Martingales

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- A filtration $\mathcal{F}_t$ is a set of σ-algebras such that $\mathcal{F}_s \subseteq \mathcal{F}_t$ for $s < t$.

  Example $\mathcal{F}_n = \sigma(X_0, \dots, X_n)$

- A process X_t is adapted to $\mathcal{F}_t$ if $\forall t$, X_t is $\mathcal{F}_t$ measurable.

- A process X_t is a martingale with respect to $\mathcal{F}_t$ if $\forall s \le t$, $\mathbb{E}[X_t \mid \mathcal{F}_s] = X_s$.

Example: $S_n = \sum_{i=1}^{n} X_i$, $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$

X_i IID, $\mathbb{E}X_i = 0$.

Then $\mathbb{E}[S_n \mid \mathcal{F}_m] = S_m$ so S_n is a martingale.

If $Q_n = \prod_{i=1}^{n} X_i$, $\mathbb{E}X_i = 1$ then

$$\begin{aligned}
\mathbb{E}[Q_n \mid \mathcal{F}_m] &= \mathbb{E}\Big[\prod_{i=1}^{m} X_i \cdot \prod_{i=m+1}^{n} X_i \mid \mathcal{F}_m\Big] \\
&= \Big(\prod_{i=1}^{m} X_i\Big) \mathbb{E}\Big[\prod_{i=m+1}^{n} X_i \mid \mathcal{F}_m\Big] \\
&= Q_m.
\end{aligned}$$

For discrete time, enough to prove that $\mathbb{E}[Y_{n+1} \mid \mathcal{F}_n] = Y_n$ to show Y_n is a martingale.

By induction suppose

$\forall n \quad \mathbb{E}[Y_n \mid \mathcal{F}_{n-k}] = Y_{n-k}$

Then

$$\mathbb{E}[Y_n \mid \mathcal{F}_{n-(k+1)}] = \mathbb{E}[\mathbb{E}[Y_n \mid \mathcal{F}_{n-1}] \mid \mathcal{F}_{n-(k+1)}]$$

$$= \mathbb{E}[Y_{n-1} \mid \mathcal{F}_{(n-1)-k}] = Y_{n-(k+1)}.$$

$\Rightarrow Y_n$ is a martingale.

<u>Example</u>: If S_n is SRW on $\mathbb{Z}$ then

$Y_n = S_n^2 - n$ is a martingale.

$$\mathbb{E}[Y_{n+1} \mid \mathcal{F}_n] = \mathbb{E}[(S_n + X_{n+1})^2 - (n+1) \mid \mathcal{F}_n]$$

$$= \mathbb{E}[(S_n^2 - n) + 2X_{n+1}S_n + X_{n+1}^2 - 1 \mid \mathcal{F}_n]$$

$$= Y_n + 2S_n \underbrace{\mathbb{E}[X_{n+1} \mid \mathcal{F}_n]}_{=0} + \underbrace{\mathbb{E}[X_{n+1}^2 - 1 \mid \mathcal{F}_n]}_{=0}$$

$$= Y_n$$

<u>Example</u>: You start with M_0 dollars.

Each round you can bet B_n dollars and your payout is $B_n W_n$ where $\mathbb{E} W_i = 0$. and W_i are IID. Your bet B_n must be $\mathcal{F}_{n-1}$ measurable i.e. you can't know future outcomes. $M_n = M_{n-1} + W_n B_n$.

$$\mathbb{E}[M_n \mid \mathcal{F}_{n-1}] = \mathbb{E}[M_{n-1} + W_n B_n \mid \mathcal{F}_{n-1}]$$
$$= M_{n-1} + B_n \mathbb{E}[W_n \mid \mathcal{F}_{n-1}]$$
$$= M_{n-1}$$

We call B_n a <u>predictable</u> sequence if $B_n \in \mathcal{F}_{n-1}$.

<u>Warning!</u> Most betting is not a martingale.

<u>Definition:</u>

If for $s<t$, $\mathbb{E}[X_t \mid \mathcal{F}_s] \geq X_s$ then X_t is a submartingale

" " " $\mathbb{E}[X_t \mid \mathcal{F}_s] \leq X_s$ " " " " supermartingale

Supermartingale = at a casino

Sub-martingale = you are the casino.

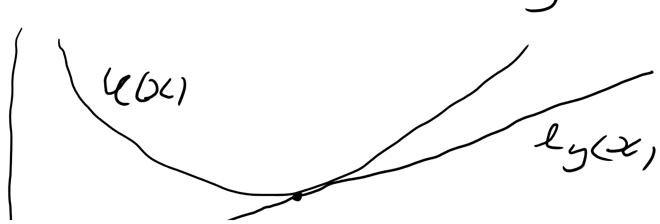
<u>Jensen's Inequality for conditional Expectation</u>

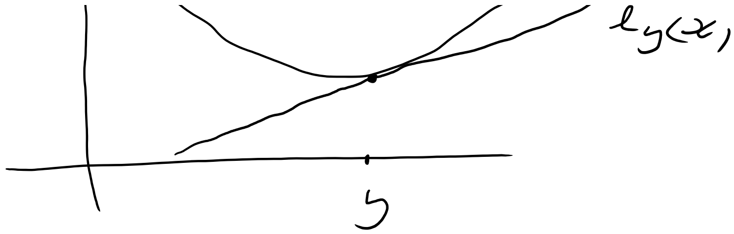
If φ is convex then

$$\mathbb{E}[\varphi(X) \mid \mathcal{F}] \geq \varphi(\mathbb{E}[X \mid \mathcal{F}]).$$

<u>Proof:</u> Exists $a(y)$, measurable, such that

$\forall x \quad a(y)(x-y) + \varphi(y) =: \ell_y(x) \leq \varphi(x)$





$$\mathbb{E}[\varphi(X)\mid\mathcal{F}] \geq \mathbb{E}[\ell_{\mathbb{E}[X\mid\mathcal{F}]}(X)\mid\mathcal{F}]$$

$$= \mathbb{E}\Big[a(\mathbb{E}[X\mid\mathcal{F}])(X-\mathbb{E}[X\mid\mathcal{F}]) + \varphi\big(\mathbb{E}[X\mid\mathcal{F}]\big)\mid\mathcal{F}\Big]$$

$$= a(\mathbb{E}[X\mid\mathcal{F}])\cdot\underbrace{\mathbb{E}[X-\mathbb{E}[X\mid\mathcal{F}]\mid\mathcal{F}]}_{=0} + \varphi(\mathbb{E}[X\mid\mathcal{F}])$$

$$= \varphi(\mathbb{E}[X\mid\mathcal{F}]).$$

<u>Corollary</u>: If X_n is a martingale and φ is convex then $Y_n = \varphi(X_n)$ is a submartingale.

$$\mathbb{E}[Y_{n+1}\mid\mathcal{F}_n] = \mathbb{E}[\varphi(X_{n+1})\mid\mathcal{F}_n]$$

$$\geq \varphi(\mathbb{E}[X_{n+1}\mid\mathcal{F}_n]) = \varphi(X_n) = Y_n$$

<u>Martingale Convergence Theorem</u>

If X_n is a martingale (or submartingale), $\sup_n \mathbb{E}X_n^+ < \infty$ then

$$X_n \xrightarrow{a.s.} X \quad \& \quad \mathbb{E}|X| < \infty$$

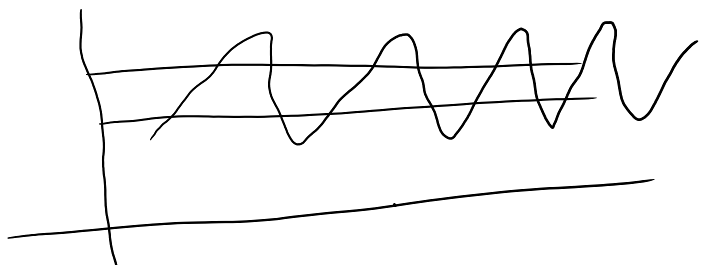
<u>Proof</u>: $\mathbb{P}[X_n > M] \leq \dfrac{\sup_n \mathbb{E}X_n^+}{M}$

so $\mathbb{P}[X_n \to \infty] = 0$

Similarly $\mathbb{P}[X_n \to -\infty] = 0$ as $\mathbb{E}X_n^- \leq \mathbb{E}|X_0| + \sup X_n^+$!

So if $X_n(\omega) \not\to X(\omega)$ then $\exists\, a, b \in \mathbb{Q}$ such that

$$\liminf_n X_n(\omega) < a < b < \limsup_n X_n(\omega)$$



It must cross $[a,b]$ infinitely many times. Let $U_{a,b}$ be # upcrossings of $[a,b]$. Enough to prove $\mathbb{P}[U_{a,b} < \infty] = 1$ by a union bound.

<u>Upcrossing Theorem</u> If X_n is a submartingale then

$$(b-a)\,\mathbb{E}\,U_{a,b}(n) \le \mathbb{E}(X_n - a)^+ - \mathbb{E}(X_0 - a)^+$$

<u>Proof</u>: "Buy low, sell high" $h(x) = (x-a)^+ + a$.

Let $Y_n = (X_n - a)^+ + a = h(X_n)$

$$\mathbb{E}[h(X_{n+1}) \mid \mathcal{F}_n] \ge h(\mathbb{E}[X_{n+1} \mid \mathcal{F}_n]) \quad h \text{ convex}$$

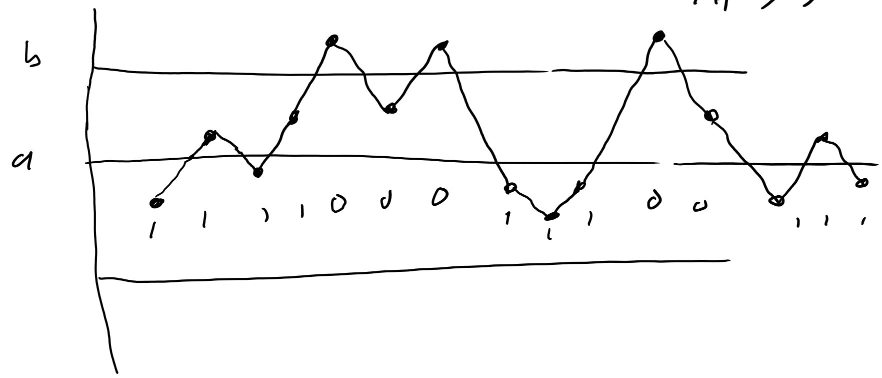
$$\ge h(X_n) \quad h \text{ increasing.}$$

So Y_n is a submartingale.

Then set $T_{n+1} = \max\{k \le n : Y_k \notin (a,b)\}$.

Then set $\tau_{n+1} = \max\{k \le n : Y_k \notin (a,b)\}$.

and $H_{n+1} = \begin{cases} 1 & \text{if } X_{\tau_{n+1}} \le a \\ 0 & \text{if } X_{\tau_{n+1}} \ge b \end{cases}$



Payoff: $Z_n = \sum_{k=0}^{n-1} H_{k+1}(Y_{k+1} - Y_k)$

$$Y_n - Y_0 = \sum_{k=0}^{n-1} (Y_{k+1} - Y_k)$$

So $Y_n - Y_0 - Z_n$

$$= \sum_{k=0}^{n-1} (1 - H_{k+1})(Y_{k+1} - Y_k)$$

$$E(1 - H_{k+1})(Y_{k+1} - Y_k)$$

$$= E\left(E\left((1 - H_{k+1})(Y_{k+1} - Y_k) \mid \mathcal{F}_k\right)\right)$$

$$= E\Big(\underbrace{(1 - H_{k+1})}_{\ge 0} \underbrace{E\left[(Y_{k+1} - Y_k) \mid \mathcal{F}_k\right]}_{\ge 0}\Big) \ge 0$$

$$\Rightarrow E Y_n - Y_0 \ge E Z_n.$$

But $Z_n \ge (b-a) U_{a,b}(n)$

So $(b-a) E U_{a,b}(n) \le E(X_n - a)^+ - E(X_0 - a)^+$ ✓

Corollary: If X_n is a non-negative martingale then $X_n \to X$ a.s.

Example: If Z_n is a branching process with $\mu = 1$ then $Z_n \to 0$ a.s.

Since Z_n is a martingale, $Z_n \to Z$ a.s.

but if $k \geq 1$, $\mathbb{P}[Z_n = k \ \forall n > N] = 0$.

Example: If Z_n is a branching process with $\mu > 1$ then $\mu^{-n} Z_n \to Z$ a.s.

$$\mathbb{E}[\mu^{-(n+1)} Z_{n+1} | \mathcal{F}_n] = \mu^{-(n+1)} \mathbb{E}\Big[\sum_{i=1}^{Z_n} X_{i,n+1} \Big| \mathcal{F}_n\Big]$$
$$= \mu^{-(n+1)} \cdot \mu Z_n$$
$$= \mu^{-n} Z_n.$$

So $\mu^{-n} Z_n$ is a martingale.

Stopping Times

We say $N \geq 0$ is a stopping time w.r.t. $\mathcal{F}_n$ if $\forall n$, $\{N \leq n\} \in \mathcal{F}_n$.

A rule to tell you to stop.

Example: [illegible]

A rule to tell you to stop.

Example: $N = \inf\{n : X_n \geq 10\}$

$$N = N_1 \vee N_2, \quad N_1 \wedge N_2 \quad \text{with } N_1, N_2 \text{ stopping times.}$$

Not a stopping time

- 2 minutes before the toast burns.
- $\inf\{n : X_n \geq 10\} - 2$.
- $\inf\{n : X_{n+1} - X_n < 0\}$ i.e. sell before market goes down.

Lemma: If X_n is a super/sub/martingale w.r.t $\mathcal{F}_n$ and $\mathcal{F}_n$ is a stopping time w.r.t. $\mathcal{F}_n$, then $Y_n = X_{N \wedge n}$ is a super/sub/martingale.

Proof: Need to show $\mathbb{E}[Y_{n+1} \mid \mathcal{F}_n] = Y_n$

$$\Leftrightarrow \mathbb{E}\left[X_{N \wedge (n+1)} - X_{N \wedge n} \mid \mathcal{F}_n\right] = 0$$

$$= \mathbb{E}[I(N \geq n+1)(X_{n+1} - X_n) \mid \mathcal{F}_n]$$

$$= \mathbb{E}[(1 - I(N \leq n+1))(X_{n+1} - X_n) \mid \mathcal{F}_n]$$

$$= I(N \geq n+1)\, \mathbb{E}[X_{n+1} - X_n \mid \mathcal{F}_n = 0.$$

Hence $\mathbb{E} X_{N \wedge n} = \mathbb{E} X_{N \wedge 0} = \mathbb{E} X_0$.

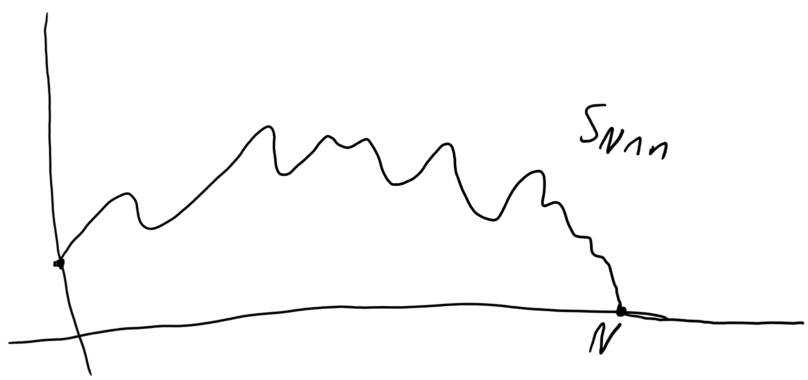
Want to prove $\mathbb{E} X_N = \mathbb{E} X_0$. but need extra conditions.

<u>Example</u> Let S_n be simple random walk on $\mathbb{Z}$,

$S_0 = 1$ and $N = \inf\{n : S_n = 0\}$.

Since S_n is recurrent, $S_n < \infty$ a.s., $\mathbb{P}[N > n] \to 0$.

$\mathbb{E} S_{N \wedge n} = \mathbb{E}[S_{N \wedge 0}] = 1$.



But $S_N = 0$ so $\mathbb{E} S_N = 0$.

The problem is $\mathbb{E} S_n I(N > n) = 1$

even though $\mathbb{P}[N > n] \to 0$.

<u>Uniform Integrability:</u>

A sequence of R.V.'s X_n is <u>uniformly integrable</u> if

$$\lim_{M \to \infty} \sup_n \mathbb{E}[|X_n| I(X_n \geq M)] = 0$$

If $\sup \|X_n\|_\infty < \infty$ then X_n is U.I.

If $\sup_n \mathbb{E} X_n^2 < \infty$ then X_n is U.I.

Proof: $Y = |X_n| I(|X_n| \geq M)$, $R = \sup_n \mathbb{E} X_n^2$.

$$\mathbb{E} Y = \int_0^\infty \mathbb{P}[Y \geq t]\, dt = \int_0^M \mathbb{P}[|X_n| \geq M]\, dt + \int_M^\infty \mathbb{P}[|X_n| > t]\, dt$$

$$\mathbb{P}[|X_n| > t] \leq \frac{\mathbb{E}|X_n|^2}{t^2} \leq \frac{R}{t^2}$$

$$\mathbb{E} Y \leq M \cdot \frac{R}{M^2} + \int_M^\infty \frac{R}{t^2}\, dt = \frac{R}{M} + \frac{R}{M}$$

So $\lim_{M \to \infty} \mathbb{E} |X_n| I(|X_n| \geq M) = 0$.

Theorem: If X_n is a martingale then the following are equivalent

i) X_n are uniformly integrable

ii) X_n converges a.s. and in L^1

iii) X_n " in L^1

iv) $\exists X \in L^1$ such that $\mathbb{E}[X \mid \mathcal{F}_n] = X_n$.

Proof: (iv) $\Rightarrow$ (i)

By Dominated Convergence Theorem if $\mathbb{P}[A_n] \to 0$ then $\mathbb{E}[X I(A_n)] \to 0$.

$\Rightarrow \forall \varepsilon>0, \exists \delta$ s.t. $\mathbb{P}[A] \leq \delta \Rightarrow \mathbb{E}[X I(A)] \leq \varepsilon$.

Pick M such that $\mathbb{E}|X|/M \leq \delta$.

$$\mathbb{E}[|X_n| I(|X_n| \geq M)] \leq \mathbb{E}[\, \mathbb{E}[|X| \mid \mathcal{F}_n]\; I(|X_n| \geq M)]$$
$$= \mathbb{E}(|X| I(|X_n| \geq M)]$$

RHS $\leq \varepsilon$ provided $\mathbb{P}[|X_n| \geq M] \leq \delta$.

$$\mathbb{P}[|X_n| \geq M] \leq \frac{1}{M} \mathbb{E}[|\mathbb{E}[X \mid \mathcal{F}_n]|]$$
$$\leq \frac{1}{M} \mathbb{E}(\mathbb{E}(|X| \mid \mathcal{F}_n)) = \frac{\mathbb{E}|X|}{M} \leq \delta.$$

Hence

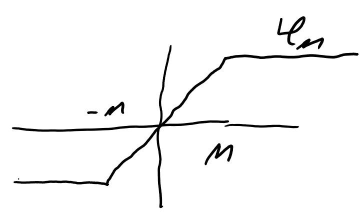
$$\lim_{M \to \infty} \sup \mathbb{E}[|X_n| I(|X_n| \geq M) = 0.$$

(i) $\Rightarrow$ (ii) U.I. $\Rightarrow \sup \mathbb{E} X_n^+ < \infty$ so by the martingale convergence theorem, $X_n \xrightarrow{a.s.} X$.

<u>Claim</u>: $Y_n \xrightarrow{P} Y$ and Y_n U.I. then $Y_n \xrightarrow{L^1} Y$.

<u>Proof</u>: Let

$$\varphi_M(x) = \begin{cases} M & x > M \\ x & -M \leq x \leq M \\ -M & x < -M \end{cases}$$



$$|X_n - X| = |X_n - \varphi_M(X_n)| + |\varphi_M(X_n) - \varphi_M(X)| + |\varphi_M(X) - X|$$

$\mathbb{E}|\varphi_M(X_n) - \varphi_M(X)| \to 0$ by D.C.T.

$$\mathbb{E}[|X_n - \varphi_M(X_n)|] = \mathbb{E}[(|X_n| - M) I(|X| - M)]$$

$$\mathbb{E}[|X_n - \varphi_M(X_n)|] = \mathbb{E}[(|X_n| - M)\, I(|X_n| > M)]$$
$$\leq \mathbb{E}[|X_n|\, I(|X_n| > M)] < \varepsilon$$

large enough M.

Hence $X_n \xrightarrow{L^1} X$.

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iii) $\Rightarrow$ (iv) Since $X_n \xrightarrow{L^1} X$, $\forall A$ $X_n I(A) \xrightarrow{L^1} X I(A)$

so $\mathbb{E}[X_n I(A)] \to \mathbb{E}[X I(A)]$.

If $A \in \mathcal{F}_n$ then

$$\mathbb{E}[X\, I(A)] = \lim_{m\to\infty} \mathbb{E}[X_m I(A)]$$

but for $m \geq n$, $\mathbb{E}[X_m | \mathcal{F}_n] = X_n$ so

$$\mathbb{E}[X_m I(A)] = \mathbb{E}[X_n I(A)]$$

$$\Rightarrow \mathbb{E}[X I(A)] = \mathbb{E}[X_n I(A)]$$

$$\Rightarrow X_n = \mathbb{E}[X | \mathcal{F}_n]$$

Optional Stopping Theorem

If X_n is a U.I. sub/martingale, N a stopping time then

$$\mathbb{E}[X_N | \mathcal{F}_n] = X_{n \wedge N} \text{ and}$$

$$\mathbb{E}[X_N] = \mathbb{E}[X_0].$$

Proof:

Claim: $\mathbb{E} X_{N\wedge n}^+ \leq \mathbb{E} X_n^+$.

$$\mathbb{E}\left(X_{n+1}^+ - X_n^+\right) - \left(X_{n+1\wedge N}^+ - X_{n\wedge N}^+\right)$$

$$= \mathbb{E}\left(\mathbb{E}\left[I(N>n)(X_{n+1}^+ - X_n^+) \mid \mathcal{F}_n\right]\right)$$

$$= \mathbb{E}\left(\underbrace{I(N>n)}_{\geq 0} \underbrace{\mathbb{E}\left[X_{n+1}^+ - X_n^+ \mid \mathcal{F}_n\right]}_{\geq 0}\right) \geq 0.$$

So $\sup_n \mathbb{E} X_{N\wedge n}^+ \leq \mathbb{E} X_n^+ < \infty$.

So by the martingale convergence theorem,

$X_{n\wedge N} \xrightarrow{a.s.} X_N$ and $\mathbb{E}|X_N| < \infty$.

Then

$$\lim_{M\to\infty} \sup_n \mathbb{E}\left[|X_{N\wedge n}| I(|X_{N\wedge n}| > M)\right]$$

$$\leq \lim_M \sup_n \mathbb{E}\left[|X_n| I(X_n > M)\right] + \mathbb{E}\left[|X_N| I(X_N > M)\right] = 0$$

So $X_{N\wedge n}$ is U.I. Hence $X_{N\wedge n} = \mathbb{E}[X_N \mid \mathcal{F}_n]$

and $\mathbb{E} X_0 = \mathbb{E}[X_N]$.

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