One Dimensional SDEs

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Let's investigate time homogeneous SDE's in one dimension,

$$dX_t = \mu(X_t)dt + \sigma(X_t)dB_t$$

Assume that μ, σ smooth, Lipschitz.

Solutions:

Step (1) Transform it into a martingale. Find φ such that $Y_t = \varphi(X_t)$ is a martingale. By Itô,

$$dY_t = \varphi'(X_t)\mu(X_t)\,dt + \varphi'(X_t)\sigma(X_t)\,dB_t + \frac{1}{2}\sigma^2(X_t)\varphi''(X_t)\,dt$$

So we want

$$\varphi'(x)\mu(x) + \frac{1}{2}\sigma^2(x)\varphi''(x) = 0$$

Set $h(x) = \varphi'(x)$

$$\frac{d}{dx}\log(h(x)) = \frac{h'(x)}{h(x)} = -\frac{2\mu(x)}{\sigma^2(x)}$$

So

$$h(x) = B\exp\left(-\int \frac{2\mu(x)}{\sigma(x)^2}\,dx\right)$$

$$\varphi(x) = A + B\int \exp\left(-\int \frac{2\mu(x)}{\sigma(x)^2}\,dx\right)$$

Example:

Brownian motion with drift.

$$X_t = \mu t + B_t$$

$$\varphi(x) = A + B\int \exp(-2\mu x)\,dx$$

$$= A + B\exp(-2\mu x)$$

Example Bessel process

$$dX_t = \frac{d-1}{2}\frac{1}{X_t}\,dt + dB_t$$

$$\varphi(x) = A + B\int \exp\left(\int -\frac{d-1}{x}\,dx\right)$$

$$= A + B\int \exp(-(d-1)\log x)$$

$$= A + B\int x^{-(d-1)}\,dx$$

$$= \begin{cases} A + B\,x^{-(d-2)} & d > 2 \\ \log(x) & d = 2 \end{cases}$$

$$\begin{cases} \log(x) & d=2 \\ A + B\,x^{-(d-2)} & d<2. \end{cases}$$

So

$$dY_t = h(Y_t)\,\sigma(\varphi^{-1}(Y_t))\,dB_t = g(Y_t)\,dB_t$$

So Y_t is a time changed Brownian motion.

Construction:

Let W_t be Brownian motion, set

$$\gamma_t = \inf\{s : \int_0^s \frac{1}{g^2(W_u)}\,du = t\},$$

set

$$Y_t = W_{\gamma_t}.$$

Then

$$\langle Y \rangle_t = \langle W_\gamma \rangle_t = \gamma_t$$

so $d\langle Y\rangle_t = d\gamma_t = g^2(W_{\gamma_t})\,dt$

$$= g^2(Y_t)\,dt$$

Let $Z_t = \int_0^t \frac{1}{g(Y_s)}\,dY_s$

so Z_t is a martingale and

$$\langle Z \rangle_t = \int_0^t \frac{1}{g(Y_s)^2}\,d\langle Y\rangle_s$$

$= t$

$$= \int_0^t ds = t$$

so Z_t is Brownian motion and Y_t solves

$$dY_t = g(Y_t)\, dZ_t.$$

Now set $X_t = \varphi^{-1}(Y_t)$ and

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dZ_t.$$

Hitting probabilities

If $x_0 \in [a,b]$ what is the probability that X_t hits a before b?

Let $T = T_{a,b} = \inf\{t : X_t \in \{a,b\}\}$

$$\mathbb{E}\,\varphi(X_T) = \mathbb{E}\,\varphi(X_0)$$
$$= \mathbb{P}[X_T = a]\,\varphi(a) + \mathbb{P}[X_T = b]\,\varphi(b).$$

So

$$\mathbb{P}[X_T = a] = \frac{\varphi(b) - \varphi(x_0)}{\varphi(b) - \varphi(a)}$$

If $dX_t = \mu(X_t)\,dt + dB_t$

under what conditions on $\mu(x)$ is

$\mathbb{P}[T_a = \infty] > 0$, i.e. X_t is transient.

If $\lim_{b \to \infty} \varphi(b) < \infty$ then

$$\lim_{b \to \infty} \mathbb{P}[X_{T_{a,b}} = a] = \lim_{b \to \infty} \frac{\varphi(b) - \varphi(x_0)}{\varphi(b) - \varphi(a)} < 1.$$

Recall

$$\varphi(x) = \int_1^x \exp\left(-2 \int_1^y \mu(u)\, du\right) dy$$

$\varphi(\infty) < \infty$ if $\mu(u)$ is positive and doesn't decay too quickly.

<u>Example:</u> Bessel Process If $\mu(x) = \frac{d-1}{2x}$

$$\varphi(x) = \int_1^x \exp(-(d-1)\log y)$$

$$= \int_1^x y^{-(d-1)}\, dy$$

$$= \begin{cases} \frac{1}{(2-d)}\left(x^{-(d-2)} - 1\right) & d \neq 2 \\ \log x & d = 2 \end{cases}$$

So $\lim_{x \to \infty} \varphi(x) < \infty$ iff $d > 2$.

<u>Example</u> $dX_t = (X_t^3 - \gamma X_t^4)\,dt + X_t^{5/2}\,dB_t$

Then

$$\varphi(x) = \int_1^x \exp\left(-2\int_1^y \frac{u^3 - \gamma u^4}{u^5}\,du\right) dy$$

$$= \int_1^x \exp\left(2u^{-1} + 2\gamma \log u\right) dy$$

$$= \int_1^x u^{2\gamma} \exp\left(2u^{-1}\right) dy$$

For any $\gamma \geq -\frac{1}{2}$ $\varphi(x) \to \infty$ as $x \to \infty$ so X_t is recurrent.

$\varphi(x) \to -\infty$ as $x \to 0$ so X_t never hits 0.

If $\sigma(x) \to 0$ as $x \to 0$ fast enough, X_t may never hit 0.

E.G. $dX_t = X_t^{\alpha}\,dB_t$.

Then $X_t = W_{\gamma_t}$, W_t B.M.,

$$\gamma_t = \inf\left\{s : \int_0^s \frac{1}{W_u^{2\alpha}}\,du = t\right\},$$

Let $\tau_k = \inf\{t : W_t = 2^{-k}\}$

Time spent in $[2^{-k-1}, 2^{-k}]$ from time

τ_k to τ_{k+1} approximately 2^{-2k},

$$\mathbb{E} \int_{\tau_k}^{\tau_{k+1}} \frac{1}{W_u^{2\alpha}} \, du \geq 2^{2k(\alpha-1)}$$

So if $\tau^* = \inf\{t : W_t = 0\} = \lim \tau_k$,

$$\int_0^{\tau^*} \frac{1}{W_u^{2\alpha}} \, du \geq \sum \int_{\tau_k}^{\tau_{k+1}} \frac{1}{W_u^{2\alpha}} \, du \to \infty$$

$\alpha > 1$ so $\gamma_t < \tau_*$ for all t.

Transition Probabilities

Define the semigroup of operators on bounded functions

$$(T_t f)(x) = \mathbb{E}[f(X_t) \mid X_0 = x]$$

$$(T_0 f)(x) = f(x) \quad \text{so} \quad T_0 f = f.$$

$$T_{t+s} f(x) = \mathbb{E}[f(X_{t+s}) \mid X_t = 0]$$

$$= \mathbb{E}[\mathbb{E}[f(X_{t+s}) \mid X_s] \mid X_t = 0]$$

$$= \mathbb{E}[T_t(f)(X_s) \mid X_t = 0]$$

$$= T_s(T_t(f))(x_0).$$

The infintesimal generator is defined as

$$Af = \lim_{h \downarrow 0} \frac{T_h f - f}{h}$$

By the semigroup property

$$\frac{d}{dt} T_t f(x) = A T_t f(x).$$

$$f(X_t) - f(X_0) = \int_0^h \mu(X_s) ds + \int_0^h \sigma(X_s) dB_s + \frac{1}{2} \int_0^h \sigma^2(X_s) ds$$

$$\frac{1}{h} \mathbb{E}[f(X_t) - f(X_0)] \to \mu(x_0) \frac{d}{dx} f(x) + \frac{1}{2} \sigma^2(x) \frac{d^2}{dx^2} f(x).$$

$$A = \mu(x) \frac{d}{dx} + \frac{1}{2} \sigma^2(x) \frac{d^2}{dx^2}.$$

If $u(x,t) = T_t f(x)$ then $\frac{du}{dt} = Au.$

Solves the PDE,

(i) $u_t = Au$ u is $C^{1,2}$ on $(0,\infty) \times \mathbb{R}$

(ii) $u(0,x) = f(x)$, u is continuous on $[0,\infty) \times \mathbb{R}$. (*)

If $u(t,x)$ solves (*) then $Y_s = u(t-s, X_s)$

$$Y_s - Y_0 = \int_0^s - u_t(t-r, X_r) dr + \int_0^s u_x(t-r, X_r) \mu(X_r) dr$$

$$-\frac{1}{2}\int_0^s u_{xx}(t-r, X_r)\,\sigma^2(X_r)\,dr$$
$$+\int_0^s u_x(t-r, X_r)\,\sigma(X_r)\,dr$$

$$= \int_0^s u_x(t-r, X_r)\,\sigma(X_r)\,dr$$

martingale

Hence $\mathbb{E}[Y_t \mid X_0 = x_0] = \mathbb{E}[f(X_t) \mid X_0 = x_0]$

$$= \mathbb{E}[Y_0 \mid X_0 = x_0] = u(t, x_0).$$

What is the density of X_t given $X_0 = x$. Call this $p_t(x,y)$.

Then

$$u(t,x) = \mathbb{E}[f(X_t) \mid X_0 = x] = \int_{\mathbb{R}} p_t(x,y)\,f(y)\,dy$$

$$0 = u_t - Au = \int f(y)\left[\frac{dp}{dt}(t,x,y) - A\,p(t,x,y)\right]dy.$$

Theorem: If μ, σ are bounded, Hölder continuous, $\inf \sigma^2(x) > 0$ then $\exists\, p_t(x,y)$ s.t.

$$u(t,x) = \int_{\mathbb{R}} p_t(x,y)\,f(y)\,dy$$

and $\dfrac{dp}{dt} = Ap$ applied in x co-ordinate.

This is the fundamental solution to the PDE and gives the transition probability

$$\mathbb{P}[X_t \in B \mid X_0 = x_0] = T_t 1_B(x_0)$$

$$= \int_B p_t(x_0, y)\, dy$$

Example: Ornstien-Ulenbeck

$$dX_t = -\theta X_t\, dt + \sigma\, dB_t,$$

Heuristic

$$\begin{aligned} X_t &= X_{t-t} - \frac{\theta}{n} X_{t-\frac{1}{n}} + \sigma N(0, \tfrac{1}{n}) \\ &= (1 - \theta/n) X_{t-\frac{1}{n}} + \sigma N(0, \tfrac{1}{n}) \\ &= (1 - \theta/n)^2 X_{t-\frac{2}{n}} + (1-\theta/n)\sigma N(0, 1/n) \\ &\quad + \sigma N(0, 1/n) \\ &= \dots \end{aligned}$$

$$X_t = e^{-\theta t} X_0 + \int_0^t e^{-\theta s} \sigma\, dB_s$$

$$N(0, \sigma^2 \int_0^t e^{-2\theta s} ds)$$

$$p_t(x,y) = \frac{1}{\sqrt{2\pi(\sigma^2/2\theta)(1-e^{-2\theta t})}} \exp\left(- \frac{(y - e^{-\theta} x_0)^2}{(\sigma^2/\theta)(1 - e^{-2\theta t})} \right)$$

$$\sim N\left(e^{-\theta} x, \frac{\sigma^2}{2\theta}(1 - e^{-2\theta t})\right)$$

Converges to $N(0, \frac{\sigma^2}{2\theta})$.

What is the stationary distribution in general?

Three approaches

(1) Find $\pi(x)$ such that

$$\int \pi(x)\, p_t(x,y)\, dx = \pi(y).$$

(2) Take limit $\lim_{t\to\infty} p_t(x,y) = \pi(y).$

(3) If π is stationary

$$\mathbb{E}_\pi f(X_0) = \mathbb{E} f(X_t)$$

so $\frac{d}{dt}\mathbb{E}_\pi f(X_t) = 0$

$$= \int \pi(x)\, A f(x)\, dx$$

$$= \int \pi(x) \left[\mu(x) \frac{d}{dx} f(x) + \frac{1}{2}\sigma^2(x) \frac{d^2}{dx^2} f(x) \right]$$

$$= \int \left[-\frac{d}{dx}\left[\pi(x)\mu(x)\right] + \frac{d^2}{dx^2} \frac{1}{2}\sigma^2(x)\pi(x) \right] f(x)\, dx$$

So $\frac{d}{dx}\left[-\pi(x)\mu(x) + \frac{d}{dx} \frac{1}{2}\sigma^2(x)\pi(x) \right] = 0$

Solve $-\pi(x)\mu(x) + \frac{d}{dx}\frac{1}{2}\sigma^2(x)\pi(x) = 0$

$$-(\mu(x) - \sigma(x)\sigma'(x))\pi(x) + \frac{1}{2}\sigma^2(x)\pi'(x) = 0$$

$$\frac{d}{dx}\log(\pi(x)) = \frac{\pi'(x)}{\quad} = \frac{2(\mu(x) - \sigma'(x)\sigma(x))}{\quad}$$

$$\frac{d}{dx}\log(\pi(x)) = \frac{\pi'(x)}{\pi(x)} = \frac{2(\mu(x) - \sigma'(x)\sigma(x))}{\sigma^2(x)}.$$

$$\pi(x) = \frac{1}{Z}\exp\left(\int_0^x \frac{2(\mu(y) - \sigma'(y)\sigma(y))}{\sigma^2(y)} dy \right)$$

For the Ornstein-Uhlenbeck process, $\mu = -\theta x$

$$\pi(x) = \frac{1}{Z}\exp\left(\frac{-\theta x^2}{\sigma^2}\right) = \frac{1}{Z}\exp\left(-\frac{x^2}{2(\sigma^2/2\theta)}\right)$$

For $dX_t = (X_t^3 - \gamma X_t^4)dt + X_t^{5/2} dB_t$,

$$\pi(x) = \exp\left(\int_1^x \frac{2(y^3 - \gamma y^4 - \frac{5}{2}y^{\frac{3}{2}} \cdot y^{\frac{5}{2}})}{y^5} dy \right)$$

$$= \exp\left(\int_1^x 2y^{-2} - (2\gamma + 5)y^{-1}\, dy \right)$$

$$= \exp\left(-2x^{-1} - (2\gamma+5)\log x \right)$$

$$= x^{-(2\gamma+5)} \exp(-2x^{-1})$$

Inverse Gamma $(2\gamma+4, 2)$.

Connection to PDEs

Dirichlet Problem on $\mathbb{R}^d$

$\Delta u = 0$ in D, u is C^2

$u = g$ on ∂D. and continuous

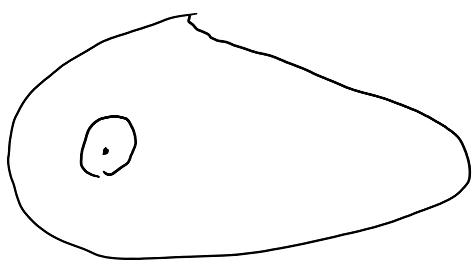
Assume D is open, bounded, ∂D is C^∞ manifold.

Let τ be first exist time of D,

$u(x) = \mathbb{E}[f(B_\tau) \mid X_0 = x]$.

Then $u(B_{t \wedge \tau})$ is a martingale.

<u>Mean value property</u>



If B ball around x_0,

$$
\begin{aligned}
u(x_0) &= \mathbb{E}[f(B_\tau) \mid \tau_B] \\
&= \mathbb{E}[u(B_{\tau_B})] \\
&= \fint_{\partial B} u(y)\, dy
\end{aligned}
$$

So by the mean value property

u is C^∞, $\Delta u = 0$.

Without PDE's

- Smoothness:

Let $\psi : [0, \infty)$ be C^∞

supported on $[\frac{\delta}{2}, \delta]$, positive on $(\frac{\delta}{2}, \delta)$.

Then $\int_{\mathbb{R}^2} u(y)\, \psi(|x-y|)\, dy = c u$

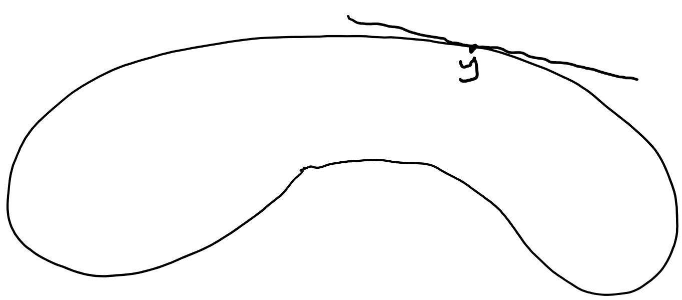
so u is C^∞.

By Itô's formula,

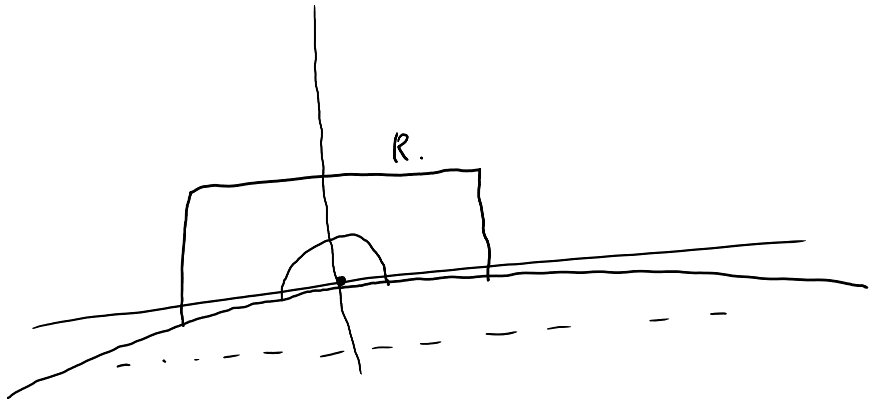
$$u(B_{t\wedge\tau}) - u(B_0) = \sum_i \int_0^\tau u_{x_i}(B_s)\, dB_s^i$$
$$+ \frac{1}{2}\int_0^\tau \Delta u(B_s)\, ds$$

and since $u(B_s)$ is a martingale.

- Need to check u is continuous on ∂D.



Suppose $y \in \partial D$.
Translate & rotate so that y is at the origin and the tangent plane is the x-axis



Suppose the surface is the graph of a function $f(x)$ locally and $|f(x)| \leq \delta |x|$ for $|x| \leq R$.

Pick R small enough so that $|f(x) - f(0)| \leq \varepsilon/2$ for $|x| \leq R$.

If $|z| < \delta R$ then, let A be the event

$$\left\{ \sup_{0 \leq s \leq \theta R^2} |B_s^{(1)}| \leq R/2, \ \inf_{0 \leq s \leq \theta R^2} B_s^{(2)} \leq -2\delta R \right\}$$

$$\mathbb{P}[A^c] \leq 4\,\mathbb{P}[|B_{\theta R^2}| \geq \tfrac{R}{2}] + \mathbb{P}[|B_{\theta R^2}| \leq 2\delta R]$$

$$= 4\,\mathbb{P}[N(0,1) \geq \tfrac{1}{2}\theta^{-1/2}] + \mathbb{P}[N(0,1) \leq 2\delta\theta^{-1/2}]$$

$$\leq \varepsilon.$$

if $\delta \ll \sqrt{\theta} \ll 1$.

On the event A,

$$\mathbb{E}[|f(B_\tau) - f(B_0)|] \leq \varepsilon\,\mathbb{P}[A] + 2\|f\|\,\mathbb{P}[A^c]$$

- What if I wanted to know $\mathbb{E}\tau$?

- Let $f(x_1, x_2) = -x_1^2$. u solves the Dirichlet problem for $\Delta u \equiv 0$, $u = f$ on ∂D.

Then if $v = u - f$, $\Delta v = -2$, $v \equiv 0$ on ∂D.

$$v(X_\tau) - v(X_0) = \sum_{x_i} \int v_{x_i}(X_s)\, dB_s^{(i)} + \frac{1}{2}\int_0^\tau \Delta v(X_s)\, ds$$

$$-v(x_0) = \mathbb{E} - \int_0^\tau ds = -\mathbb{E}\tau$$

$$v(x_0) = \mathbb{E}\tau.$$

Hitting Measure

What is the density h on ∂D?

Half plane

$$H := \{(x,y) : x \in \mathbb{R}^{d-1},\ y \in \mathbb{R},\ y > 0\}$$

$H := \{(x,y) : x \in \mathbb{R}^{d-1}, y \in \mathbb{R}, y > 0\}$

Let $h_\theta(x,y) = \dfrac{C_d\, y}{\left((x-\theta)^2 + y^2\right)^{d/2}}$ for $\theta \in \mathbb{R}^{d-1}$

such that $\int_{\mathbb{R}^{d-1}} h_\theta(0,1)\, d\theta = 1.$

$$\Delta h_\theta = \sum_i \frac{d^2}{dx_i^2} h_\theta(x,y) + \frac{d}{dy^2} h_\theta(x,y) = 0$$

Set

$$u(x,y) = \int_{\mathbb{R}^{d-1}} h_\theta(x,y) f(\theta)\, d\theta.$$

$\Delta u = 0,$

If $x_n \to x$, $y_n \to 0$ then

$u(x_n, y_n) \to f(x)$

So

$$u(x,y) = \mathbb{E}\left[f(B_{\tau_H}) \mid B_0 = (x,y) \right]$$

and the hitting density from $(x,y) \in H$ to $\theta \in \partial H$ is $h_\theta(x,y)$.

$h_\theta(0,1) = \dfrac{1}{\pi(1+\theta^2)}$ Cauchy Distribution

Also,

$h_\theta \sim B^{(1)}_\tau$

where $\tau = \inf\{t : B^{(2)}_t - B^{(2)}_0 = -1\}$.

By Brownian Scaling

$h_\theta \sim \sqrt{\tau}\, N(0,1)$

$\mathbb{P}[\tau \le t] = \mathbb{P}[\min_{0 \le s \le t} B^{(2)}_s \le -1]$

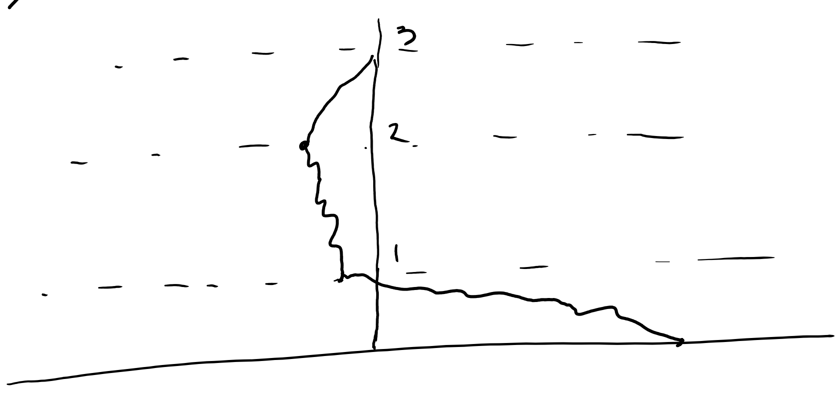
$= \mathbb{P}[\max_{0 \le s \le t} B_s \geq 1] = \mathbb{P}[|B_t| \geq 1]$

$= \mathbb{P}[|N(0,1)| \geq 1/\sqrt{t}] = \mathbb{P}[\frac{1}{|N(0,1)|^2} \le t]$

So $h_\theta \overset{d}{=} W/W'$ ratio of independent normals.

Another reason it must by Cauchy.

The Cauchy distribution is the only symmetric 1-self similar process. (up to scaling)



Starting from $(0,h)$ let $\tau_h = \inf\{t : B^{(2)}_t = h\}$.

Then set $Y_j = B^{(1)}_{\tau_j} - B^{(1)}_{\tau_{j-1}} \sim h_\theta(0,1)$

independent

$B^{(1)}_{\tau_k} - B^{(1)}_{\tau_0} \sim h_\theta(0,k)$

So h_θ is 1-self similar.

Other Domains

A map $f : D \to D'$, $D, D' \in \mathbb{C}$ is conformal if it is holomorphic and has derivative that is no-where vanishing.

$\Delta f = 0$ so $f(B_t)$ is a martingale.

Let $f(z) = u(z) + i v(z)$, $u(B_t) = U_t$, $v(B_t) = V_t$.

Thm: If M_t is a d-dimensional martingale

$$\langle M^{(i)}, M^{(j)} \rangle_t = \begin{cases} 0 & \text{if } i \neq j \\ t & \text{if } i = j \end{cases}$$

then M_t is d-dimensional B.M.

Proof: $\sum a_i M^{(i)}_t$ is a one dimensional martingale with $\langle \sum a_i M^{(i)}_t \rangle = t \sum a_i^2$

So is B.M. $\times \sqrt{\sum a_i^2}$. Follows that jointly

Gaussian with same covariance as B.M. ∎

$$U_t - U_0 = \int_0^t u_x(B_s)\, dB_s^{(1)} + \int_0^t u_y(B_s)\, dB_s^{(2)}$$

$$V_t - V_0 = \int_0^t v_x(B_s)\, dB_s^{(1)} + \int_0^t v_y(B_s)\, dB_s^{(2)}$$

Cauchy - Riemann Relations

$$\boxed{u_x = v_y \qquad u_y = -v_x}$$

So $\langle U \rangle_t = \int_0^t |u_x(B_s)|^2 + |u_y(B_s)|^2\, ds$

$= \langle V \rangle_t$

$$\langle U+V \rangle_t = \int_0^t (u_x + v_x)^2 + (u_y + v_y)^2\, ds$$

$$= \int_0^t u_x^2 + v_x^2 + u_y^2 + v_y^2\, ds$$

$$= \langle U \rangle_t + \langle V \rangle_t$$

So $\langle U, V \rangle_t = \frac{1}{2}\left(\langle U+V \rangle_t - \langle U \rangle_t - \langle V \rangle_t \right)$

$= 0.$

If $\gamma_t = \inf\{ s : \langle U \rangle_s = t \}$ then

$W_t = f(B_{\gamma_t})$ is 2-d B.M.

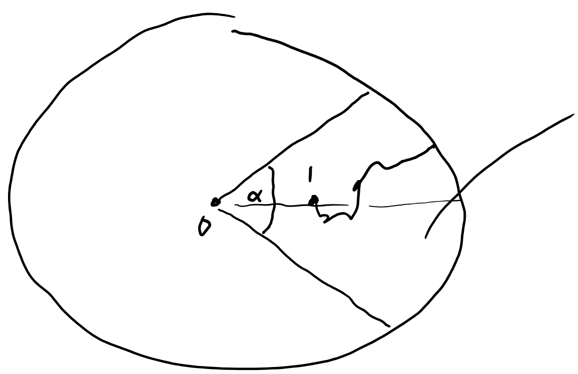
So $f(B_t)$ is time changed Brownian Motion.

By the Riemann - Mapping Theorem there is a conformal map between any simply connected open subdomains of $\mathbb{C}$

$$f : H \to D'.$$

and the hitting measure on D' starting from z is the push forward of $h_\theta(f^{-1}(z))$.

Example



D cone of angle α.

Ball of radius r. $B_0 = 1$.

$T_r = \inf\{s : B_s = r\}$

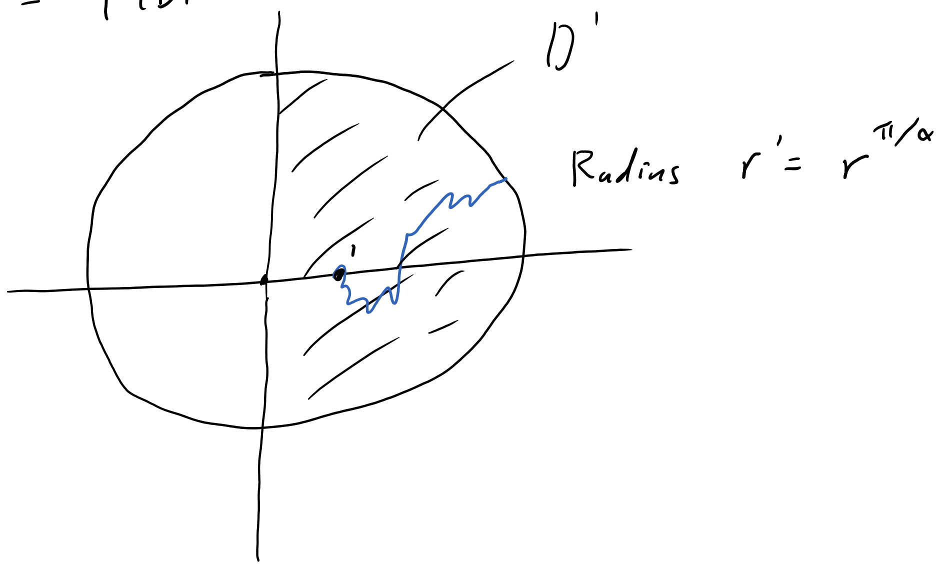
$$\mathbb{P}\{ T_{\partial D} = T_r \}$$

i.e. exits cone through circle.

Set $f(x) = x^{\pi/\alpha}$

Set $f(x) = x^{\cdots}$

$D' = f(D)$



$$\mathbb{P}[T_{\partial D'} = T_{r'}] = \mathbb{P}[T_{\partial D} = T_r]$$

Let $S = \inf\{t : \mathrm{Re}(B_t) = 0\}$

$$\tilde{B}_t = \begin{cases} B_t & t \le S \\ B_t - 2\,\mathrm{Re}(B_t) & t \ge S \end{cases}$$ Reflection Principles

$$B_t \overset{d}{=} \tilde{B}_t.$$

$$\mathbb{P}[\mathrm{Re}(B_{T_{r'}}) > 0]$$

$$= \mathbb{P}[\mathrm{Re}(B_{T_{r'}}) > 0,\ S > T_{r'}] + \mathbb{P}[\mathrm{Re}(B_{T_{r'}}) > 0,\ S < T_{r'}]$$

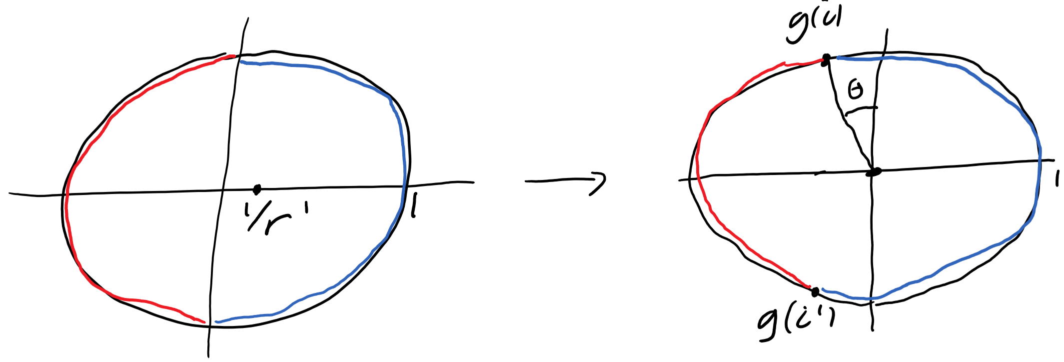
$$= \mathbb{P}[S > T_{r'}] + \mathbb{P}[\mathrm{Re}(\tilde{B}(\tilde{T}_{r'})) < 0]$$

$$\mathbb{P}[S > T_{r'}]$$

$$= \mathbb{P}[\mathrm{Re}(B_{T_{r'}}) > 0] - \mathbb{P}[\mathrm{Re}(B_{T_{r'}}) < 0]$$

$$= \mathbb{P}[\operatorname{Re}(B_{T_{r'}}) > 0] - \mathbb{P}[\operatorname{Re}(B_{T_{r'}}) < 0]$$

Now map to unit disc with $B_0 = 1/r'$.



Let g be the Moebius Transform

$$g(z) = \frac{z - 1/r'}{1 - z/r'}$$

which maps $B(1)$ to itself and 1 to 0.

$$g(i) = \frac{i - 1/r'}{1 - i/r'} = \frac{(r'i - 1)(r' + i)}{(r' - i)(r' + i)}$$

$$= \frac{((r')^2 - 1)i - 2r'}{(r')^2 + 1}$$

$$|\text{Blue} - \text{Red}| = 4\theta$$

$$\theta = \arctan\left(\frac{2r'}{(r')^2 - 1}\right)$$

$$\mathbb{P}[S > T_{r'}] = \frac{1}{2\pi} \cdot 4 \arctan\left(\frac{2r'}{(r')^2 - 1}\right)$$

$$= \frac{2}{\pi} \arctan\left(\frac{2 r^{\pi/\alpha}}{r^{2\pi/\alpha} - 1}\right)$$

Hausdorff Dimension of Brownian Motion

α-Hausdorff content

$$\mathcal{H}^\alpha(E) = \inf\{ \sum_i |E_i|^\alpha : E \subseteq \bigcup E_i \}$$

$|E_i|$ = diameter of E.

Hausdorff Dimension

$$\mathrm{Dim}(E) = \inf\{\alpha \geq 0 : \mathcal{H}^\alpha(E) > 0\}.$$

A box in $\mathbb{R}^d$ dimension d, line dim 1, point dim 0.

What about the range of 2 dimensional B.M?

Upper bound:

If $f(x)$ is β-Hölder continuous then

$$E = \mathrm{Range}(f) := \{ f(x) : x \in [0,1] \}$$

has Hausdorff dimension at most $\frac{1}{\beta}$.

Proof: $|f(x) - f(y)| \leq C|x-y|^\beta$

So let $E_i^{(n)} = B_{Cn^{-\beta}}(f(i/n))$ for $1 \leq i \leq n$.

So let $E_i^{(n)} = B_{Cn^{-\beta}}(f(i/n))$ for $1 \le i \le n$.

Then $f(x) \in E_i^{(n)}$ for $\frac{i-1}{n} \le x \le i/n$,

$|E_i^{(n)}| \le 2Cn^{-\beta}$.

For $\alpha > 1/\beta$,

$$\sum |E_i^{(n)}|^\alpha \le (2Cn^{-\beta})^\alpha \cdot n$$

$$\to 0.$$

Hence $\mathcal{H}^\alpha(E) = 0$ so $\dim(E) \le 1/\beta$.

<u>Lemma:</u> Brownian motion on $[0,1]$ is α-Hölder continuous for all $\alpha < 1/2$.

<u>Proof:</u>

Let $A_i^n = [(i-1)2^{-n}, (i+1)2^{-n}]$.

If $x, y \in [0,1]$, $2^{-(n+1)} \le x - y \le 2^{-n}$

then $x, y \in A_i^n$ for some $1 \le i \le 2^n$.

For $L > 0$, let $D_i^n(L)$ be the event

$$D_i^n(L) := \{ \max_{x,y \in A_i^n} |B(x) - B(y)| \le L\, 2^{-(n+1)\alpha} \}.$$

If $\bigcap_{n \ge 0} \bigcap_{i=1}^{2^n} D_i^n(L)$ holds then

$B(t)$ is α-Hölder with constant L.

By Markov property $\mathbb{P}[D_i^n] = \mathbb{P}[D_1^n]$.

By Brownian Scaling,

$$\mathbb{P}[D_1^n(L)] = \mathbb{P}\Big[\max_{0 \le x, y \le 2 \cdot 2^{-n}} |B(x) - B(y)| \le L\, 2^{-\alpha(n+1)}\Big]$$

$$= \mathbb{P}\Big[\max_{0 \le x, y \le 2} |B(x) - B(y)| \cdot 2^{-n/2} \le L\, 2^{(\frac{1}{2}-\alpha)n}\, 2^{-\alpha}\Big]$$

$$= \mathbb{P}\big[D_1^0(L\, 2^{(\frac{1}{2}-\alpha)n})\big]$$

$$\mathbb{P}[D_1^0(L)]$$

$$= \mathbb{P}\Big[\max_{0 \le x \le 2} B(x) - \min_{0 \le x \le 2} B(x) \le L\, 2^{-\alpha}\Big]$$

$$\le 2\, \mathbb{P}\Big[\max_{0 \le x \le 2} B(x) \le L\, 2^{-\alpha-1}\Big]$$

$$\le \exp(-C L^2).$$

$$\mathbb{P}\Big[\bigcap_n \bigcap_i D_i^n(L)\Big]$$

$$= \sum_n 2^{n+1} \exp(-C_2 L^2\, 2^{(\frac{1}{2}-\alpha)n})$$

$$< \infty$$

$$\to 0 \text{ as } L \to \infty.$$

So $B(t)$ is a.s. α-Hölder.

Corollary: $\text{Range}(B(t)) \leq 2$ for $d \geq 2$.

Lower Bound:

Define the α-energy of a measure μ as

$$I_\alpha(\mu) := \iint |x-y|^{-\alpha} \, d\mu(x) \, d\mu(y).$$

Theorem: If μ is a measure supported on E,

$$\mathcal{H}^\alpha(E) \geq \frac{\mu(E)^2}{I_\alpha(E)}$$

Proof: If A_n is a pairwise disjoint covering of E, such that $\mathcal{H}^\alpha(E) \leq \sum_n |A_n|^\alpha \leq \mathcal{H}^\alpha(E)+\delta$,

$$I_\alpha(E) \geq \sum_n \int_{A_n}\int_{A_n} |x-y|^{-\alpha} \mu(dx)\mu(dy)$$

$$\geq \sum_n \frac{\mu(A_n)^2}{|A_n|^\alpha}$$

$$\mu(E) \leq \sum_n \mu(A_n) = \sum_n |A_n|^{\alpha/2} \frac{\mu(A_n)}{|A_n|^{\alpha/2}}$$

By Cauchy-Schwartz,

$$\mu(E)^2 \le \left(\sum |A_n|^\alpha\right) \sum_n \frac{\mu(A_n)^2}{|A_n|^\alpha}$$

$$\le (\mathcal{H}(E)+\delta) \cdot I_\alpha(\mu)$$

Take the occupation measure of d-dimensional Brownian motion

$$\mu(A) = \int_0^1 I(B(t) \in A)\,dt.$$

Then

$$\mathbb{E}\, I_\alpha(\mu) = \int_0^1 \int_0^1 \mathbb{E}\,|B(t) - B(s)|^{-\alpha}\,ds\,dt$$

$$= \int_0^1 \int_0^1 \mathbb{E}\,|B(t-s)|^{-\alpha}\,ds\,dt$$

$$\le 2\int_0^1 \mathbb{E}\,|B(t)|^{-\alpha}\,dt$$

$$= 2\,\mathbb{E}\,|B(1)|^{-\alpha} \int_0^1 t^{-\frac{\alpha}{2}}\,dt$$

If $d \ge 2$, $\alpha < 2$ then

$$\mathbb{E}\,|B(1)|^{-\alpha} = \int_{\mathbb{R}^d} |x|^{-\alpha} \frac{1}{(\sqrt{2\pi})^d} e^{-|x|^2/2}\,dx < \infty$$

So $\mathbb{E}\, I_\alpha(\mu) < \infty$.

Hence

$$\mathcal{H}^{\alpha}(E) \geq \frac{1}{I_{\alpha}(\mu)} > 0 \quad \text{a.s.}$$

So $\dim(E) \geq \alpha$.

Taylor's Theorem The range of Brownian motion is 2 for $d \geq 2$.

Dimension of Zero set and record times.

Let $Z = \{t : B(t) = 0\}$.

Let $R = \{t : B(t) = M(t)\}$ $\quad M(t) = \max\limits_{0 \leq s \leq t} B(s)$.

For $\alpha < \frac{1}{2}$, B.M. is α-Hölder a.s. and hence so is $M(t)$. If $\{A_n\}$ is a union of disjoint intervals covering E, $A_n = (a_n, b_n)$

$$\begin{aligned}
\mathcal{H}^{\alpha}(R) &\geq \sum_n |A_n|^{\alpha} - \delta \\
&\geq \sum_n |b_n - a_n|^{\alpha} - \delta \\
&\geq \sum_n \frac{M(b_n) - M(a_n)}{L} - \delta \\
&\geq \frac{1}{L}\left[M(1) - M(0)\right] - \delta \\
&> 0.
\end{aligned}$$

So $\dim(R) \geq \frac{1}{2}$.

Let $Y_t = M_t - B_t$.

Theorem: $\{Y_t\} \stackrel{d}{=} \{B_t\}$ and is a Markov process.

Proof:

(M_t, B_t) is a markov process.

Since if $X = \max_{0 \le u \le s} B_{t+u} - B_t$,

$$Z = B_{t+s} - B_t$$

Then (X, Z) is independent of $\mathcal{F}_t$

and

$$(M_{t+s}, B_t) = (M_t \vee B_t + X, \; B_t + Z).$$

Now

$$Y_{t+s} = (M_t \vee B_t + X) - (B_t + Z)$$

$$= Y_t \vee X - Z$$

which only involves (M_t, B_t) through Y_t so

Y_t is also a Markov chain.

To show that $Y_t \stackrel{d}{=} |B_t|$ it suffices to show that

$$\mathbb{P}[Y_{t+s} > a \mid Y_t = y] = \mathbb{P}[|B_{t+s}| > a \mid |B_t| = y]$$

$$= \mathbb{P}[|N(y, s)| > a].$$

As this will imply equal finite dimensional distributions and they both have continuous paths.

Let $\hat{M}_s = y \vee M_s$, $\hat{Y}_s = \hat{M}_s - B_s$

$\mathbb{P}[\ Y_{t+s} > a \mid Y_t = y]$

$= \mathbb{P}[\ \hat{Y}_s > a]$

Now

$\mathbb{P}[\ \hat{Y}_t > a]$

$= \mathbb{P}[y - B_s > a]$

since $M_s > y$

$+ \mathbb{P}[\ y - B_s \le a,\ M_s - B_s > a]$

$= (I) + (II)$

$(I) = \mathbb{P}[N(y, s) > a)$

For (II) set $W_u = B_{s-u} - B_s$ also B.M.

Let $N_t = \max\limits_{0 \le u \le t} N_u = (\max\limits_{0 \le u \le t} N_{s-u}) - B_s$

so $N_s = M_s - B_s$

So applying reflection principle to W_s at a,

$(II) = \mathbb{P}[\ y + W_s \le a,\ N_s > a]$

$= \mathbb{P}[\ y + 2a - W^*_{s} \le a,\ N^*_s > a]$

$= \mathbb{P}[\, y - W_s^{*} \le -a]$

$= \mathbb{P}[\, N(y, s) \le -a]$

So $(\mathrm{I}) + (\mathrm{II}) = \mathbb{P}[\,|N(y,s)| > a]$.

Stronger Result

$$\{(|B_t|, L_t^0)\}_{t\ge 0} \stackrel{d}{=} \{(M_t - B_t, M_t)\}_{t\ge 0}.$$

Lemma: $\dim(Z) \le \frac{1}{2}$.

Let $A_i^{(n)} = [in, (i+1)n]$.

Then $\mathbb{P}[\, Z \cap A_i \ne \emptyset]$

$$= \mathbb{P}\Big[\max_{\frac{i}{n} \le t \le \frac{i+1}{n}} B_t - B_{i/n} > |B_{i/n}|\Big]$$

$$= \mathbb{P}[\,|N(0, \tfrac{1}{n})| > |N(0, \tfrac{i}{n})|]$$

$$\approx C\, i^{-1/2}.$$

So

$$\mathbb{E}\, \mathcal{H}^{\alpha}(Z) \le \mathbb{E} \sum_i \left(\tfrac{1}{n}\right)^{\alpha} I(A_i^{(n)} \cap Z)$$

$$\le C \sum_i n^{-\alpha} i^{-1/2} \approx C' n^{\frac{1}{2} - \alpha}$$

$$\to 0 \quad \text{if } \alpha > \tfrac{1}{2}.$$