*Random Variables

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- Measurable map: [KS Def 1.10]

  $f : (\Omega, \mathcal{F}) \to (S, \mathcal{S})$

  such that $\forall A \in \mathcal{S}$, $f^{-1}(A) \in \mathcal{F}$.

- A random variable is a measurable map on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$

  $X : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathbb{B})$

  $X : (\Omega, \mathcal{F}) \to (\mathbb{R}^d, \mathbb{B}^d)$ random vector

* Could be other set, $C[0,1]$, graphs e.t.c.

We write $\mathbb{P}[X \in A] := \mathbb{P}(\{X^{-1}(A)\})$

<u>Lemma</u> [D Thm 1.3.1]

- If $f : (\Omega, \mathcal{F}) \to (S, \mathcal{S})$ and $\mathcal{A} \subset \mathcal{S}$ generates $\mathcal{S}$, $\forall A \in \mathcal{A}$ $\{X \in A\} \in \mathcal{F}$ then f is measurable.

<u>Pf</u>: $\mathcal{D} = \{D : \{X \in D\} \in \mathcal{F}\}$

$$\{X \in \cup_i D_i\} = \cup_i \{X \in D_i\} \in \mathcal{F}$$

$$\{X \in D^c\} = \{X \in D\}^c \in \mathcal{F}.$$

So $\mathcal{D}$ σ-algebra, $\mathcal{S} = \sigma(\mathcal{A}) \subseteq \mathcal{D}$.

$\Rightarrow$ f is measureable

<u>Lem</u> [D Thm 1.3.2]

If $f:(\Omega, \mathcal{F}) \to (S, \mathcal{S})$, $g:(S,\mathcal{S}) \to (S', \mathcal{S}')$ measurable then $g(f(\omega))$ is measurable.

[D Thm 1.3.3/1.3.4]

<u>Lemma</u>: If $X_1, \ldots X_n$ are random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ then so are

$$X_1 + X_2, \quad X_1 X_2, \quad e^{X_1} + 2\cos(X_2), \ldots \text{ etc.}$$

<u>P.f.</u> $X_1, \ldots X_n$ is measurable since

$$\{(X_1, \ldots, X_n) \in (a_1, b_1] \times \ldots \times (a_n, b_n]\}$$
$$= \bigcap_{i=1}^{n} \{X_i \in (a_i, b_i]\} \in \mathcal{F}.$$

$g(x_1, x_2) = x_1 + x_2$ is measurable

check $\{g^{-1}((-\infty, a])\} = \{(x_1, x_2) : x_1 + x_2 < a\}$ open and so in $\mathcal{B}(\mathbb{R}^2)$.

Almost any function you can think of on $\mathbb{R}^n$ is measurable.

[D Thm 1.3.5]

<u>Lemma</u>: If $X_1 \ldots$ are R.V.

$\inf_n X$, $\sup X_n$, $\liminf X_n$, $\limsup X_n$ are random variables.

* Review: sup or supremum is like the

* Review: sup or supremum is like the maximum, but it may not be achieved

$\sup_n x_n$ is smallest a such that $\forall n \quad x_n \leq a$.

$\inf x_n$ is largest a s.t. $\forall n \quad x_n \geq a$

Pf: $\{\sup_n X_n < a\} = \bigcap_{n=1}^{\infty} \{X_n < a\} \in \mathcal{F}$.

$$\limsup_n X_n := \lim_{n\to\infty} \sup_{m\geq n} X_m$$

$$= \inf_n \sup_{m\geq n} X_m \quad \text{measureable.}$$

Distributions

Discrete

Uniform $\{1, \dots, n\}$

$\mathbb{P}[X = k] = 1/n$

Binomial $Bin(n, p)$

n trials, success p

$\mathbb{P}[X = k]$

$$= \binom{n}{k} p^k (1-p)^{n-k}$$

$$0 \le k \le n$$

Geometric $Geom(p)$

$\mathbb{P}[X = k] = (1-p)^k p$

Poisson $Pois(\lambda)$

$$\mathbb{P}[X = k] = \frac{\lambda^k e^{-\lambda}}{k!}$$

Continuous

Uniform $U[a, b]$

Density $f(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{o.w.} \end{cases}$

Normal / Gaussian

$N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$$

Exponential $Exp(\lambda)$

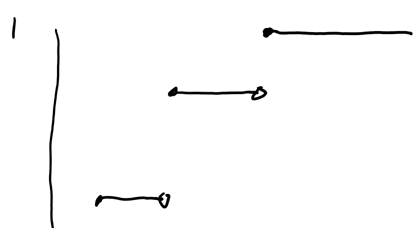
$f(x) = \lambda e^{-\lambda x}$

The distribution function of X is

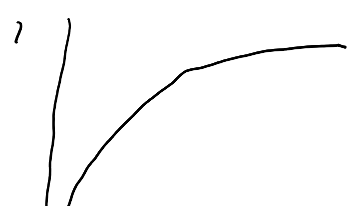
$$F(x) = F_X(x) = \mathbb{P}[X \le x]$$

$*$ Defined for any R.V.

Discrete



Continuous



[D Thm 1.2.1]

Properties:

i) Increasing: $x < y \Rightarrow F(x) \le F(y)$

ii) $\lim_{x \to -\infty} F(x) = 0, \quad \lim_{x \to \infty} F(x) = 1$

iii) Right Continuous: $\lim_{y \downarrow x} F(y) = F(x)$

iv) $F(x-) := \lim_{y \uparrow x} F(y) = \mathbb{P}[X < x]$

v) $\mathbb{P}[X = x] = F(x) - F(x-)$.

Lemma: If $F(x)$ satisfies (i)-(iii) then it is the distribution function of a R.V.

Pf: 2 Proofs

[KS Sec 3.2]

a) The function $m((a,b]) = F(b) - F(a)$ is additive so by Caratheodory extends to a measure μ on $(\mathbb{R}, \mathbb{B})$.

We say that F induces this measure on $\mathbb{R}$.

- $\mu(\mathbb{R}) = \lim_{\substack{a \to -\infty \\ b \to \infty}} F(b) - F(a) = 1$.

Let $X(x) = x$ be identity on $\mathbb{R}$.

$\mathbb{P}[X \leq x] =$

$\mu(X \leq x) = \lim_{a \to 0} F(x) - F(a) = F(x)$

[D Thm 1.2.2]

b) For Lebesgue measure on $[0,1]$,

$U(x) = x$ is a uniform R.V.

Let $X(x) = \sup\{y : F(y) < x\}$.

Homework: Show that F is the distribution function of X.

* This is useful as an algorithm to construct a R.V. with distribution F.

- If X, X' have the same distribution function we say they are identically distributed and write $X \overset{d}{=} X'$.

Then $\mathbb{P}[X \in A] = \mathbb{P}[X' \in A]$.

Distributions from densities

If $f : \mathbb{R} \to \mathbb{R}$ is non-negative, integrable &

If $f: \mathbb{R} \to \mathbb{R}$ is non-negative, integrable &

$\int_{\mathbb{R}} f(x)\, dx = 1$

Then

$$F(x) = \int_{-\infty}^{x} f(y)\, dy \text{ is a distribution function.}$$

Ex: Exp (1) $f(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & x \le 0 \end{cases}$

$$F(x) = \int_0^x e^{-x}\, dx = 1 - e^{-x} \quad \text{when } x \ge 0.$$

Properties: • $F'(x) = f(x)$

• F is continuous

• We interpret $f(x) \cdot dx \approx \mathbb{P}[X \in (x, x+dx)]$

• $\mathbb{P}[X \in A] = \int_A f(x)\, dx = \int f(x)\, I(x \in A)\, dx$