Gaussian Processes and the construction of Brownian Motion

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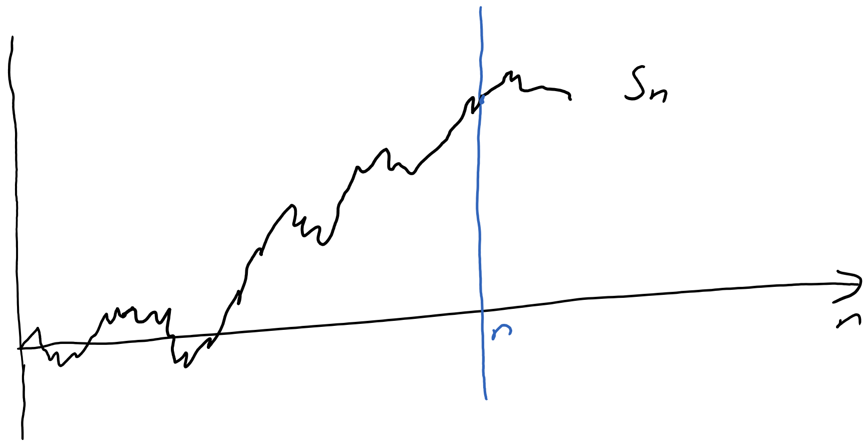
Let $X_1, \ldots$ be IID, mean 0, variance 1 and let $S_n = \sum_{i=1}^{n} X_i$ be a random walk.



By the CLT,

$$\frac{S_n}{\sqrt{n}} \xrightarrow{d} N(0,1).$$

This gives us convergence at a particular time. But we may want path properties E.G.

- Maximum of $Y_n(t)$, time of maximum
- Time $Y_n(t)$ is positive

So at time n, the value is approximately normal. What about the trajectory of the path from 0 to n? For $t_1 < t_2 < \ldots < t_k$

does $Y_n = \left(\frac{S_{nt_1}}{\sqrt{n}}, \frac{S_{nt_2}}{\sqrt{n}}, \ldots \frac{S_{nt_k}}{\sqrt{n}} \right)$ converge?

does $Y_n = \left(\frac{S_{nt_1}}{\sqrt{n}}, \frac{S_{nt_2}}{\sqrt{n}}, \dots \frac{S_{nt_k}}{\sqrt{n}} \right)$ converge?

<u>Yes</u>: Note that $S_{nt_i} - S_{nt_{i-1}}$ are independent. Also

$$\frac{S_{nt_i} - S_{nt_{i-1}}}{\sqrt{n}} = \sqrt{t_i - t_{i-1}} \; \frac{S_{nt_i} - S_{nt_{i-1}}}{\sqrt{n(t_i - t_{i-1})}}$$

$$\xrightarrow{d} \sqrt{t_i - t_{i-1}} \; N(0,1)$$

$$= N(0, t_i - t_{i-1})$$

So Y_n converges in distribution to a random Gaussian vector. If we let

$$Y_n(t) = \frac{S_{nt}}{\sqrt{n}}$$ then we will show

$$Y_n \xrightarrow{d} B$$

where $B(t)$ is Brownian motion, a Gaussian process taking values in $C([0,\infty))$.

A collection of random variables $\{X_i\}_{i \in A}$ is <u>jointly Gaussian / Normal</u> or a <u>Gaussian process</u> if

$\forall S \subseteq A$, S finite, $\forall \{a_i\}_{i \in S}$,

$$\sum_{i \in S} a_i X_i \quad \text{is normal.}$$

If A is finite $\underline{X}$ is a <u>Gaussian vector</u>.

If $\underline{X}, \underline{Y}$ are Gaussian Vectors in $\mathbb{R}^n$

such that $\mathbb{E} X_i = \mathbb{E} Y_i = \mu_i$

$$Cov(X_i, X_j) = Cov(Y_i, Y_j) = V_{ij}$$

then $\underline{X} \overset{d}{=} \underline{Y}$.

<u>Proof</u>: If $\theta \in \mathbb{R}^n$,

$$\mathbb{E}\, \theta \cdot \underline{X} = \sum \theta_i \mu_i = \mathbb{E}\, \theta \cdot \underline{Y}$$

$$Var(\theta \cdot \underline{X}) = \sum_{ij} Cov(\theta_i X_i, \theta_j X_j)$$

$$= \theta^T V \theta = Var(\theta \cdot \underline{Y})$$

$$\text{So} \quad \theta \cdot \underline{X} \overset{d}{=} N(\theta \cdot \underline{\mu}, \theta^T V \theta) \overset{d}{=} \theta \cdot \underline{Y}$$

and so

$$\varphi_{\underline{X}}(\theta) = \mathbb{E}\, e^{i\theta \cdot \underline{X}} = \mathbb{E}\, e^{i\theta \cdot \underline{Y}} \implies \underline{X} \overset{d}{=} \underline{Y}.$$

<u>Corollary</u>: If X_i uncorrelated then X_i independent.

A $n \times n$ matrix V is <u>positive semi-definite</u> if $\forall \theta \in \mathbb{R}^n$, $\theta^T V \theta \geq 0$.

and <u>positive definite</u> if

$\forall \theta \in \mathbb{R}^n \setminus \{0\}$ $\quad \theta^T V \theta > 0$.

If $V = (V_{ij})$ $\quad V_{ij} = Cov(X_i, X_j)$ then

V is symmetric positive semi-definite.

Let $Y = \sum \theta_i X_i$. Then

$$0 \leq Var(Y) = \sum_{ij} Cov(\theta_i X_i, \theta_j X_j) = \sum_{ij} \theta_i V_{ij} \theta_j$$

$$= \theta^T V \theta.$$

<u>Theorem</u>

For each $\underline{\mu} \in \mathbb{R}^n$, V $n \times n$ PSD Matrix there is a unique Gaussian Vector distribution distribution $N_d(\underline{\mu}, V)$ such that for $Y \sim N_d(\underline{\mu}, V)$,

$\mathbb{E} Y = \underline{\mu}$, $\quad Cov(Y_i, Y_j) = V_{ij}$

In particular a Gaussian vector is determined by its mean and co-variance

If V is PD (and so has full rank) then

$$f_Y(y) = \frac{1}{\sqrt{(2\pi)^n \operatorname{Det} V}} \exp\left(-\tfrac{1}{2}(y-\mu)^T V^{-1} (y-\mu)\right).$$

Construction of Y given $\underline{\mu}$ and V

Let $Z = (Z_1, \dots, Z_d)$ be an IID vector of $N(\underline{0}, 1)$.

Since V is symmetric PD there exists U, D such that U is orthonormal $U^T U = I$, and D is diagonal such that

$$V = U^T D^2 U.$$

Let $A = DU \qquad Y = A^T Z + \mu, \quad \mathbb{E}Y = \mu.$

$$\begin{aligned} \mathrm{Cov}(Y_i, Y_j) &= \mathbb{E}\left[\left(\sum_e A^T_{ie} Z_e\right)\left(\sum_{e'} A^T_{je'} Z_{e'}\right)\right] \\ &= \mathbb{E}\left[\sum_e A^T_{ie} A^T_{je}\right] \\ &= (A^T A)_{ij} = V_{ij}. \end{aligned}$$

If V is full rank, so is D and so is A.

If V is not full rank then $\exists\, \theta \neq 0$ such that $V\theta = 0$ so $\mathrm{Var}(\theta \cdot Y) = \theta^T V \theta = 0 \Rightarrow \theta \cdot Y$ constant

If $\theta_j \neq 0$ then $Y_j = \theta_j^{-1} \sum \theta_i Y_i + C$ determined by other co-ordinates. So Y is supported on a lower dimensional hyperplane.

Properties of Gaussian Vectors

<u>Marginals</u> If Y is jointly Gaussian, so is Y_S.

Conditional Distribution:

If S, T disjoint subsets of co-ordinates,

Given $Y_T = y_T$

$Y_S \mid Y_T = y_T$

$$\sim N(\mu_S - V_{ST} V_{TT}^{-1} (y_T - \mu_T), V_{SS} - V_{ST} V_{TT}^{-1} V_{TS}).$$

Multivariate Central Limit Theorem

If $Y_1, \ldots$ IID random vectors $\mathbb{E} Y = \mu \in \mathbb{R}^d$,

$V_{ij} = \mathrm{Cov}(Y(i), Y(j))$ then

$$\frac{\sum_{i=1}^{n} (Y_i - \mu)}{\sqrt{n}} \xrightarrow{d} N_d(0, V)$$

Hilbert Space

We say H, a normed linear space is a Hilbert space if it has an inner product

a) $\langle ax + y, z \rangle = a \langle x, z \rangle + \langle y, z \rangle$ $(a \in \mathbb{R})$

b) $\|x\| = \langle x, x \rangle > 0$ for $x \neq 0$.

c) H is a complete space,

if $\limsup_{m} \sup_{n>m} \|x_n - x_m\| = 0$

then $x_n \to x \in H$.

Example: $\mathbb{Z}^d$, $L^2([0,1])$ or $L^2(\mathbb{R})$, $\langle f, g \rangle = \int f(x) g(x) dx$

Square Integrable R.V., $\langle X, Y \rangle = \mathbb{E}[XY]$.

All Hilbert Spaces have an orthonormal basis and two Hilbert spaces with same dimension are isomorphic.

Let H be a Hilbert space, then there exists a Gaussian Process $\{X_h\}_{h \in H}$ such that

$$X_h \sim N(0, \|h\|^2),$$

$$Cov(X_h, X_{h'}) = \langle h, h' \rangle.$$

Proof: Let h_i be an orthonormal basis for H an Z_i IID $N(0,1)$.

Then if $h = \sum a_i h_i$,

$$X_h := \sum a_i Z_i.$$

$$\langle h, h' \rangle = \sum a_i a_i'$$

$$Cov(X_h, X_{h'}) = \sum_{i,j} a_i a_j' Cov(Z_i, Z_j)$$

$$= \sum_i a_i a_i'.$$

Let $H = L^2([0,1])$,

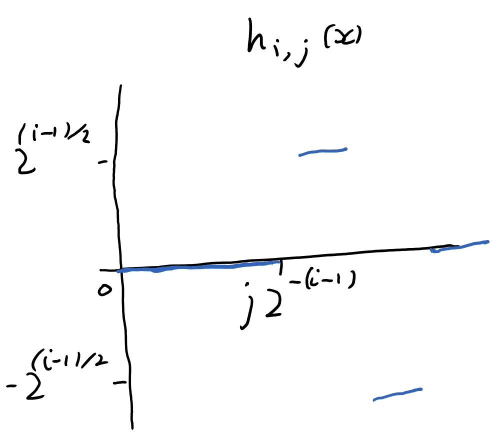
let $f(x) = \begin{cases} 1 & x \in [0,1) \\ -1 & x \in [1,2) \\ 0 & \text{o.w.} \end{cases}$

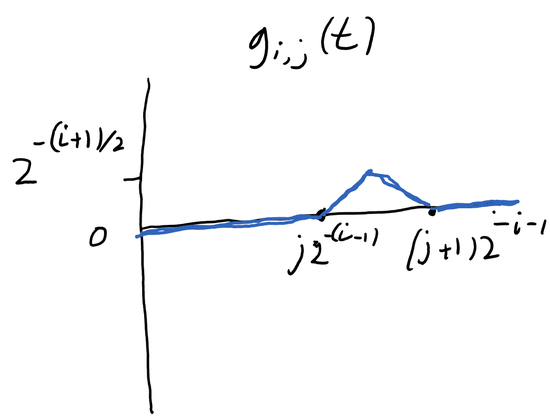
$$h_{i,j}(x) = 2^{(i-1)/2} \, h(2^i(x - j2^{-(i-1)})$$

for $i \geq 1$, $0 \leq j \leq 2^{i-1}$

and $h_{0,0}(x) = 1$. Let

$$g_{i,j}(t) = \int_0^t h_{i,j}(x)\, dx$$





Then $\{h_{i,j}\}$ are an orthonormal basis for H.

Let $\xi_t = I(0 \leq x \leq t)$ and define

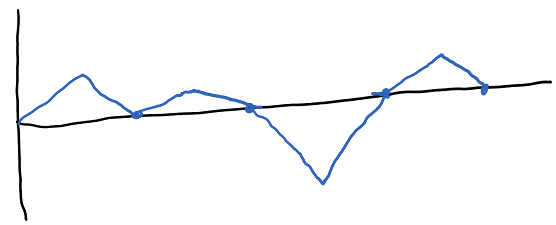
$$R.:= X_t = \sum Z_{i,j} \langle \xi_t, h_{i,j} \rangle$$

$$B_t := X_{\xi_t} = \sum_{i,j} Z_{i,j} \langle \xi_t, h_{i,j} \rangle$$

$$= \sum_{i,j} Z_{i,j} \cdot \int_0^t h_{i,j}(x)\, dx.$$

$$= \sum_i \sum_j Z_{i,j}\, g_{i,j}(t)$$

For $i = 2$



Let $R_i(t) = \sum_j Z_{i,j}\, g_{i,j}(t)$

Then $\|R_i\|_\infty = \max_j 2^{-(i-1)/2} |Z_{i,j}|$

If $\sum_i \|R_i\|_\infty < \infty$ then $\sum R_i$ converges uniformly and B_t is continuous.

$$\mathbb{E}[\|R_i\|_\infty] = 2^{-(i-1)/2}\, \mathbb{E} \max_j |Z_{i,j}|$$

$$\leq 2^{-(i-1)/2} \int_0^\infty \mathbb{P}[\max_j |Z_{i,j}| > x]\, dx$$

$$\leq 2^{-(i-1)/2} \int_0^\infty \left(2^i \exp(-x^2/2) \wedge 1\right) dx$$

$$\le 2^{-(i-1)/2}\left(i + \int_i^\infty \exp(x - x^2/2)\, dx \right)$$

$$\le C\, 2^{-i/3}.$$

So $\mathbb{E}\left[\sum_i \|R_i\|_\infty\right] < \infty.$

Hence B_t is continuous almost surely.

Covariance of B_t:

- $\mathrm{Cov}(B_t, B_s) = \langle \xi_t, \xi_s \rangle = s \wedge t.$

  $\mathrm{Var}(B_t) = t.$

- If $0 = t_0 < t_1 < \dots < t_n$,

$$\mathrm{Cov}(B_{t_i} - B_{t_{i-1}}, B_{t_j} - B_{t_{j-1}})$$

$$= \langle \xi_{t_i} - \xi_{t_{i-1}}, \xi_{t_j} - \xi_{t_{j-1}} \rangle$$

$$= \int I(t_{i-1} \le t < t_i)\, I(t_{j-1} \le t < t_j)\, dt$$

$$= 0.$$

So the increments of B_t are independent.

Brownian Motion

A random function $B:\Omega \to C[0,\infty)$.
is a Brownian motion if:

- $B(t)$ is a Gaussian process.
- $B(0)=0$, $\mathbb{E}B(t)=0$, $Cov(B(t), B(s)) = t \wedge s$.
- $B(t)$ is continuous almost surely.

Properties of Brownian Motion

a) Independent Increments

$B(t_1)$, $B(t_2)-B(t_1), \dots, B(t_n)-B(t_{n-1})$ are independent.

Furthermore $W_s = B_{t+s} - B_t$
is a Brownian motion and is independent of $\mathcal{F}_t$.
Strong Markov Property says this also holds for S a stopping time.

b) $B(t)$ is a martingale

$$\mathbb{E}[B_t \mid \mathcal{F}_s] = \mathbb{E}[B_s + (B_t - B_s) \mid \mathcal{F}_s]$$
$$= B_s.$$

c) $B(t)$ is $\frac{1}{2}$ - self similar (fractal)

If $Y(t) = s^{-1/2} B(ts)$

then $Y(t)$ is Brownian Motion.

$$
\begin{aligned}
\mathrm{Cov}(Y(t), Y(t')) &= s^{-1}\,\mathrm{Cov}(B(st), B(st')) \\
&= s^{-1}(st \wedge st') \\
&= t \wedge t'
\end{aligned}
$$

d) <u>Non-differentiable</u>

<u>At one point</u>

$$\frac{d}{dt} B(t) = \lim_{h \to 0} \frac{B(t+h) - B(t)}{h} \sim \frac{N(0,h)}{h} \sim N(0, \tfrac{1}{h})$$

$$\mathbb{P}[|N(0, 1/h)| < M] = \mathbb{P}[|N(0,1)| < \tfrac{M}{\sqrt{h}}] \to 0.$$

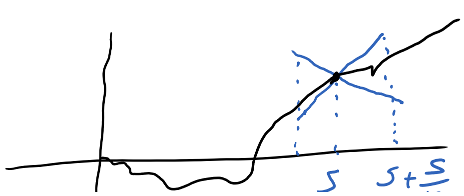
<u>At all points</u>

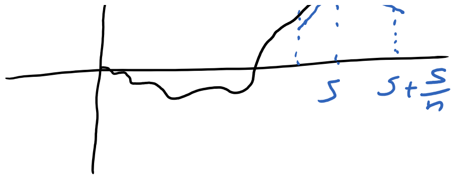
$$\mathbb{P}[\exists s \in [0,1],\ B(t) \text{ differentiable at } s] = 0.$$

Fix $C > 0$.

Let $A_n = \{ \exists s \in [0,1] : |B(t) - B(s)| \le 2C|t-s|$ for $t \in [s - \frac{5}{n}, s + \frac{5}{n}]. \}$

Must hold for all large n if $B(t)$ is differentiable at s with $|B'(s)| \le C$.





Let $Y_{k,n} = \max\limits_{j\in\{0,1,2\}} \left\{ \left| B\left(\frac{k+j+1}{n}\right) - B\left(\frac{k+j}{n}\right) \right| \right\}.$

Let

$$B_n = \left\{ \min_{0\le k\le n-2} Y_{k,n} \le \frac{100C}{n} \right\}.$$

<u>Claim</u> $A_n \subseteq B_n$.

If s satisfies A_n and $\left(\frac{k+j}{n}, \frac{k+j+1}{n}\right) \subseteq \left[s-\frac{5}{n}, s+\frac{5}{n}\right]$

then by Δ inequality

$$\left|B\left(\tfrac{k+j+1}{n}\right) - B\left(\tfrac{k+j}{n}\right)\right| \le \left|B\left(\tfrac{k+j+1}{n}\right) - B(s)\right| + \left|B\left(\tfrac{k+j}{n}\right) - B(s)\right|$$

$$\le 2C\left|\frac{k+j+1}{n} - s\right| + 2C\left|\frac{k+j}{n} - s\right|$$

$$\le 2C\left(\frac{5}{n}+\frac{5}{n}\right) = \frac{20C}{n}.$$

We can pick k such that

<u>$\left[\frac{k+j}{n}, \frac{k+3}{n}\right] \subseteq \left[s-\frac{5}{n}, s+\frac{5}{n}\right]$</u> so $Y_{k,n} \le \frac{20C}{n}$.

$$\mathbb{P}\left[Y_{k,n} \le \frac{100C}{n}\right] = \mathbb{P}\left[\left|N\left(0,\tfrac{1}{n}\right)\right| \le \frac{100C}{n}\right]^3$$

$$= \mathbb{P}\left[|N(0,1)| \le \frac{100C}{\sqrt{n}}\right]^3$$

$$\le \left(\frac{200C}{\sqrt{n}}\cdot\frac{1}{\sqrt{2\pi}}\right)^3 \quad \text{density} \le \frac{1}{\sqrt{2\pi}}$$

$$\le D\, n^{-3/2}$$

$$\leq D n^{-3/2}.$$

$$\mathbb{P}[B_n] \leq n \cdot D n^{-3/2} \leq D/\sqrt{n}.$$

So $\mathbb{P}[A_n] \leq \mathbb{P}[B_n] \leq D/\sqrt{n} \to 0.$

But A_n is increasing in n so $\mathbb{P}[A_n] = 0$.

$\Rightarrow \mathbb{P}[\,B(t)$ is nowhere differentiable$\,] = 0.$

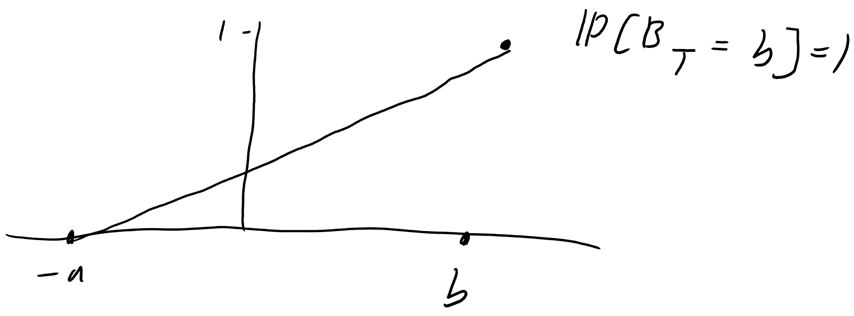
Hitting Probabilities:

Let T be the first hitting time of $-a$ or b. Since B_t is a martingale,

$$\mathbb{E}\, B_T = \mathbb{E} B_0 = 0$$

$$= -a\,\mathbb{P}[B_T = -a] + b\,\mathbb{P}[B_T = b]$$

So $$\mathbb{P}[B_T = b] = \frac{a}{b+a}$$



A *stochastic process* on $\mathbb{R}$ is a function valued random variable.

$$X : \Omega \times [0,\infty) \to \mathbb{R}.$$

Convergence of Processes

We say $X_n(t)$ converges in finite dimensional distributions if $\forall\, t_1 < t_2 < \ldots < t_n$

$$(X_n(t_1), \ldots, X_n(t_n)) \xrightarrow{d} (X(t_1), \ldots, X(t_n)).$$

Stronger Notion of Convergence

If $X_n(t), X(t) \in C[0,1]$ we say

$X_n(t) \to X(t)$ in the sup-norm topology if

for all bounded continuous functions

$f : (C[0,1], \|\cdot\|_\infty) \to \mathbb{R}$, (E.G. $f(X) = \max_{0 \le x \le 1} X(x)$).

$$\mathbb{E}\, f(X_n) \to \mathbb{E}\, f(X).$$

Functional Central Limit Theorem

Theorem: If $S_n = \sum_{i=1}^{n} X_i$, X_i IID, $\mathbb{E} X_i = 0$, $\mathrm{Var}\, X_i = 1$,

$Y_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{nt} X_i$ then

$$(Y_n(t_1),\ Y_n(t_2) - Y_n(t_1), \ldots,\ Y_n(t_k) - Y_n(t_{k-1}))$$

$$\xrightarrow{d} (B_{t_1}, B_{t_2} - B_{t_1}, \dots, B_{t_k} - B_{t_{k-1}})$$

so $(Y_n(t_1), \dots, Y_n(t_k)) \xrightarrow{d} (B_{t_1}, \dots, B_{t_k})$ and hence

$$Y_n(t) \xrightarrow{f.d.d.} Y(t).$$

Donsker's Theorem

With the same conditions as above,

$$Y_n(t) \longrightarrow B(t)$$

in the sup-norm.

Proof: Idea: Embed S_n into $B(t)$.

Claim: We can find a stopping time T such that $X_1 \overset{d}{=} B_T$, $\mathbb{E}\, T = 1$.

Proof: Motivating Case: $X_1 \in \{-a, b\}$, $a, b > 0$

so $\mathbb{P}[X_1 = -a] = \frac{b}{a+b}$, $\mathbb{P}[X_1 = b] = \frac{b}{a+b}$.

Let $T_{-a,b}$ be the first hitting time of $\{-a, b\}$.

Then $B_{T_{-a,b}} \sim \mu_{a,b} \sim X$

$$\mu_{a,b}(-a) = \frac{b}{a+b}, \quad \mu_{a,b}(b) = \frac{a}{a+b}, \quad \mu(\mathbb{R} \setminus \{-a, b\}) = 0.$$

HW: $\mathbb{E}\, T_{-a,b} =$

General Case:

If $\mathbb{P}[X = 0] = p > 0$ then set $\mathbb{P}[T = 0] = p$, the case $T > 0$ reduces to the case $p = 0$.

So assume $\mathbb{P}[X=0]=0$.

Set $M = \mathbb{E}[X I(X>0)] = \mathbb{E}[-X I(-X>0)]$.

Define ν on $\mathbb{R}^2$ with support in $(0,\infty)^2$ as

$$\nu(S) := M^{-1} \mathbb{E}[(X'-X) I((-X,X')\in S,\ X<0<X')]$$

$$M\nu(\mathbb{R}^2) = M\nu((0,\infty)^2)$$
$$= \mathbb{E}[(X'-X) I(X<0<X')]$$
$$= \mathbb{E}[X' I(X'>0) \cdot I(-X>0)]$$
$$+ \mathbb{E}[-X I(X<0) I(X'>0)]$$
$$= M\,\mathbb{P}[-X>0] + M\,\mathbb{P}[X'>0] = M$$

so ν is a probability measure.

Pick $(A,B)\sim\nu$ then for $\Delta \subseteq (0,\infty)$

$$\mathbb{P}[T_{-A,B} \in \Delta] = \mathbb{E}\left[I(B\in\Delta)\cdot \frac{A}{A+B}\right]$$
$$= M^{-1} \mathbb{E}\left[(X'-X) I(X'\in\Delta,\ X<0<X')\cdot \frac{-X}{-X+X'}\right]$$
$$= M^{-1} \mathbb{E}[-X I(X<0) \cdot I(X'\in\Delta)]$$
$$= M^{-1}\cdot M\cdot \mathbb{P}[X'\in\Delta] = \mathbb{P}[X\in\Delta]$$

Similarly for $\Delta \subseteq (-\infty,0)$

$\mathbb{P}[T_{-A,B}\in\Delta] = \mathbb{P}[X\in\Delta]$.

So $T_{-A,B} \overset{d}{=} X$.

Embedding, let $(A_i, B_i) \sim \nu$ be IID

and set

$$\tau_n = \min\{ t \geq \tau_{n-1} : B_t - B_{\tau_{n-1}} \in \{-A_n, B_n\} \}$$

Then $B_{\tau_n} - B_{\tau_{n-1}} \sim X$ and are IID, so

$$\{ S_n \}_{n \geq 1} \stackrel{d}{=} \{ B_{\tau_n} \}_{n \geq 1}.$$

By SLLN, $\frac{1}{n} \tau_n \to 1$ since $\{ \tau_n - \tau_{n-1} \}$ also IID.

Let
$$S(t) = \begin{cases} S_n & \text{if } t \in \mathbb{Z} \\ (t-n) S_{n+1} + (n+1-t) S_n & \text{if } t \in (n, n+1) \end{cases}$$

Let D_n, G_n be the events that

$$D_n = \left\{ \max_{1 \leq m \leq n} |\tau_m - m| < \delta n \right\}$$

$$G_{n,\delta} = \left\{ \max_{0 \leq t \leq n} \max_{s : |s-t| < 2\delta n} |B(t) - B(s)| \leq \varepsilon \sqrt{n} \right]$$

By SLLN $\mathbb{P}[D_n] \to 1$.

$\mathbb{P}[G_{n,\delta}] = \mathbb{P}[G_{1,\delta}] \to 1$ as $\delta \to 1$, by continuity of B.M. and Brownian scaling.

If $0 \leq t \leq n$ and $t = m + \alpha$, $m \in \mathbb{Z}$, $\alpha \in (0,1)$ and $D_n, G_{n,\delta}$ hold then

$$|S(t) - B(t)| = |(1-\alpha) B(\tau_m) + \alpha B(\tau_{m+1}) - B(t)|$$

$$\leq (1-\alpha)\,|B(\tau_m) - B(t)| + \alpha\,|B(\tau_{m+1}) - B(t)|$$

$$\leq \varepsilon$$

since $|\tau_m - t| \leq |\tau_m - m| + \alpha \leq 2\delta n.$

So

$$\mathbb{P}\Big[\max_{0\leq t\leq 1} \Big|\underbrace{\tfrac{1}{\sqrt{n}} S(tn)}_{= Y_n(t)} - \underbrace{\tfrac{1}{\sqrt{n}} B(tn)}_{\overset{d}{=} B(t)}\Big| \leq \varepsilon\Big] \to 1.$$

Let $\varphi : C[0,1] \to \mathbb{R}$ be bounded and continuous.

Then if

$$H_\varepsilon = \{w : \max\{|\varphi(w') - \varphi(w)| : |w' - w| \leq \varepsilon\} \leq \gamma\}$$

then

$$\big|\mathbb{E}[\varphi(Y_n(t))] - \mathbb{E}[\varphi(B(t))]\big|$$

$$= \Big|\mathbb{E}\Big[\varphi(Y_n(t)) - \varphi\big(\tfrac{1}{\sqrt{n}} B(nt)\big)\Big]\Big|$$

$$\leq \gamma + 2\|\varphi\|_\infty \Big(\underbrace{\mathbb{P}[H_\varepsilon^c]}_{\to 0 \text{ as } \varepsilon \to 0} + \underbrace{\mathbb{P}\big[\|Y_n(t) - \tfrac{1}{\sqrt{n}} B(tn)\|_\infty > \varepsilon\big]}_{\to 0}\Big)$$

Hence $Y_n(t) \to B(t)$ weakly in the sup norm

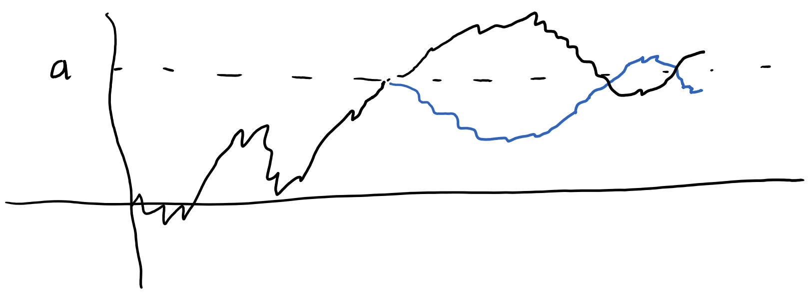
Reflection Principle

Let $M = \max B(s)$

Let $M_t = \max_{0 \le s \le t} B(s)$.

Let T be the first hitting time of a.

Let $B^*(t) = \begin{cases} B(t) & t \le T \\ 2a - B(t) & t > T \end{cases}$



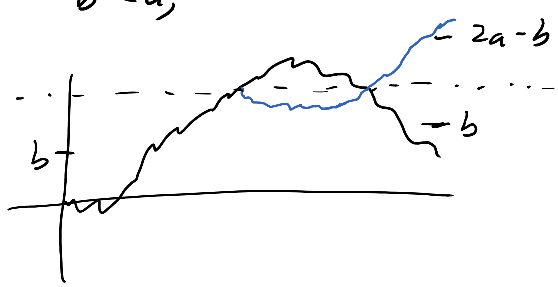
Since $-B(t) \stackrel{d}{=} B(t)$,

$$B(t+T) - a \stackrel{d}{=} -(B(t+T) - a) = a - B(t+T)$$

is B.M independent of $\mathcal{F}_T$.

So $B^*(t)$ is also B.M, let $M_t^* = \max_{s \le t} B^*(s)$,

for $b < a$,



$$\mathbb{P}[M(t) \ge a,\ B(t) \le b] = \mathbb{P}[M^*(t) \ge a,\ B^*(t) \ge 2a - b]$$

$$= \mathbb{P}[B^*(t) \ge 2a - b]$$

$$= \mathbb{P}[B(t) \ge 2a - b]$$

Set $a=b$,

$$\mathbb{P}[M(t)\geq a] = \mathbb{P}[M(t)\geqslant a,\ B(t)\geqslant a]$$
$$+\ \mathbb{P}[M(t)\geqslant a,\ B(t)\leq a]$$
$$= 2\,\mathbb{P}[B(t)\geqslant a]$$

So $M(t)\stackrel{d}{=}|B(t)|$.

Let $T_s = \inf\{t : B(t)=s\}$.

Then T_{s_1}, $T_{s_2}-T_{s_1}$ are independent,

$T_1 \stackrel{d}{=} T_2-T_1$ are independent.

$$\mathbb{P}[T_1\leqslant t] = \mathbb{P}[M_t\geqslant 1] = \mathbb{P}[\,|N(0,t)|\geqslant 1]$$
$$= \mathbb{P}[\,|N(0,1)|\geqslant 1/\sqrt{t}\,]$$

so density of T_1 is

$$f_{T_1}(t) = \frac{d}{dt}\, 1-\int_{-1/\sqrt{t}}^{1/\sqrt{t}} e^{-x^2/2}\,dx$$
$$= t^{-3/2}\, e^{-1/2t}$$

$$T_2 = \inf\{t : B(t)=2\}$$
$$\stackrel{d}{=} \inf\{t : 2B(t/4)=2\} \quad \text{Brownian Scaling}$$
$$= 4\inf\{t : B(t)=1\}$$
$$= 4T_1.$$

So $X\sim T_1$, X, X' IID then

$$X + X' \stackrel{d}{=} 4X \quad \text{so } X \text{ is } \tfrac{1}{2}\text{-stable}$$

$$T_1 \stackrel{d}{=} \sum_{x \in \Pi} x$$

where Π Poisson Process with intensity $\lambda(x) = x^{-3/2}$.

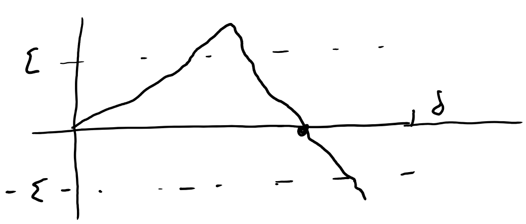
No isolated zero's

Let $Z = \{ t : B(t) = 0 \}$.

Theorem: Z has no isolated points almost surely.

Claim: $\inf\{ t > 0 : B(t) = 0 \} = 0$ a.s.

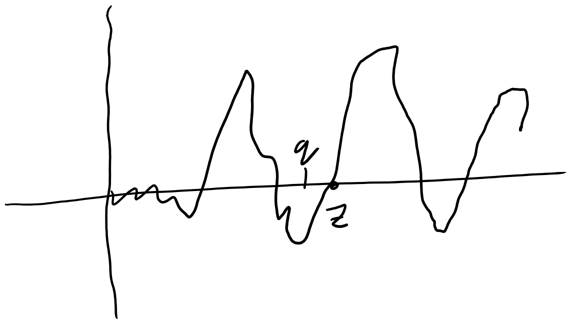
If $T_\varepsilon, T_{-\varepsilon} < \delta$ then $\exists\, 0 < t < \delta$ such that $B(t) = 0$



Since
$$\mathbb{P}[T_\varepsilon > \delta] = \mathbb{P}[T_{-\varepsilon} > \delta]$$
$$= \mathbb{P}[\varepsilon^2 T_1 > \delta]$$
$$= \mathbb{P}[T_1 > \delta/\varepsilon^2] \to 0$$
as $\varepsilon \to 0$.

$\Rightarrow \mathbb{P}[\inf\{ t > 0 : B(t) = 0 \}] = 1.$

Suppose $z \in Z$ is isolated. Then $\exists q \in \mathbb{Q}$, $q < z$ such that $[q, z) \cap Z = \emptyset$.



Let S_q be the stopping time $S_q = \inf\{t \geq q : B(t) = 0\}$.

Then $B(S_q) = 0$ and $\inf\{t > S_q : B(t) = 0\} > S_q$.

$$\mathbb{P}[S_q \text{ isolated on the right}]$$

$$= \mathbb{P}[\inf\{t > S_q : B(t) = 0\} > S_q]$$

$$= \mathbb{P}[\inf\{t > 0 : B(t) = 0\} > 0] = 0$$

since $\{B(t + S_q) - B(S_q)\}_{t \geq 0} \overset{d}{=} \{B(t)\}_{t \geq 0}$.

Taking a union bound over $q \in \mathbb{Q}$

$$\mathbb{P}[\exists \text{ isolated point in } Z]$$

$$= \mathbb{P}[\exists q \in \mathbb{Q} : S_q \text{ right isolated}]$$

$$\leq \sum_{q \in \mathbb{Q}} \mathbb{P}[S_q \text{ right isolated}] = 0.$$

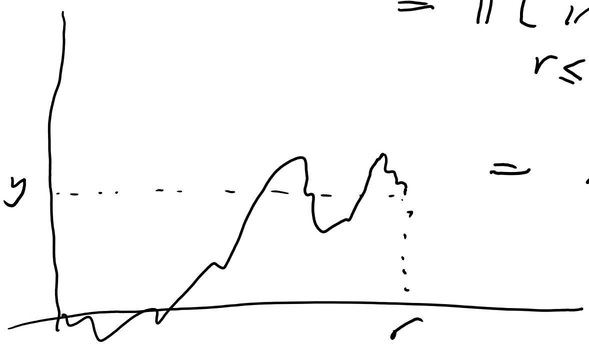
<u>Last zero</u>: Let $R = \sup\{t \in [0, 1] : B(t) = 0\}$.

$$\mathbb{P}[R \leq r] = \mathbb{E}[\mathbb{P}[R \leq r \mid \mathcal{F}_r]]$$

$$= \mathbb{E}[\mathbb{P}[\text{no zeros in } (r, 1] \mid B(r)]]$$

$$\mathbb{P}[\text{ no zero's in } (r,1] \mid B(r)=y] \quad \text{for } y>0$$

$$= \mathbb{P}\Big[\inf_{r\le t\le 1} B(t)-B(r) \geq -y \mid B(r)=y\Big]$$



$$= \mathbb{P}\Big[\inf_{0\le t\le 1-r} B(t) \geq -y\Big]$$

$$= \mathbb{P}[M_{1-r} \le y]$$

$$= \mathbb{P}[|N(0,1-r)| \le |y|]$$

$$= 2\int_0^{|y|/\sqrt{1-r}} \frac{1}{\sqrt{2\pi(1-t)}} e^{-\frac{x^2}{2(1-t)}}\, dx$$

also by symmetry for negative y.

So $\mathbb{P}[R\le r] = \mathbb{E}\Big[2\int_0^{|B_r|/\sqrt{1-r}} \frac{1}{\sqrt{2\pi(1-t)}} e^{-\frac{x^2}{2(1-t)}}\, dx\Big]$

$$= \int_{-\infty}^{\infty} 2\int_{-|y|/\sqrt{1-r}}^{|y|/\sqrt{1-r}} \frac{1}{\sqrt{2\pi(1-t)}} e^{-\frac{x^2}{2(1-t)}}\, dx\, \frac{1}{\sqrt{2\pi t}} e^{-\frac{y^2}{2t}}\, dy$$

$$= \ldots$$

$$= \frac{1}{2\pi} \arcsin(\sqrt{t}).$$