalize the line spanned by N. Hence we may write
$$\sigma(\gamma) = \begin{pmatrix} \chi_1(\gamma) & * \\ 0 & \chi_2(\gamma) \end{pmatrix},$$
where $\chi_1, \chi_2\colon F^\times \to \mathbb{C}^\times$ (actually characters of the Weil group) are smooth characters.

We have $\sigma(\gamma) \cdot N = \|\gamma\| \cdot N \cdot \sigma(\gamma)$, which implies that $\chi_1(\gamma) = \|\gamma\| \cdot \chi_2(\gamma)$.

Conversely, suppose $\chi_1, \chi_2 \colon F^\times \to \mathbb{C}^\times$ are smooth and satisfy the property $\chi_1/\chi_2 = \|\cdot\|$. Then we get a two-dimensional Weil-Deligne representation (σ, N), where $N = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $\sigma(\gamma) = \begin{pmatrix} \chi_1(\gamma) & 0 \\ 0 & \chi_2(\gamma) \end{pmatrix}$. In this case, $\rho(\sigma, N) = \mathrm{St}_{\mathrm{GL}_2(F)} \otimes (\omega \circ \det)$, where $\omega \colon F^\times \to \mathbb{C}^\times$ is defined by $\chi_1 = \omega \cdot \|\cdot\|^{1/2}$, $\chi_2 = \omega \cdot \|\cdot\|^{-1/2}$. ◇

30.2 The Local Langlands Correspondence for Principal Series Irreducible Representations of $\mathrm{GL}_2(F)$

Consider a non-irreducible two-dimensional Frobenius-semisimple Weil-Deligne representation (σ, N). What is the corresponding $\rho(\sigma, N)$?

We have some cases:

1. $N \neq 0$ was done in the previous remark.
2. Now assume $N = 0$. We may write $\sigma = \chi_1 \otimes \chi_2$ where $\chi_1, \chi_2 \colon F^\times \to \mathbb{C}^\times$ are smooth.
3. Suppose the above characters satisfy $\chi_1/\chi_2 = \|\cdot\|^{\pm 1}$. Then $\rho(\chi_1 \oplus \chi_2, 0) = \omega \circ \det$, where $\omega = \sqrt{\chi_1 \cdot \chi_2}$ (ish).
4. Otherwise,
$$\rho(\chi_1 \oplus \chi_2, 0) = \mathrm{Ind}_B^{\mathrm{GL}_2(F)}(\delta_B^{-1/2} \otimes \widetilde{\chi}),$$
where $B = \left\{\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)\right\} \twoheadrightarrow F^\times \times F^\times \xrightarrow{\chi_1 \cdot \chi_2} \mathbb{C}^\times$ and $\delta_B\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) = \|d\|/\|a\|$.

The correspondence is much harder for supercuspidal representations. It is particularly hard when the characteristic of the residue field is 2.

30.3 Remarks on the Local Langlands Correspondence for Supercuspidal Representations of $\mathrm{GL}_n(F)$

Take $F \subseteq L \subseteq F^{\mathrm{sep}}$ and take L to be a degree-n extension of F. Let $\Theta \colon L^\times \to \mathbb{C}^\times$ be a smooth homomorphism. Then reciprocity map gives $\Theta \circ \mathrm{rec}_L \colon \mathscr{W}_L \to \mathbb{C}^\times$. $\mathscr{W}_F$ is an open subgroup of index n in $\mathscr{W}_L$.

$\mathrm{Ind}_{\mathscr{W}_L}^{\mathscr{W}_F}(\Theta \circ \mathrm{rec}_L) = \sigma_\Theta$ is a smooth n-dimensional representation of $\mathscr{W}_F$.

Sometimes, σ_Θ is irreducible.

On the other hand, for certain Θ, we can construct an irreducible supercuspidal representation ρ_Θ of $\mathrm{GL}_n(F)$. Roughly:

1. Embed $L^\times \hookrightarrow \mathrm{GL}_n(F)$

2. Construct a compact open subgroup $K_\Theta \subseteq \mathrm{GL}_n(F)$
3. Construct a smooth irreducible representation ρ'_Θ of $L^\times \cdot K_\Theta$
4. Check that $\rho_\Theta := \mathrm{Ind}_{L^\times \cdot K_\Theta}^{\mathrm{GL}_n(F)}(\rho'_\Theta)$ is irreducible. (If it is irreducible, it is automatically supercuspidal.)

QUESTION: If σ_Θ and ρ_Θ are irreducible, do they correspond to each other under the local Langlands correspondence?

ANSWER: No, but they almost do. That is, σ_Θ corresponds to $\rho_{\Theta \cdot \xi_\Theta}$, where $\xi_\Theta \colon L^\times \to \mathbb{C}^\times$ is a "rectifier."

(Bushnell-Henniart)

Example 30.1. If $L \supseteq F$ is unramified, then $\xi_\Theta|_{\Theta_L^\times} \equiv 1$ and $\xi_\Theta(\pi) = (-1)^{n-1}$.

Remark. If $p \nmid n$ (p is the characteristic of the residue field), every smooth n-dimensional irreducible representation of $\mathscr{W}_F$ has the form σ_Θ for some suitable choice of L and Θ.

The characteristic 2 story is due to Kutzko.

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