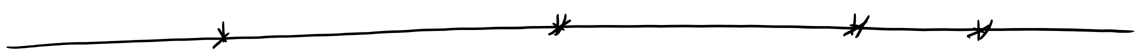
Poisson Processes

Friday, November 17, 2017 1:14 PM

A rate λ <u>Poisson process</u> on a line is a random measure N such that

a) $N(A) \sim \text{Pois}(\lambda |A|)$

b) If $A_1, \dots, A_n$ disjoint, $N(A_1), \dots, N(A_n)$ are independent.



We will write $N_t = N([0,t]) \sim \text{Pois}(\lambda t)$.

We denote the set of points as Π.

<u>Alterate formulation</u>:

- $\{N_{t+s} - N_t\}_{s \geq 0}$ independent of $\mathcal{F}_t$

- $\mathbb{P}[N_{t+\Delta} - N_t = 1] = \lambda \Delta + o(\Delta)$

  $\mathbb{P}[N_{t+\Delta} - N_t \geq 2] = o(\Delta)$ as $\Delta \to 0$

$$N_t \approx \sum_{i=1}^{t/\Delta} N_{i\Delta} - N_{(i-1)\Delta} \approx \text{Bin}(t/\Delta, \lambda\Delta)$$

$$\to \text{Pois}(t).$$

Let $T_k = \inf\{t : N_t = k\}$. - stopping times.

$\mathbb{P}[T_1 > t] = \mathbb{P}[\text{Pois}(\lambda t) = 0] = e^{-\lambda t}$

So $T_1 \sim \text{Exp}(\lambda)$.

- Memoryless Property: If $T \sim \text{Exp}(\lambda)$

$\mathbb{P}[T > t+s \mid T > t] = \mathbb{P}[T > s]$.

That is given that T is at least t it takes another $\text{Exp}(\lambda)$ time to occur.

<u>Strong Markov Property</u>

If S is a stopping time

$\mathbb{P}[N_{t+S} - N_S \in A \mid N_S] = \mathbb{P}[N_t \in A]$

i.e. still Poisson process after S.

$\Rightarrow \quad \mathbb{P}[T_k - T_{k-1} > t \mid T_1, \dots, T_{k-1}] = \mathbb{P}[T_1 > t]$

so $J_i = T_i - T_{i-1}$ are IID $\text{Exp}(1)$.

<u>Lemma</u>: If $M_1, \dots, M_k$ are independent $\text{Pois}(\lambda_i)$

then if $s_1 + \dots + s_k = s$, $\lambda = \sum \lambda_i$

$$\mathbb{P}\left[M_1 = s_1, \dots, M_k = s_k \mid \sum_{i=1}^{k} M_i = s\right]$$

$$= \frac{s!}{\prod_{i=1}^{k} s_i!} \prod_{i=1}^{k} \left(\frac{\lambda_i}{\lambda}\right)^{s} \sim \mathbb{P}\left[\text{Mult}\left(s, \frac{\lambda_1}{\lambda}, \dots, \frac{\lambda_k}{\lambda}\right) = (s_1, \dots, s_k)\right]$$

Mutinomial distribution.

So $\mathbb{P}[N_t = k \mid N_{t+s} = \ell]$

$$= \mathbb{P}[N_t = k, N_{t+s} - N_t = \ell - k \mid N_{t+s} = \ell]$$

$$= \mathbb{P}[\text{Bin}(\ell, \tfrac{t}{t+s}) = k] = \binom{\ell}{k}\left(\frac{t}{t+s}\right)^k \left(\frac{s}{t+s}\right)^{\ell-k}$$

Thinning:

Lemma: If $Z \sim \text{Pois}(\lambda)$, $X \sim \text{Bin}(Z, p)$ $Y = Z - X$ then $X \sim \text{Pois}(\lambda p)$, $Y \sim \text{Pois}(\lambda(1-p))$, X and Y are independent.

Proof: By above $Z' = N_1$, $X' = N_p$, $Y' = N_1 - N_p$ then $X' \sim \text{Bin}(Z', p)$ and so

$$(X, Y, Z) \overset{d}{=} (X', Y', Z') \Rightarrow X, Y \text{ independent}$$

$$X \sim \text{Pois}(\lambda p),\ Y \sim \text{Pois}(\lambda(1-p)).$$

For a rate λ Poisson process, assign each point v a Mark Y_v IID. The set of points $\Pi^y = \{v \in \Pi : Y_v = y\}$ is a rate $\lambda \mathbb{P}[Y_v = y]$ Poisson process. The Π^y are independent.

Superposition: If $\Pi^1, \ldots, \Pi^\ell$ are independent rate λ_i Poisson processes then $\Pi = \bigcup_{i=1}^{k} \Pi^i$ is a rate $\lambda = \sum \lambda_i$ P.P.

If $Y_v = \{ j : v \in \Pi^j \}$ then Y_v are IID, on Π and $\mathbb{P}[Y_v = j] = \lambda_j / \lambda$.

<u>Example</u>: Blue cars go past rate λ_1 P.P

Red " " " " λ_2 P.P.

i) Distribution of ~~#~~ red cars before first blue,

$\text{Geom}(\lambda_1/\lambda) \qquad \lambda = \lambda_1 + \lambda_2$

ii) Number of blue cars in first 10,

$\text{Bin}(10, \frac{\lambda_1}{\lambda})$.

iii) Time to first car, $\text{Exp}(\lambda)$

If $X_1, \dots, X_n$ independent $\text{Exp}(\lambda_i)$, $X = \min_{1 \le i \le n} X_i$

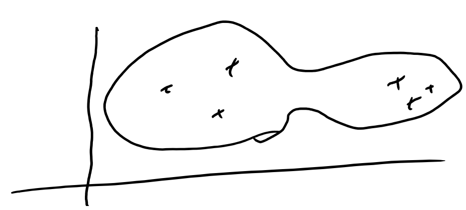
Distribution of X?

Let Π^i rate λ_i - P.P., $\Pi = \bigcup \Pi^i$,

$X_i' = \min\{v : v \in \Pi^i\} \sim \text{Exp}(\lambda_i)$

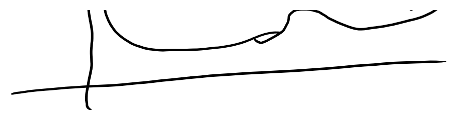
$\min X_i' = \min\{v : v \in \Pi\} \sim \text{Exp}(\sum \lambda_i)$.

<u>Spatial Poisson Process</u>



Intensity $\lambda : D \to \mathbb{R}$.

Intensity $\lambda : D \to \mathbb{R}$.
or measure μ,



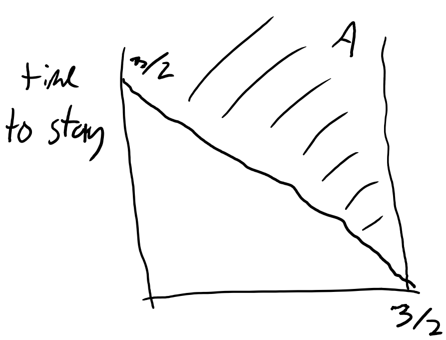
$N(A) \sim \text{Pois}(\mu(A))$ or $\text{Pois}\left(\int_A \lambda(x)dx\right)$

Y = distance to closest place to origin (constant λ)

$$\mathbb{P}[Y > r] = \mathbb{P}[N(B_r) = 0] = \exp(-\mu(B_r))$$
$$= e^{-\pi \lambda r^2}$$
$$\mathbb{P}[Y \le r] = 1 - e^{-\pi\lambda r^2}$$

Marked point process. If Π P.P. on D and $\{X_v\}_{v \in \Pi}$ are IID marks with law ν then
$$\tilde{\Pi} = \{(v, X_v) : v \in \Pi\}$$
is a P.P. on $D \times \mathbb{R}$ intensity $\mu \times \nu$.

<u>Example</u>: Students come to office hours at rate λ per hour and stay for time $W_i \sim \text{Exp}(1)$. S = # students present at the end



$$\mu = \lambda dx \otimes \nu$$
$$\mu(A) = \int_A \lambda \, dx \otimes d\nu$$
$$= \int_0^{3/2} \int_{\frac{3}{2}-x}^{\infty} \lambda e^{-y} \, dy \, dx$$
$$= \lambda \int_0^{3/2} e^{-(\frac{3}{2}-x)} \, dx$$

$$= \lambda \int_0 e^{-(\frac{3}{2}-x)}\, dx$$

$$= \lambda\left(1 - e^{-3/2}\right).$$

$$S \sim \mathrm{Pois}\left(\lambda\left(1-e^{-3/2}\right)\right).$$

Construction of Π a P.P. with intensity μ on D.
Let $N \sim \mathrm{Pois}(\mu(D))$. Set $\tilde{\mu} = \mu / \mu(D)$ and $X_1, \dots$ IID distribution $\tilde{\mu}$.

$$\Pi = \{X_1, \dots, X_N\}.$$

Campbell's Formula

If Π is PP with intensity μ, $f: D \to \mathbb{R}$.

$$Y = \sum_{x \in \Pi} f(x)$$

$$\mathbb{E}\, Y = \int f\, d\mu, \quad \mathrm{Var}\, Y = \int f^2\, d\mu.$$

$$\mathbb{E}\, Y = \mathbb{E} \sum_{i=1}^{N} f(X_i) = \mathbb{E}\left\{\mathbb{E}\left[\sum_{i=1}^{N} f(X_i) \mid N\right]\right\}$$

$$= \mathbb{E}\left[N \cdot \int f\, d\tilde{\mu}\right] = \mu(D) \int f\, d\tilde{\mu}$$

$$= \int f\, d\mu.$$

<u>Or</u> (assuming $f \geq 0$), $\Delta_{k,n} = \{x : k 2^{-n} \leq x < (k+1) 2^{-n}\}$.

$$f_n(x) = \sum_k k 2^{-n} I(x \in \Delta_{k,n}) = 2^{-n} \lfloor 2^n f(x) \rfloor$$

$$Y_n = \sum_{x \in \Pi} f_n(x) \uparrow Y$$

$$= \sum_k \sum_{x \in \Pi \cap \Delta_{k,n}} k 2^{-n}$$

$$= \sum_k k 2^{-n} N(\Delta_{k,n})$$

$$\mathbb{E} Y = \lim_{n \to \infty} \sum_k \mathbb{E}\, k 2^{-n} \mu(\Delta_{k,n})$$

$$= \lim \int f_n \, d\mu = \int f \, d\mu$$

$$\lim_n \operatorname{Var} Y_n = \lim \operatorname{Var} \sum_k k 2^{-n} N(\Delta_{k,n})$$

$$= \lim \sum_k (k 2^{-n})^2 \cdot \mu(\Delta_{k,n})$$

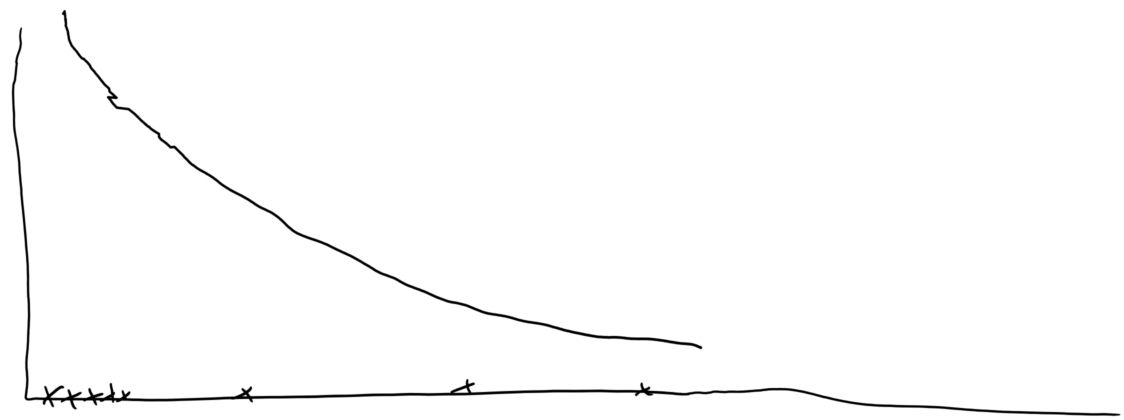
$$= \lim \int f_n^2 \, d\mu = \int f^2 \, d\mu$$

Stable Laws

Let μ have intensity $x^{-\alpha-1}$ on $[0,\infty)$

and $f(x) = x$,

Set $Y = \sum_{x \in \Pi} f(x)$



If $0 < \alpha < 1$ then

$$\mathbb{E}\, Y = \int_0^\infty x \cdot x^{-\alpha-1}\, dx = \infty$$

But $\mathbb{E} \sum_{x \in \pi \cap [0,1]} x = \int_0^1 x^{-\alpha}\, dx = \frac{1}{1-\alpha}$

and $N([1,\infty)) = \int_1^\infty x^{-\alpha-1}\, dx = \frac{1}{\alpha} < \infty$

so $Y = \sum_{x \in \pi} x$ exists.

Theorem: If Y_1, Y_2 are independent copies of Y,

$$Y_1 + Y_2 \overset{d}{=} 2^{1/\alpha}\, Y \qquad Y \text{ is } \alpha\text{-stable.}$$

Claim: If π is P.P on D with intensity μ, $h : D \to \tilde{D}$ then

$$\tilde{\pi} = \{ h(v) : v \in \pi \} \quad \text{with multiplicity.}$$

Then $\tilde{\pi}$ is P.P with intensity

$$\tilde{\mu}(A) = \mu(h^{-1}(A)).$$

Proof: $\tilde{N}(A) = N(h^{-1}(A)) \sim \text{Pois}(\mu(h^{-1}(A)))$

$$= \text{Pois}(\tilde{\mu}(A)).$$

If $A_1, \dots, A_n$ disjoint so are $h^{-1}(A_1), \dots h^{-1}(A_n)$.

$$2^{-\frac{1}{\alpha}}(Y_1 + Y_2) = \sum_{x \in \Pi^*} 2^{-\frac{1}{\alpha}} x.$$

$\Pi^* = \Pi_1 + \Pi_2$ density $2x^{-\alpha-1} \sim \mu^*$

Let $h(x) = s\, x$, $s = 2^{-\frac{1}{\alpha}}$ $A = [a,b]$

$\tilde{\Pi} = \{ h(\pi^*) : v \in \Pi^* \}.$

$$\tilde{\mu}(A) = \mu^*(h^{-1}(A))$$

$$= \int_{a/s}^{b/s} 2x^{-\alpha-1}\,dx \qquad y = x/s$$

$$= \int_a^b 2y^{-\alpha-1} s^{\alpha+1} \cdot s^{-1}\,dy$$

$$= 2s^{\alpha} \mu(A) = \mu(A)$$

$$\sum_{x \in \pi^*} 2^{-\frac{1}{\alpha}} x = \sum_{x \in \tilde{\Pi}} x \overset{d}{=} Y.$$
