Markov Chains

Sunday, October 8, 2017 11:02 PM

We will start with the simplest case, discrete time & space Markov Chains.

We say $X_0, X_1, \dots$ is a <u>Markov chain</u> if

$$\mathbb{P}[X_n = a_n \mid X_0 = a_0, \dots, X_{n-1} = a_{n-1}] = \mathbb{P}[X_n = a_n \mid X_{n-1} = a_{n-1}].$$

Future | Present & Past = Future | Present.

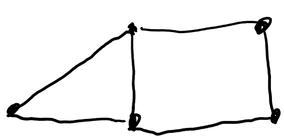
<u>Examples</u>

1) X_n IID

2) $X_n = \sum_{i=1}^{n} Y_i$, Y_i IID

3) Card shuffle

4) Random walks on a graph



5) $M_n = \max_{1 \le i \le n} X_i$ X_i IID.

It is <u>time homogeneous</u> if

$$\mathbb{P}[X_n = y \mid X_{n-1} = x] = \mathbb{P}[X_1 = y \mid X_0 = x] = P_{xy}$$

and we call P_{xy} the transition matrix.

Its rows sum to 1,

$$\sum_y P_{xy} = 1 \quad \text{called a stochastic matrix.}$$

Lemma: $\mathbb{P}[X_n = y \mid X_0 = x] = P^n_{xy}$ ← matrix power.

Assume true up to n.

$$\mathbb{P}(X_{n+1} = y \mid X_0 = x] = \sum_z \mathbb{P}[X_{n+1} = y, X_n = z \mid X_0 = x]$$

$$= \sum_z \mathbb{P}[X_{n+1} = y \mid X_n = z, X_0 = x] \cdot \mathbb{P}(X_n = z \mid X_0 = x]$$

$$= \sum_z P^n_{xz} P_{zy} = P^{n+1}_{xy}$$

Does it ever return?

A state x for a Markov chain X_n is recurrent if

$$\mathbb{P}[\exists n \ X_n = x \mid X_0 = x] = 1 \quad \text{always returns}$$

otherwise it is transient.

Claim: A markov chain is recurrent iff

$$\mathbb{E} \sum_{n=1}^{\infty} \mathbb{P}_x[X_n = x] = \infty$$

Proof: Let N be the number of returns to x.

If recurrent,

$$\mathbb{P}_x[X_n = x, \forall m > n \ X_m \neq x]$$

$$= \mathbb{P}_x[X_n = x] \cdot \mathbb{P}[\forall m > n \ X_m \neq x \mid X_n = 0]$$

$$= \mathbb{P}_x[X_n = x] \cdot \mathbb{P}[\forall m > 0 \ X_m \mid X_0 = x] = 0$$

If transient and $q = \mathbb{P}_x[X_n \text{ returns to } x]$ then $N \sim \text{Geom}(1-q)$ so $\mathbb{E}N < \infty$.

<u>Theorem</u>: If X_n is simple random walk on $\mathbb{Z}^d$ then it is recurrent if $d = 1, 2$ and transient if $d \geq 3$.

"The drunk man always returns home but the drunk bird may not."

<u>Proof</u>: If $d=1$ then $X_n = W_n - (n - W_n)$ where W_n is # moves right by time n.

$W_n \sim \text{Bin}(n, \frac{1}{2})$ and

$$\begin{aligned}
\mathbb{P}[X_{2n} = 0] &= \mathbb{P}[W_{2n} = n] \\
&= \frac{(2n)!}{n!\,n!} \cdot 2^{-n} 2^{-n} \\
&= (1+o(1)) \frac{\sqrt{2\pi \cdot 2n}\,(2n)^{2n} \cdot e^{-2n}}{(\sqrt{2\pi n}\, n^n e^{-n})^2} \cdot 2^{-2n} \\
&= \frac{1+o(1)}{\sqrt{\pi n}}
\end{aligned}$$

So $\sum_n \mathbb{P}[X_{2n} = 0] \sim \sum_n \frac{1}{\sqrt{\pi n}} = \infty$.

<u>Stirling's Formula</u>:

$$\boxed{n! \approx \sqrt{2\pi n}\, n^n e^{-n}}$$

Let $Y_{n,i} = \#\{\text{steps in direction } i\}$

$Y_{n,i} \sim \mathrm{Bin}(n, \frac{1}{d})$.

$Z_{n,i} = \#$ positive steps in direction d.

$Z_{n,i} \mid Y_{n,i} \sim \mathrm{Bin}(Y_{n,i}, \frac{1}{2})$

$X_n = 0$ iff $Z_{n,i} = \frac{1}{2} Y_{n,i} \quad 1 \le i \le d$.

$$\mathbb{P}[X_{2n} = 0]$$

$$\geq \sum_{\substack{s_1, s_2 \\ \text{even}}} \mathbb{P}\left[Z_{2n,1} = \frac{s_1}{2}, Z_{2n,2} = \frac{s_2}{2}, Y_{2n,1} = s_1, Y_{2n,2} = s_2\right]$$

$$= \sum \mathbb{P}\left[Z_{2n,1} = \frac{s_1}{2}, Z_{2n,2} = \frac{s_2}{2} \,\middle|\, Y_{2n,1} = s_1, Y_{2n,2} = s_2\right] \cdot \mathbb{P}[Y_{2n,1} = s_1, Y_{2n,2} = s_2]$$

$$\geq \sum_{\substack{s_1, s_2 \\ \text{even}}} \frac{C}{\sqrt{s_1}} \cdot \frac{C}{\sqrt{s_2}} \cdot \mathbb{P}[Y_{2n,1} = s_1, Y_{2n,2} = s_2]$$

$$\geq \frac{C^2}{n} \sum_{\substack{s_1, s_2 \\ \text{even}}} \mathbb{P}[Y_{2n,1} = s_1, Y_{2n,2} = s_2]$$

$$= \frac{C^2}{n} \mathbb{P}[Y_{2n,1}, Y_{2n,2} \text{ even}] = \frac{C^2}{2n}.$$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \overset{\|}{\frac{1}{2}}$, last step has $\frac{1}{2}$ prob for even

So $\sum_n \mathbb{P}[X_{2n} = 0] = \infty$.

<u>$d \geq 3$:</u>

Let $A_n = \bigcap_{i=1}^{d} \{ Y_{n,i} \geq \frac{n}{2d} \}$

$$\mathbb{P}[A_n] \geq 1 - d e^{-cn}$$

$$\mathbb{P}[Z_{n,i} = \tfrac{s}{2} \mid Y_{n,i} = s] \leq \begin{cases} 0 & s \text{ is odd} \\ \frac{C}{\sqrt{s}} & s \text{ is even} \end{cases}$$

$$\mathbb{P}[X_n = 0] \leq \mathbb{P}[A_n^c] + \mathbb{P}[X_n = 0, A_n^c]$$

$$\leq d e^{-cn} + \sum_{\substack{s_1, \dots, s_d \\ \min s_i \geq \frac{n}{2d}}} \mathbb{P}[\forall i, \; Y_{n,i} = \tfrac{s_i}{2}, Z_{n,i} = s_i]$$

$$= d e^{-cn} + \sum \mathbb{P}[\forall i \; Y_{n,i} = \tfrac{s_i}{2} \mid Z_{n,i} = s_i] \cdot \mathbb{P}[\forall i \; Z_{n,i} = s_i]$$

$$\leq d e^{-cn} + \sum \left(\frac{C}{\sqrt{n/2d}} \right)^d \mathbb{P}[\forall i \; Z_{n,i} = s_i]$$

$$\leq d e^{-cn} + \frac{C'}{n^{d/2}}$$

So $\sum_{n=1}^{\infty} \mathbb{P}[X_n = 0] \leq \sum_{n=1}^{\infty} d e^{-cn} + \frac{C'}{n^{d/2}} < \infty$.

Hence X_n is recurrent.