# Martingales 2

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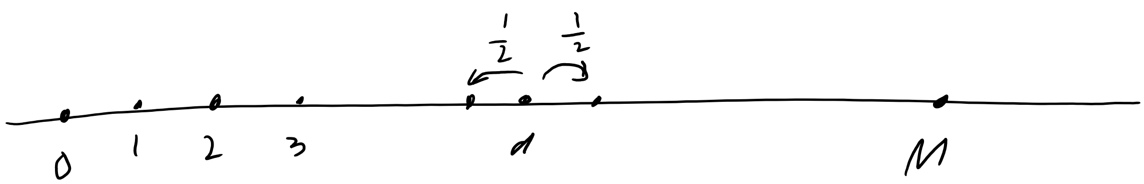
## Exit Probability

Let $S_n$ be SRW with $S_0 = a$

Let $T = \inf\{n : X_n \in \{0, M\}\}$




Find $\mathbb{P}[X_T = M]$

$\mathbb{E} X_T = \mathbb{E} X_0 = a$ since $X_{n \wedge T}$ is bounded.

$= M \, \mathbb{P}[X_T = M] + 0 \cdot \mathbb{P}[X_T = 0]$

$\mathbb{P}[X_T = M] = \frac{a}{M}, \quad \mathbb{P}[X_T = 0] = 1 - \frac{a}{M}.$

Time to exit, $\mathbb{E} T$.

$R_n = X_n^2 - n$ martingale

$\mathbb{E} R_T = \mathbb{E} R_0^2 = a^2$ ✱

$= \mathbb{E} X_T^2 - \mathbb{E} T$

$\mathbb{E} T = 0 \cdot \mathbb{P}[X_T = 0] + M^2 \cdot \mathbb{P}[X_T = M] - a^2$

$= aM - a^2 = a(M - a)$

✱ Check $R_n$ is U.I.

$$R_{n\wedge T} = \underbrace{X^2_{n\wedge T}}_{\text{bounded}} - T$$

Enough to check $T$ is U.I. i.e. $\mathbb{E}\, T < \infty$.

$\mathbb{P}[T > kM] \leq k \cdot \left(\tfrac{1}{2}\right)^M$ $M$ right in a row.

so $\frac{1}{M} T \preceq \mathrm{Geom}\left(\left(\tfrac{1}{2}\right)^n\right)$ finite expectation.

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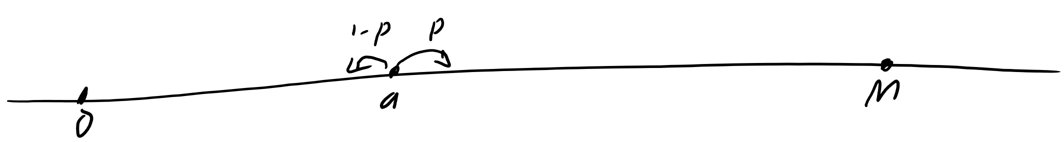
Biased RW:

$$\mathbb{P}[S_{n+1} = k+1 \mid S_n = k] = 1 - \mathbb{P}[S_{n+1} = k-1 \mid S_n = k] = p$$




$$\mathbb{E}[S_{n+1} \mid S_n] = S_n + \mathbb{E}[S_{n+1} - S_n] = S_n + p - (1-p) = S_n + (2p-1).$$

Martingale iff $p = \frac{1}{2}$. Set $q = 1-p$.

$$W_n = \left(\frac{q}{p}\right)^{S_n}$$

$$\begin{aligned}
\mathbb{E}[W_{n+1} \mid W_n] &= \mathbb{E}\left[W_n \left(\frac{q}{p}\right)^{X_{n+1}} \,\Big|\, W_n\right] \\
&= W_n\, \mathbb{E}\left[\left(\frac{q}{p}\right)^{X_{n+1}}\right] \\
&= W_n \left[\frac{q}{p} \cdot p + \left(\frac{q}{p}\right)^{-1} \cdot q\right] = W_n (p+q) \\
&= W_n
\end{aligned}$$

$$\mathbb{E}\, W_T = \mathbb{E}\, W_0 = \left(\frac{q}{p}\right)^a$$

$$= 1 \cdot \mathbb{P}[S_T = 1] + \left(\frac{q}{p}\right)^M \mathbb{P}[S_T = M]$$

$$= 1 + \left(\left(\frac{q}{p}\right)^M - 1\right) \mathbb{P}[S_T = M]$$

$$\mathbb{P}[S_T = M] = \frac{1 - \left(\frac{q}{p}\right)^a}{1 - \left(\frac{q}{p}\right)^M}$$

$\mathbb{P}[$ Ever returns to $0]$

$$= \lim_{M \to \infty} 1 - \frac{1 - \left(\frac{q}{p}\right)^a}{1 - \left(\frac{q}{p}\right)^M} = \begin{cases} (q/p)^a & p > \frac{1}{2} \\ 1 & p < \frac{1}{2}. \end{cases}$$

## Birth and death chains

A Markov Chain on $\mathbb{Z}$ such that $X_{n+1} - X_n \in \{-1, 0, 1\}$

with transition probabilities $P_{i,i-1} = p_i$, $P_{i,i} = q_i$, $P_{i,i+1} = r_i$

$p_i + q_i + r_i = 1$. Assume irreducible, $p_i, r_i > 0$.

Stationary distribution on $\{0, \dots, n\}$.

- Reversible since # left to right crossings of $i$ to $i+1$ = # crossings $i+1$ to $i$ over long term

So $\pi_i P_{i,i+1} = \pi_{i+1} P_{i+1,i}$.

$$\frac{\pi_{i+1}}{\pi_i} = \frac{r_i}{p_{i+1}}, \quad \pi_j = \pi_0 \prod_{i=0}^{j-1} \frac{\pi_{i+1}}{\pi_i}$$

$$= \pi_0 \prod_{i=0}^{j-1} \frac{r_i}{p_{i+1}}$$

$n$ $\quad$ $n$ $\quad$ $j-1$ $\quad$ $r_i$

$$1 = \sum_{j=0}^{n} \pi_j = \pi_0 \sum_{j=0}^{n} \prod_{i=0}^{j-1} \frac{r_i}{p_{i+1}}$$

$$\pi_k = \frac{\prod_{i=0}^{k-1} \frac{r_i}{p_{i+1}}}{\sum_{j=0}^{n} \prod_{i=0}^{j-1} \frac{r_i}{p_{i+1}}}$$

On $\{0,1,\dots\}$ it has a stationary distribution iff $\sum_{j=0}^{\infty} \prod_{i=0}^{j-1} \frac{r_i}{p_{i+1}} < \infty$.

On $\{0,1,\dots\}$ is it recurrent if

$$\lim_{M\to\infty} \mathbb{P}[X_{T_M} = M \mid X_0 = 1] = 0$$

where $T_M = \inf\{t : X_t \in \{0, M\}\}$.

Construct a martingale $Y_t = h(X_t)$.

$$\mathbb{E}[Y_{t+1} - Y_t \mid Y_t = i]$$

$$= p_i h(i-1) + q_i h(i) + r_i h(i+1) - h(i)$$

$$= r_i (h(i+1) - h(i)) - p_i (h(i) - h(i-1))$$

$$h(i+1) - h(i) = \frac{p_i}{r_i} (h(i) - h(i-1))$$

$$= \left(\prod_{j=1}^{i} \frac{p_j}{r_j}\right)(h(1) - h(0))$$

So $h(i) = h(0) + \sum_{k=1}^{i} h(k) - h(k-1)$

So $\quad h(i) = h(0) + \sum_{k=1} h(k) - h(k-1)$

$$= h(0) + (h(1) - h(0)) \cdot \left( \sum_{k=1}^{i} \prod_{j=1}^{k-1} \frac{p_j}{r_j} \right)$$

$$= A + B \left( \sum_{k=1}^{i} \prod_{j=1}^{k-1} \frac{p_j}{r_j} \right)$$

Set $\qquad J_M = \left( \sum_{k=1}^{M} \prod_{j=1}^{k-1} \frac{p_j}{r_j} \right)$

and $\quad h_M(i) = B_M^{-1} \cdot B_i$

Then $h_M(X_{n \wedge T_M})$ is a bounded martingale

$$\mathbb{E}\, h_M(X_{T_M}) = \mathbb{E}\, h_M(a) = B_M^{-1} B_a$$

$$= h_M(0) \cdot \mathbb{P}[X_{T_M} = 0] + h_M(M) \cdot \mathbb{P}[X_{T_M} = M]$$

$$\Longrightarrow \mathbb{P}[X_{T_M} = M] = \frac{B_a}{B_M}$$

So $X_n$ is recurrent $\Longleftrightarrow \mathbb{P}[X_{T_M} = M]$

$$\Longleftrightarrow \sum_{k=1}^{\infty} \prod_{j=1}^{k-1} \frac{p_j}{r_j} = \infty.$$

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First step analysis

$X_n$ is a Markov chain

$D$ is a set of exit points

$T = \min\{t : X_t \in D\}$.

Find $\mathbb{P}[X_T = d \mid X_0 = a]$ and $\mathbb{E}[T \mid X_0 = a]$.

Let $h(x) = \mathbb{P}[X_T = d \mid X_0 = x]$

Then $\mathbb{E}[I(X_T = d) \mid \mathcal{F}_n]$

$$= \mathbb{E}[I(X_T = d) \mid X_{T \wedge n}]$$

$$= h(X_{T\wedge n}) \text{ - martingale.}$$

So for $x \notin D$,

$$h(x) = \mathbb{E}[h(X_1) \mid X_0 = x] = Ph(x).$$

Solve for $h(x) = Ph(x) \quad x \notin D$

$h(x) = I(x = d) \quad x \in D$
Boundary condition.

Set of linear equations to solve.

If $g(x) = \mathbb{E}[T \mid X_0 = x]$,

then $g(x) = Pg(x) + 1 \quad x \notin D$

$g(x) = 0 \quad x \in D.$

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Concentration via Martingales

Azuma-Hoeffding Inequality

Let $X_n$ be a martingale such that

$\|X_i - X_{i-1}\|_\infty \leq k_i$. Then

$$\mathbb{P}[X_n - X_0 \geq t] \leq \exp\left(-\frac{t^2}{2\sum_{i=1}^{n} k_i^2}\right)$$

$< \sum_{i=1} k_i$

Proof:

Claim If $\|Y\|_\infty \leq M$, $\mathbb{E}\, Y = 0$, then $\mathbb{E}\, e^{\theta Y} \leq \exp\left(\frac{1}{2}\theta^2 M^2\right)$.

Proof: Reversing Jensen $\mathbb{E}\, e^{\theta Y} \leq \dfrac{e^{\theta M} + e^{-\theta M}}{2} = \cosh \theta M$.

E.G. $U \sim \mathrm{Unif}[0,1]$, $\tilde{Y} = \begin{cases} M & U < \frac{Y+M}{2M} \\ -M & U > \frac{Y+M}{2M} \end{cases}$

$$\cosh(\theta M) = \mathbb{E}\left[e^{\theta \tilde{Y}}\right] = \mathbb{E}\left[\mathbb{E}\left[e^{\theta \tilde{Y}} \mid Y\right]\right]$$

$$\geq \mathbb{E}\, e^{\mathbb{E}[\theta \tilde{Y} \mid Y]} = \mathbb{E}\, e^{\theta Y}$$

$$\cosh x = \sum_{i=0}^{\infty} \frac{x^i + (-x)^i}{2 \cdot i!}$$

$$= \sum_{i=0}^{\infty} \frac{x^{2i}}{(2i)!} \leq \sum_{i=0}^{\infty} \frac{x^{2i}}{2^i\, i!} = \exp\left(\frac{x^2}{2}\right)$$

So $$\mathbb{E}\, e^{\theta(X_n - X_0)} = \mathbb{E}\left[e^{\theta(X_{n-1} - X_0)}\, \mathbb{E}\left[e^{\theta(X_n - X_{n-1})} \mid \mathcal{F}_n\right]\right]$$

$$\leq \mathbb{E}\, e^{\theta(X_n - X_0)} \cdot \exp\left(\theta^2 k_n^2 / 2\right)$$

$$\overset{\cdots}{\leq} \exp\left(\frac{1}{2}\theta^2 \sum_{i=1}^{n} k_i^2\right)$$

Set $\theta = t / \sum_{i=1}^{n} k_i^2$

$$\mathbb{P}[X_n - X_0 \geq t] = \mathbb{P}\left[e^{\theta(X_n - X_0)} \geq e^{\theta t}\right]$$

$$\leq \frac{\mathbb{E}\, e^{\theta(X_n - X_0)}}{e^{\theta t}}$$

$$= \exp\left(\frac{\frac{1}{2} t^2}{\sum k_i^2} - \frac{t^2}{\sum k_i^2}\right) = \exp\left(\frac{-t^2}{2 \sum_{i=1}^{n} k_i^2}\right)$$

Example Bounded random variables.

If $|Y_i| \le M$, independent

$$\mathbb{P}\left[\sum_{i=1}^{n}(Y_i - \mathbb{E}Y_i) \ge t\right] \le \exp\left(-\frac{t^2}{2Mn}\right)$$

For $t \approx \alpha\sqrt{n}$, $\exp(-\alpha^2/2M)$

Chebyshev $\mathbb{P}[\cdots] \le \frac{\mathrm{Var}[\sum Y_i]}{\alpha^2 n} \le \frac{M^2}{\alpha^2}$

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Lipschitz Functions

If $W_1,\dots,W_n$ independent $0 \le W_i \le M$ and $f(w_1,\dots,w_n)$ is $R$-Lipschitz then for

$$Y = f(W_1,\dots,W_n)$$

$$\mathbb{P}[|Y - \mathbb{E}Y| \ge t] \le 2\exp\left(\frac{-t^2}{2R^2M^2}\right).$$

Proof: Let $\mathcal{F}_m = \sigma(W_1,\dots,W_m)$.

$Y_m = \mathbb{E}[Y \mid \mathcal{F}_m]$ Doob martingale.

- $\mathbb{E}[Y_{m'} \mid \mathcal{F}_m] = \mathbb{E}[\mathbb{E}[Y \mid \mathcal{F}_{m'}] \mid \mathcal{F}_m] = Y_m$ for $m' \ge m$.

$Y_n = Y$, $Y_0 = \mathbb{E}Y$.

Let $(W_1',\dots,W_n')$ be independent copy of $(W_1,\dots,W_n)$.

$$Y_m = \mathbb{E}[f(W_1,\dots,W_n) \mid \mathcal{F}_m]$$

$$= \mathbb{E}[f(W_1,\dots,W_m,W'_{m+1},\dots,W'_n) \mid \mathcal{F}_m]$$

$$= \mathbb{E}[f(W_1,\dots,W_m,W'_{m+1},\dots,W'_n) \mid \mathcal{F}_n]$$

$$|Y_{m+1} - Y_m| \le \Big|\mathbb{E}[f(W_1,\dots,W_m,W_{m+1},W'_{m+2},\dots,W'_n) - f(W_1,\dots,W_m,W'_{m+1},\dots W'_n) \mid \mathcal{F}_n]\Big|$$

$$\le \mathbb{E}\, R|W_{m+1} - W_m| \le RM.$$

By A-H

$$\mathbb{P}[|Y_n - Y_0| \ge t] \le 2\exp\left(-\frac{t^2}{2R^2M^2}\right)$$

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Let $Z_1,\dots,Z_n$ independent, $0 \le Z_i \le M$,
Let $X$ be sum of $n/2$ largest $Z_i$.

$$\mathbb{P}[Z - \mathbb{E}Z > t] \le \exp\left(-\frac{t^2}{2M^2}\right).$$

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- <u>Concentration of the chromatic number of a random graph</u>

Let $G$ be an Erdos Renyi random graph $G(n,p)$. with $n$ vertices and each edge independently with probability $p$.

$\chi$ = chromatic number of $G$.

$X_i = \mathbb{E}[X \mid \mathcal{F}_i]$ - Doob martingale

$\mathcal{F}_i$ - edges connected to vertices $1, \dots, i$.

$X$ monotone in edge set.

Let $G^{i,+}$ be graph with all edge from $i$ to $\{i+1, \dots, n\}$ present.

$G^{i,-}$ - no edges $i$ to $\{i+1, \dots, n\}$

and $X^{i,\pm}$ chromatic number of $G^{i,\pm}$.

$\mathcal{F}_{i-1}$ measurable

$$\mathbb{E}[X^{i,-} \mid \mathcal{F}_i] \le \mathbb{E}[X \mid \mathcal{F}_i] = X_i \le \mathbb{E}[X^{i,+} \mid \mathcal{F}_i]$$

$$\shortparallel \qquad\qquad\qquad\qquad\qquad\qquad \shortparallel$$

$$\mathbb{E}[X^{i,-} \mid \mathcal{F}_{i-1}] \le X_{i-1} \le \mathbb{E}[X^{i,+} \mid \mathcal{F}_{i-1}]$$

Since $0 \le X^{i,+} - X^{i,-} \le 1$

$\Rightarrow \quad |X_i - X_{i-1}| \le 1.$

$$\Rightarrow \quad \mathbb{P}[X_n - X_0 > t\sqrt{n}] \le \exp\left(\frac{-t^2 n}{2n}\right)$$

$$\mathbb{P}[|X - \mathbb{E}X| > t\sqrt{n}] \le 2e^{-t^2/2}$$