Stochastic Integral

Sunday, September 29, 2019 6:56 PM

<u>Goal</u> Define $\int_0^t X(s)\,dB_s$

If G_s was C^1 then we would write

$$\int_0^t X(s)\,dG_s = \int_0^t X(s)\,G'(s)\,ds$$
$$= \lim_{n\to\infty} \sum_{i=1}^n X(\tfrac{i}{n})\,G(i/n) - G(\tfrac{i-1}{n})$$

But Brownian motion is <u>not</u> differentiable and

$$\sum_{i=1}^n |B_{i/n} - B_{(i-1)/n}|$$
$$\overset{d}{=} \frac{1}{\sqrt{n}} \cdot \sum_{i=1}^n |B_i - B_{i-1}| \longrightarrow \infty.$$

so we need to define things carefully.

X_t is a submartingale / supermartingale if $\forall\, t' > t > 0$,

$$\mathbb{E}[X_{t'} \mid \mathcal{F}_t] \geq X_t \quad \text{resp. } \mathbb{E}[X_{t'} \mid \mathcal{F}_t] \leq X_t.$$

If φ is convex, X_t is a martingale then $\varphi(X_t)$ is a submartingale.

If T is a stopping time with $0 \leq T \leq t$ then

$$\mathbb{E} X_0 \leq \mathbb{E} X_T \leq \mathbb{E} X_t.$$

<u>Dooh's Inequality</u> If X_t is a submartingal and $\bar{X}_t = \max_{0\leq s\leq t} X_s^+$ where $a^+ = \max\{0, a\}$

then $\forall \lambda \geq 0$

$$\lambda \mathbb{P}[\bar{X}_t \geq \lambda] \leq \mathbb{E} X_t I(\bar{X}_t \geq \lambda) \leq \mathbb{E} X_t^+ I(\bar{X}_t \geq \lambda)$$

LHS follows since if $T = t \wedge \inf\{s \le t : X_s \geq \lambda\}$ then

$\mathbb{E}[X_t \mid \mathcal{F}_T] \geq X_T \geq M$ on $\{\bar{X}_t \geq \lambda\}$.

Doob's Maximal Inequality

If X_n is a (sub)martingale then

$$\mathbb{E}[\bar{X}_t^2] \leq 4\,\mathbb{E}[(X_t^+)^2]$$

Proof:

$$\mathbb{E}[(\bar{X}_t \wedge M)^2] = \int 2\lambda\, \mathbb{P}[\bar{X}_t \wedge M \geq \lambda]\, d\lambda$$

$$\leq \int 2\lambda \left(\lambda^{-1} \mathbb{E}[X_t^+ I(\bar{X}_t \wedge M \geq \lambda)]\right) d\lambda$$

$$= \mathbb{E}\left[2X_t^+ \int_0^{\bar{X}_t \wedge M} 1 \cdot d\lambda\right] \quad \text{Fubini}$$

$$= 2\,\mathbb{E}[X_t^+ (\bar{X}_t \wedge M)]$$

$$\leq 2\sqrt{\mathbb{E}[(X_t^+)^2]\,\mathbb{E}[(\bar{X}_t \wedge M)^2]}$$

Cauchy Schwartz

$\Rightarrow$ for all M,

$$\mathbb{E}[(\bar{X}_t \wedge M)^2] \leq 4\,\mathbb{E}[(X_t^+)^2]$$

If Y_t is a martingale and $X_t = |Y_t|$ then

$$\mathbb{E}\left[\left(\max_{0 \le s \le t} |Y_s|\right)^2\right] \leq 4\,\mathbb{E}\,|Y_t|^2.$$

Quadratic Variation

Let $t_i^n = i 2^{-n}$ and

let $\langle X \rangle_t = \lim_n \sum_{i=1}^{t2^n} (B_{t_i^n} - B_{t_{i-1}^n})^2$.

<u>Lemma:</u> The limit exists and

$\langle X \rangle_t = t$ a.s.

<u>Proof:</u>

Let $D_\varepsilon = \{ \max_{0 \le m \le n} \sum_{i=1}^{m} (B_{t_i^n} - B_{t_{i-1}^n})^2 - \frac{m}{2^n} > \varepsilon \}$

$\mathbb{P}[D_\varepsilon] \le 2^n \exp(-c 2^n)$

So by Borel-Cantelli, limit exists almost surely for all dyadic points. Since t is monotone and dyadic points are dense

$$\lim_n \sum_{i=1}^{t2^n} (B_{t_i^n} - B_{t_{i-1}^n})^2 \to t \quad \text{a.s.}$$

Note that $d\langle X \rangle_t = dt$.

We will define the Itô integral

$$\int_0^t X(s)\, dB_s$$

first when $X(s) = f(B_s)$ where $f: \mathbb{R} \to \mathbb{R}$ is a C^1 function. Let $t_i^n = i2^{-n}$. Define

$$\int_0^t f(B_s)\, dB_s := \sum_{i=1}^{2^n t} f(B_{t_i^n})(B_{t_{i+1}^n} - B_{t_i^n}) = I_t^n$$

- First assume that $\|f\|_\infty, \|f'\| \le K$.

Then for $n > m$, set $A_i^{n,m} = f(B_{t_i^n})$, $D_i^{n,m} = f(B_{t^m_{\lfloor i2^{m-n} \rfloor}})$:

$$\mathbb{E}(I_t^n - I_t^m)^2 = \mathbb{E}\left(\sum (A_i - D_i)(B_{t_{i+1}^n} - B_{t_i^n}) \right)^2$$

$$= \sum_i \mathbb{E}\,(A_i - D_i)^2 (B_{t_{i+1}^n} - B_{t_i^n})^2$$

$$+ \sum_{i,j} \underbrace{\mathbb{E}\,(A_i - D_i)(A_j - D_j)(B_{t_{i+1}} - B_{t_i})(B_{t_{j+1}} - B_{t_j})}_{= 0 \quad (*)}$$

$$= \sum_i \mathbb{E}\Big[(A_i - D_i)^2\, \mathbb{E}\big((B_{t_{i+1}} - B_{t_i})^2 \,|\, \mathcal{F}_{t_i}\big]\Big]$$

$$= \sum_i \mathbb{E}\,[(A_i - D_i)^2 \cdot 2^{-n}]$$

$$\le K^2 t\, \mathbb{E} \max_i (B_{t_i^n} - B_{t^m_{i 2^{m-n}}})^2$$

$$\le K^2 t\; C \cdot 2^{-m/3}.$$

Since

$$(*) = \mathbb{E}\Big[(A_i - D_i)(A_j - D_j)(B_{t^n_{i+1}} - B_{t_i})\,\mathbb{E}\big[\underbrace{B_{t_{j+1}} - B_{t_j} \,|\, \mathcal{F}_{t_j}}_{=0}\big]\Big]$$

$$= 0$$

So $I_t^n \xrightarrow{a.s.} I_t$ so the limit exists.

To reduce to the case of general $f \in C^1$, take $T_M = \inf\{t : |B_t| = M\}$, define

$I_{t \wedge T_M}$ up to time T_M, then let $M \to \infty$.

So $\int_0^t f(B_s)\, dB_s := I_t$.

- Properties of Stochastic Integral

Martingale: If $I_t = \int_0^t f(B_s)\,dB_t$ then I_t is a martingale.

Proof: Let $0 < t < t'$ Then

$$\mathbb{E}[I_{t'} \mid \mathcal{F}_t] = \mathbb{E}\Big[\lim_n \sum_{i=0}^{t'2^n-1} f(B_{t_i^n})(B_{t_{i+1}^n} - B_{t_i^n}) \mid \mathcal{F}_t\Big]$$

$$= \lim_n \sum_{i=0}^{t2^n-1} f(B_{t_i^n})(B_{t_{i+1}^n} - B_{t_i})$$

$$+ \lim_n \sum_{i=t2^n}^{t'2^n-1} \underbrace{\mathbb{E}\big[f(B_{t_i^n})(B_{t_{i+1}^n} - B_{t_i}) \mid \mathcal{F}_t\big]}_{=0.}$$

$$= I_t.$$

Continuity: By above estimates

$$\mathbb{E}[(I_t^n - I_t^{n+1})^2] \le C\,2^{-n/3}$$

so $\mathbb{E}\max_t (I_t^n - I_t^{n+1})^2 \le 4C\,2^{-n/3}$

Since I_t^n are continuous martingales, I_t^n converges uniformly to I_t. a.s.

Itô's Formula (1)

If f is C^2 then

$$f(B_t) - f(B_0) = \underbrace{\int_0^t f'(B_s)\,dB_s}_{\text{Martingale}} + \underbrace{\frac{1}{2}\int_0^t f''(B_s)\,ds.}_{\text{Bounded Variation}}$$

By Taylor Series if $f \in C^2$

$$f(x+z) = f(x) + f'(x)z + \frac{1}{2}f''(y)z^2$$

for some $y \in [x, x+z]$. Hence

$$f(B_t) - f(B_0) = \sum^{2^n t-1} f(B_{t_{i+1}^n}) - f(B_{t_i^n})$$

$$f(B_t) - f(B_0) = \sum_{i=0}^{2^n t-1} f(B_{t_{i+1}^n}) - f(B_{t_i^n})$$

$$= \sum_{i=0}^{2^n t-1} f'(B_{t_i^n})(B_{t_{i+1}^n} - B_{t_i^n}) \to \int_0^t f'(B_s)\,dB_s$$

$$+ \sum_{i=0}^{2^n t-1} f(B_{y_i^n}) \cdot (B_{t_{i+1}^n} - B_{t_i^n})^2$$

where $y_i^n \in [t_i^n, t_{i+1}^n]$

Claim: If $\mu_n \to \mu$ weakly on $[0,t]$ and $f_n \to f$ uniformly then

$$\int f_n\,d\mu_n \to \int f\,d\mu.$$

Proof: Write X_n R.V. with law $\frac{\mu_n}{\mu[0,t]}$

$$X_n \xrightarrow{d} X.$$

$$\int f_n\,d\mu_n = \mathbb{E}[f_n(X_n)]$$

$$\left|\int f_n - f\,d\mu_n\right| \le \|f - f_n\|_\infty \mu_n[0,t] \to 0.$$

$$\left|\int f\,d\mu_n - \int f\,d\mu\right| = |\mathbb{E} f(X_n) - \mathbb{E} f(X)| \to 0.$$

So writing $f_n(t_i^n) = f(B_{y_i^n})$, $f_n \to f$ uniformly

So $$\sum_{i=0}^{2^n t-1} f(B_{y_i^n})(B_{t_{i+1}^n} - B_{t_i^n})^2 \to \int_0^t f(B_s)\,ds.$$

General Stochastic Integral.

Definition: $X(t)$ is adapted (w.r.t $\mathcal{F}_t$) if $\forall t$, $X(t)$ is $\mathcal{F}_t$ measurable.

Then $\int_0^t f(B_s)\,dB_s$ is adapted.

Definition: X_t is progessively measurable if for all t, $X : \Omega \times [0,t]$.

is $\mathcal{F}_t \otimes \mathcal{B}[0,t]$ measurable.

(Right continuous + Adapted) $\Rightarrow$ Progressively measurable

$\Rightarrow$ Adapted

Stochastic Integral for a step function

$$H_t = \sum H_i \, I(t_i \le t < t_{i+1})$$

where $\mathbb{E}(H_i^2) < \infty$.

Write $\|H\|_2^2 = \int \mathbb{E} H_t^2 \, dt$. and

$\langle H, H' \rangle = \int \mathbb{E} H_t H'_t \, dt$.

Then

$$I_t^H = \int_0^t H_s \, dB_s := \sum_i H_i \left(B_{t_{i+1}\wedge t} - B_{t_i \wedge t}\right).$$

With this definition I_t^H is a martingale and

$$\| I_t^H \|^2 := \mathbb{E}\left[(I_t^H)^2\right], \quad \langle I_t^H, I_t^{H'} \rangle = \mathbb{E}\left[I_t^H I_t^{H'}\right].$$

then

$$\| I_t^H \|^2 = \mathbb{E} \sum_{i,j} H_i (B_{t_{i+1}} - B_{t_i}) H_j (B_{t_{j+1}} - B_{t_j})$$

$$= \mathbb{E} \sum_i H_i^2 (B_{t_{i+1}} - B_{t_i})^2$$

$$= \sum \mathbb{E} H_i^2 (t_{i+1} - t_i) = \int \mathbb{E} H_i^2.$$

$$= \|H\|^2$$

So $H_t \mapsto I_t^H$ is an isometry.

Thus for a general progressively measurable H, if we can find step functions $\|H_n - H\| \to 0$ then $\| I^{H_n} - I^H \| \to 0$ for some limit I^H.

Approximation

Progressively measurable

$\downarrow$ (A)

Bounded & P.M

$\downarrow$ (B)

Bounded continuous & P.M.

$\downarrow$ (C)

Bounded Simple Function P.M.

For (A) set $H_t^n = H_t I(|H_t| \le n)$

then $\|H_t^n\|_\infty \le n$ and $\|H^n - H\| \to 0$.

For (B). set

$$H_t^n = n \int_{t-\frac{1}{n}}^{t} H_s \, ds$$

then H^n has cts paths and

$H^n(t, w) \to H(t,w)$ t-almost everywhere for all w so

$\|H^n - H\| \to 0$.

For (C) take

$$H_t^n = \sum H_{t_i^n} I(t_i^n \le t < t_{i+1}^n).$$

Do these definitions coincide?

- Yes since they are both the limit of the same simple functions.

What about replacing B_t with something else?

$\int X_s \, dM_s$.

If M_s is an L^2 martingale define

$$\langle M \rangle_{t,n} := \sum_i \left(M_{t_{i+1}^n \wedge t} - M_{t_i^n \wedge t}\right)^2.$$

We will show that

$$\langle M \rangle_t := \lim_n \langle M \rangle_{t,n} \text{ exists.}$$

and that

$$M_t^2 - \langle M \rangle_t \text{ is a martingale.}$$

If M_t is a martingale and $i \le j \le n$ then

$$\mathbb{E}\left[(M_{t_n} - M_{t_j})^2 \mid \mathcal{F}_{t_i}\right]$$

$$= \mathbb{E}\left[M_{t_n}^2 - 2M_{t_n} M_{t_j} + M_{t_j}^2 \mid \mathcal{F}_{t_i}\right]$$

$$= \mathbb{E}\left[M_{t_n}^2 - 2M_{t_j}\mathbb{E}[M_{t_n} \mid \mathcal{F}_{t_j}] + M_{t_j}^2 \mid \mathcal{F}_{t_i}\right]$$

$$= \mathbb{E}\left[M_{t_n}^2 - M_{t_j}^2 \mid \mathcal{F}_{t_i}\right].$$

Claim: If $\|M\|_\infty \le K$ then

$$\mathbb{E}[\langle M\rangle_t^2] \le 6K^4.$$

Proof:

$$\mathbb{E}\Big(\sum_{i<j} (M_{t_{i+1}} - M_{t_i})^2 (M_{t_{j+1}} - M_{t_j})^2\Big)$$

$$= \sum_i \mathbb{E}\Big[(M_{t_{i+1}} - M_{t_i})^2\, \mathbb{E}\Big(\sum_j (M_{t_{j+1}} - M_{t_j})^2 \,\Big|\, \mathcal{F}_{t_{i+1}}\Big)\Big]$$

$$= \sum_i \mathbb{E}\Big[(M_{t_{i+1}} - M_{t_i})^2\, \mathbb{E}\big(M_t^2 - M_{t_{i+1}}^2 \,\big|\, \mathcal{F}_{t_{i+1}}\big)\Big]$$

$$\le K^2 \sum_i \mathbb{E}(M_{t_{i+1}} - M_{t_i})^2 \le K^2\, \mathbb{E} \sum_i M_{t_{i+1}}^2 - M_{t_i}^2$$

$$\le K^4$$

$$\mathbb{E} \sum_i (M_{t_{i+1}} - M_{t_i})^4 \le 4K^2\, \mathbb{E} \sum (M_{t_{i+1}} - M_{t_i})^2$$

$$\le 4K^4$$

$$\mathbb{E}(\langle M\rangle_t^2) = \sum_i \mathbb{E}(M_{t_{i+1}} - M_{t_i})^4$$

$$+ 2\sum_{i<j} (M_{t_{i+1}} - M_{t_i})^2 (M_{t_{j+1}} - M_{t_j})^2$$

$$\le 6K^4.$$

Claim: $M_t^2 - \langle M\rangle_{t,n}$ is a martingale.

Proof: Enough to prove that for $t_i^n \le s \le t \le t_{i+1}^n$ that

$$\mathbb{E}[M_t^2 - \langle M\rangle_{t,n} - (M_s^2 - \langle M\rangle_{s,n}) \mid \mathcal{F}_s] = 0.$$

$$= \mathbb{E}[M_t^2 - (M_t - M_{t_i})^2 - M_s^2 - (M_s - M_{t_i})^2 \mid \mathcal{F}_s]$$

$$= \mathbb{E}[M_{t_i}(2M_t - M_{t_i}) - M_{t_i}(2M_s - M_{t_i}) \mid \mathcal{F}_s]$$

$$= M_{t_i}\, \mathbb{E}[2M_t - M_s \mid \mathcal{F}_s] = 0.$$

Let $m<n$ and

$$J_t = \langle M \rangle_{t,n} - \langle M \rangle_{t,m}$$

$$= \left(M_t^2 - \langle M \rangle_{t,m}\right) - \left(M_t^2 - \langle M \rangle_{t,n}\right)$$

is a martingale since it is a difference of two martingales.

Let $\Delta_m = \sup_j \sup_{s \in [t_j^m, t_{j+1}^m]} |M_s - M_{t_j^m}|$

Then $\mathbb{E}\, J_t^2 = \mathbb{E} \sum_i \left(J_{t_{i+1}} - J_{t_i}\right)^2$

If $t_j^m \le t_i^n < t_{i+1}^n \le t_{j+1}^m$ then

$$J_{t_{i+1}} - J_{t_i} = \left(M_{t_{i+1}^n} - M_{t_i^n}\right)^2 - \left(\left(M_{t_{i+1}^n} - M_{t_j^m}\right)^2 - \left(M_{t_i^n} - M_{t_j^m}\right)^2\right)$$

$$= 2\left(M_{t_{i+1}^n} - M_{t_i^n}\right)\left(M_{t_j^m} - M_{t_i^n}\right)$$

So $\mathbb{E}\left(J_{t_{i+1}} - J_{t_i}\right)^2 \le 4\, \mathbb{E}\left(\left(M_{t_{i+1}^n} - M_{t_i^n}\right)^2 \cdot \Delta_m^2\right)$

$$\mathbb{E}\, J_t^2 \le 4\, \mathbb{E}\left[\langle M \rangle_{t,n} \cdot \Delta_m^2\right]$$

$$\le 4 \sqrt{\mathbb{E}\left[\langle M_{t,n} \rangle^2\right]} \sqrt{\mathbb{E}\, \Delta_m^4}$$

$$= 4 \sqrt{6K^4} \sqrt{\mathbb{E}\, \Delta_m^4} \to 0. \text{ as } m \to \infty$$

Hence $\langle M \rangle_{t,n} \xrightarrow{L^2} \langle M \rangle_t$.

When M is unbounded we let

$T_K = \inf\{s : |M_s| = K\}$ and set M_t

$$\langle M \rangle_t = \lim_{K \to \infty} \langle M_{t \wedge T_K} \rangle_t .$$

The quadratic variation process is continuous, increasing and

$M_t^2 - \langle M \rangle_t$ is a martingale and so

$$\mathbb{E}[M_{t_{i+1}}^2 - M_{t_i}^2 \mid \mathcal{F}_{t_i}] = \mathbb{E}[\langle M\rangle_{t_{i+1}} - \langle M\rangle_{t_i}].$$

Example (Homework)

If $M_t = \int_0^t H_s \, dB_s$ then

$$\langle M_t \rangle = \int_0^t H_s \, ds.$$

General Integrals with respect to continuous martingales $\int_0^t H_s \, dM_s$

- H_s P.M. w.r.t. σ-algebra generated by M_s
- For a step function $H = \sum H_i \, I(t_i \le t < t_{i+1})$

$$I_t^H = \sum_i H_i (M_{t_{i+1} \wedge t} - M_{t_i \wedge t})$$

which is a martingale so

$$\mathbb{E}[I_t^H]^2 = \sum_i \mathbb{E}(H_i(M_{t_{i+1}} - M_{t_i}))^2$$

$$= \sum_i \mathbb{E}(H_i^2 \mathbb{E}[(M_{t_{i+1}} - M_{t_i})^2 \mid \mathcal{F}_{t_i}])$$

$$= \sum_i \mathbb{E}(H_i^2 \mathbb{E}[M_{t_{i+1}}^2 - M_{t_i}^2 \mid \mathcal{F}_{t_i}])$$

$$= \sum_i \mathbb{E}[H_i^2 (\langle M_{t_{i+1}} \rangle - \langle M \rangle_{t_i})]$$

$$= \mathbb{E}\left[\int_0^t H_s^2 \, d\langle M\rangle_s\right]$$

If $\|H\|_M^2 = \mathbb{E}[\int_0^t H_s^2 \, d\langle M\rangle_s]$

then $H \mapsto I_t^H$ is an isometry

and we define

$$I_t^H = \lim_n I_t^{H_n}.$$

$F(t)$ is Function of Finite variation

if $F(t) = F^+(t) - F^-(t)$ where

$F(t)$ is function of [illegible]

if $\quad F(t) = F^+(t) - F^-(t)$ where

$F^{\pm}(t)$ are increasing.

Equivalently:

$$\sup_{\{t_i\}} \sum_i |F(t_i) - F(t_{i-1})| < \infty$$

over all partitions t_i.

Proof: Set $\displaystyle F^+(t) = \sup_{0 \le t_1 \le \dots \le t_n = t} \sum_i \left(F(t_i) - F(t_{i-1})\right)^+$

and $F^-(t) = F(t) - F^+(t)$.

Brownian motion is not of finite variation.

For any bounded H_s and Y_s of finite variation we can define

$$\int_0^t H_s \, dY_s = \int_0^t H_s \, dY_s^+ - \int_0^t H_s \, dY_s^-.$$

So if $X_t = M_t + Y_s$ then

$$\int_0^t H_s \, dX_s := \int_0^t H_s \, dM_s + \int_0^t H_s \, dY_s$$

We call X_t a semi-martingale.

If $f \in C^2$ then

$$f(X_t) - f(X_0) = \underbrace{\int_0^t f'(B_s)\, dB_s}_{\text{martingale}} + \underbrace{\int_0^t \tfrac{1}{2} f''(B_s)\, ds}_{\text{finite variation}}$$

So $f(X_t)$ is a semi-martingale and we can write Itô's formula as

$$d f(X_s) = f'(B_s)\, dB_s + \tfrac{1}{2} f''(B_s)\, ds.$$

Itô's Formula for general martingales

If M is a continuous L^2 martingale

$$f(M_t, t) - f(M_0, 0) = \int_0^t f_x(M_s)\,dM_s + \int_0^t f_t(M_s)\,ds + \frac{1}{2}\int_0^t f_{xx}(M_s)\,ds$$

Itô's Formula II

Let $f(x,t) \in C^2$ be such that

$\mathbb{E}\int (f_x(B_s))^2\,ds < \infty$

Then

$$f(B_t, t) - f(B_0, 0) = \int_0^t f_x(B_s, s)\,dB_s + \int_0^t f_t(B_s, s)\,ds + \frac{1}{2}\int_0^t f_{xx}(B_s, s)\,ds.$$

Proof:

Write

$$f(B_{t_{i+1}}, t_{i+1}) - f(B_{t_i}, t_i)$$
$$= f_x(B_{t_i}, t_i)(B_{t_{i+1}} - B_{t_i}) + f_t(B_{t_i}, t_i)(t_{i+1} - t_i)$$
$$+ \frac{1}{2} f_{xx}(B_{t_i}, t_i)(B_{t_{i+1}} - B_{t_i})^2 + f_{xt}(B_{t_i}, t_i)(B_{t_{i+1}} - B_{t_i})(t_{i+1} - t_i)$$
$$+ \frac{1}{2} f_{tt}(B_{t_i}, t_i)(t_{i+1} - t_i)^2 + \ldots$$

Check that last two terms are negligible.

Multi-dimensional Brownian Motion

- We say B_t is <u>d-dimensional Brownian motion</u> if $B_t = (B_t^{(1)}, \dots, B_t^{(d)})$ is an $\mathbb{R}^d$ valued stochastic process where $B_t^{(i)}$ are independent Brownian motions.

- <u>Properties</u>:
  - Gaussian process
  - Independent Increments

    $\{B_{t_{i+1}} - B_{t_i}\}_{i \geq 0} \sim N(0, (t_{i+1} - t_i) I_d)$.

  - Continuous paths.

  It is uniquely defined by these properties.

<u>Also</u> - Martingale
- Scaling $B_t \overset{d}{=} a^{-1/2} B_{at}$
- Strong Markov Property

- If Q is an orthogonal matrix, $QQ^T = I_d$.

$$Q B_t \overset{d}{=} B_t$$

- check covariances

$$\mathrm{Cov}(x^T B_t \; y^T B_s) = \mathrm{Cov}\left(\sum_i x_i B_t^{(i)}, \sum y_i B_s^{(i)}\right)$$
$$= (t \wedge s)\, x^T y$$

$$\mathrm{Cov}(x^T Q B_t, y^T Q B_s)$$
$$= (t \wedge s)\, x^T Q Q^T y$$
$$= (t \wedge s)\, x^T y.$$

Higher Dimensional Itô's Formula

If B_t is d-dimensional B.M.

$f: \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$, f is C^2 and

$\int_0^t |\nabla f(B_s, s)|^2 ds < \infty$ then

$$f(B_t, t) - f(B_0, 0) = \sum_{i=1}^{d} \int_0^t f_{x_i}(B_s, s)\, dB^{(i)}$$
$$+ \int_0^t f_t(B_s, s)\, ds$$
$$+ \sum_{i=1}^{d} \int_0^t f_{x_i x_i}(B_s, s)\, ds.$$
$$= \int_0^t \nabla f(B_s, s) \cdot dB_s$$
$$+ \int_0^t f_t(B_s, s)\, ds$$
$$+ \int_0^t \Delta f(B_s, s)\, ds$$

Proof: Expand out

$$f(B_{t_{i+1}}, t_{i+1}) - f(B_{t_i}, t_i)$$
$$= \sum_i f_{x_i}(B_{t_i}, t_i)(B_{t_{i+1}} - B_{t_i})$$
$$+ f_t(B_{t_i}, t_i)(t_{i+1} - t_i)$$
$$+ \frac{1}{2} \sum_i f_{x_i x_i}(B_{t_i}, t_i)\left(B^{(i)}_{t_{i+1}} - B^{(i)}_{t_i}\right)^2$$
$$+ \frac{1}{2} \sum_{i \neq j} f_{x_i x_j}(B_{t_i}, t_i)\left(B^{(i)}_{t_{i+1}} - B^{(i)}_{t_i}\right)\left(B^{(j)}_{t_{i+1}} - B^{(j)}_{t_i}\right)$$
$$+ \sum_i f_{x_i t}(B_{t_i}, t)\left(B^{(i)}_{t_{i+1}} - B_{t_i}\right)(t_{i+1} - t_i) + \dots$$

Recurrence and transience.

For $d \geq 3$ let $\varphi(x) = |x|^{2-d}$

Then $\varphi_{x_i x_i}(x) = \frac{d^2}{dx_i^2}\left(\sum_j x_j^2 \right)^{1-d/2}$

$$= \frac{d}{dx_i}(1-\tfrac{d}{2})\, 2x_i \left(\sum_i x_i^2 \right)^{-d/2}$$

$$= -\tfrac{d}{2}(1-\tfrac{d}{2})\, 4x_i^2 \left(\sum_i x_i^2 \right)^{-d/2-1}$$

$$+ (1-\tfrac{d}{2})\, 2 \left(\sum_i x_i^2 \right)^{-d/2}$$

$$\sum_{i=1}^{d} \varphi_{x_i x_i} = (1-\tfrac{d}{2})|x|^{-d}(-2d+2d) = 0,$$

So $\Delta \varphi = 0$.

Is $\varphi(B_t)$ a martingale? No. Why?

Well $\mathbb{E}[\varphi(B_t)] = \mathbb{E}[|B_t|^{2-d}]$

$$= \mathbb{E}[|t^{1/2} B_1|^{2-d}]$$

$$= t^{\frac{2-d}{2}}\, \mathbb{E}|B_1|^{2-d}$$

$$\mathbb{E}|B_1|^{2-d} = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^3} |x|^{2-d} \exp(-|x|^2/2)\, dx$$

$$= C \int_0^\infty r^{d-2}\, r^{2-d} \exp(-r^2/2)\, dr$$

$$= C_1$$

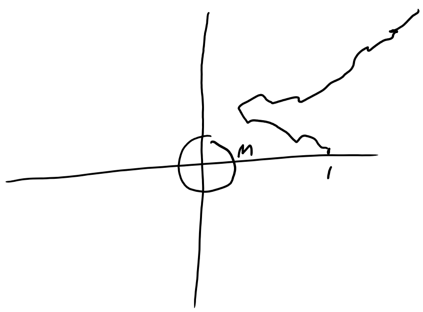
So $\mathbb{E}[\varphi(B_t)] \to 0$.

It is, however, a <u>local martingale</u>.

$$\boxed{\text{Defn: } X_t \text{ is a local martingale if there exists stopping times } T_n \text{ such that } T_M \uparrow \infty \text{ a.s. and } X_{t\wedge T_M} \text{ is a martingale.}}$$

Set $B_0 = 1$, $T_M = \inf\{t : |B_t| = M\}$.

For $M < 1$,



$\varphi(B_{t\wedge T_M})$ is a martingale.

$$\mathbb{E}\,\varphi(B_{t\wedge T_M}) = \mathbb{E}[\varphi(B_0)] = 1$$

$$\lim_{t\to\infty} \mathbb{E}\,\varphi(B_{t\wedge T_M}) = M^{2-d}\,\mathbb{P}[T_M < \infty]$$

so $\mathbb{P}[T_M < \infty] = M^{d-2}$ for $M < 1$.

Hence $\mathbb{P}[\exists t : B_t = 0] = 0$, B_t is transient.

Furthermore $\varphi(B_t)$ is a local martingale w.r.t. $T_{n^{-1}}$.

<u>What about $d=2$</u>

Set $\varphi(x) = \log|x|$, $\Delta(\varphi) = 0$.

If $|B_0| = x > 0$, if B_t were recurrent then for some M, $|B_t|$ hits 0 before M with positive probability $p > 0$.

Set $T_n = \inf\{t : |B_t| \in \{\frac{1}{n}, M\}\}$.

Then $\varphi(B_{t\wedge T_n})$ is a martingale and

$$\begin{aligned}
\log x &= \varphi(B_0) \\
&= \mathbb{E}\,\varphi(B_{t\wedge T_n}) \\
&\to \mathbb{E}\,\varphi(B_{T_n}) \\
&= (-\log n)\,\mathbb{P}[B_{T_n} = \tfrac{1}{n}] + \mathbb{P}[B_{T_n} = M](\log M) \\
&\leq \underbrace{-p\log n}_{\to -\infty} + (1-p)\log M
\end{aligned}$$

as $n \to \infty$.

Hence not recurrent.

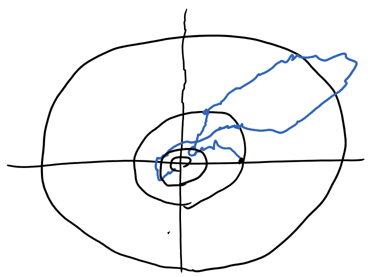
Now let $T = \inf\{t : |B_t| \in \{\frac{x}{2}, 2x\}\}$.

Then $\varphi(B_{t \wedge T})$ is a martingale and

$$\mathbb{E}\,\varphi(B_T) = \mathbb{E}(B_0) = \log x$$

$$= \mathbb{P}[|B_T| = \tfrac{x}{2}](\log x - \log 2)$$
$$+ \mathbb{P}[|B_T| = 2x](\log x + \log 2)$$

So $\mathbb{P}[|B_T| = 2x] = \mathbb{P}[\underline{B}_T = \frac{x}{2}]$



Random walk on $\{x : x = 2^k\}$.

Let's apply Ito's formula to

$f(x) = |x|$.

Then $f_{x_i}(x) = \dfrac{x_i}{|x|}$

$$f_{x_i x_i}(x) = \frac{1}{|x|} - \frac{x_i^2}{|x|^3}$$

So $\quad |\nabla f(x)|^2 = \sum_i \dfrac{x_i^2}{|x|^2} = 1.$

$$\Delta f(x) = -(d-1)/|x|$$

$$\nabla f(B_{t_i}) \cdot (B_{t_{i+1}^n} - B_{t_i^n}) \overset{d}{=} N(0, 2^{-n})$$

independent of $\mathcal{F}_{t_i^n}$.

Consider

$$\sum_{i=0}^{2^n t} \nabla f(\underline{B}_{t_i}) \cdot (\underline{B}_{t_{i+1}^n} - \underline{B}_{t_i})$$

$$\sum_{i=0}^{2^n t} \nabla f(\underline{B}_{t_i}) \cdot (\underline{B}_{t_{i+1}^n} - \underline{B}_{t_i})$$

$$\stackrel{d}{=} \sum_{i=0}^{2^n t} (B_{t_{i+1}} - B_{t_i}) \to \text{Brownian motion}$$

So if $Y_t = |B_t|$ then

$$Y_t - Y_0 = \underbrace{\int_0^t \sum f_{x_i}(B_s)\, dB_s^{(i)}}_{\text{Brownian motion}} - \frac{d-1}{2} \int_0^t Y_s^{-1}\, ds$$

A solution to the stochastic differential equation (SDE)

$$dY_s = dB_s - \frac{d-1}{2} Y^{-1}\, ds$$

called a Bessel Process of order d.

We will see later that Brownian motion, conditioned to be non-negative, has the law of a Bessel-3.