PROBLEM SET 1: DUE MARCH 17

Do as many of the problems as you can. You may work in groups of up to 3 if you like - it's fine to submit a single set of solutions for the group. Some of the problems are not so easy, feel free to come and see me if you want to discuss any of them or some hints - I'm mainly interested in you really thinking about them.

Problem 1: Let G be a random d-regular graph with n vertices ($d \geq 3$ is fixed). Show that the probability that G is connected tends to 1 as n tends to infinity.

Problem 2: Let G_n be a sequence of Erdos Renyi random graphs $G(n, d_n)$. Let u and v be two uniformly chosen vertices, $\ell \geq 2$ a fixed positive integer and A_ℓ the event that there is a path from u to v of length ℓ. Show that
(a) If $d_n n^{1-1/\ell} \to 0$ then $\mathbb{P}[A_\ell] \to 0$ as $n \to \infty$.
(b) If $d_n n^{1-1/\ell} \to \infty$ then $\mathbb{P}[A_\ell] \to 1$ as $n \to \infty$.
(c) What if $d_n n^{1-1/\ell} \to c \in (0, \infty)$?

Problem 3: You have n bins and $2n$ balls. Each ball is placed independently in one of the bins chosen uniformly. Let B_n be the event that there are no empty bins. The coupon collector problem says that $\mathbb{P}[B_n] \to 0$, we want to know how unlikely B_n is. Find

$$\lim_n \frac{1}{n} \log \mathbb{P}[B_n].$$

A more precise question one could ask is to find a function $g(n)$ such that $\mathbb{P}[B]/g(n) \to 1$.

Problem 4: Let X_i be IID random variables with $\mathbb{E}X_i = \mu$ and density $f_X(x)$ satisfying $f(x)x^\alpha \to 1$ as $x \to \infty$ for some $\alpha > 3$. Let $a > \mu$.
(a) Suppose that M_n is a sequence of real numbers such that $M_n \to \infty$ and $M_n = o(n)$, find the best upper bound you can for

$$\mathbb{P}\left[\sum_{i=1}^n \min\{X_i, M_n\} > an\right]$$

(b) Let $X_{(1)} > X_{(2)} > \ldots > X_{(n)}$ be the order statistics of $X_1, \ldots, X_n$, i.e. the random variables reordered in decreasing order. Show that for any $\epsilon > 0$,

$$\mathbb{P}\left[X_{(1)} > (a - \mu - \epsilon)n \mid \sum_{i=1}^n X_i > an\right] \to 1,$$

and
$$\mathbb{P}\left[X_{(2)} > \epsilon n \mid \sum_{i=1}^{n} X_i > an\right] \to 0.$$

Problem 5: Large deviations for Brownian motion. Let $B(t)$ be standard Brownian motion (see below) and let $f : [0,1] \to \mathbb{R}$ be a smooth function with $f(0) = 0$. We want to understand the probability that $B(t)$ rescaled is approximately $f(t)$ which is a large deviation event. Find
$$\lim_{\epsilon \searrow 0} \lim_{n \to 0} \frac{1}{n} \log \mathbb{P}\left[\max_{0 \le t \le 1} \left|\frac{1}{n} B(tn) - f(t)\right| < \epsilon\right]$$

If you haven't studied Brownian motion before (a great subject) there are only a few simple properties that you need to know for this problem,

- Brownian motion is a continuous Gaussian Markov process such $B(t) \sim N(0,t)$
- Covariance: $B(0) = 0$ and $\mathrm{Cov}(B(t), B(s)) = \min\{t, s\}$.
- Independent Increment: If $0 = t_0 < t_1 < t_2 < \ldots < t_k$ then the increments $\{B(t_i) - B(t_{i-1})\}_{1 \le i \le k}$ are independent.
- Interpolation: For any $a, b \in \mathbb{R}$ and $0 \le s < t$,

$$\mathbb{P}\left[\max_{s \le u \le t} \frac{1}{\sqrt{t-s}} \left|B(u) - \frac{a(t-u) + b(u-s)}{t-s}\right| > x \middle| B(s) = a, B(t) = b\right] \le 4e^{-x^2/2}.$$