Limits and the law of large numbers

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3 Types of convergence

Almost Sure (Almost everywhere)

$X_n \xrightarrow{a.s.} X$ if $\mathbb{P}[\{w : X_n(w) \to X(w)\}] = 1.$

In probability

$X_n \xrightarrow{P} X$ if $\forall \varepsilon > 0,\ \mathbb{P}[|X_n - X| > \varepsilon] \to 0.$

In L^p, $\|X\|_p = (\mathbb{E}|X|^p)^{1/p}$ $p \geq 1.$

A metric & norm. If $p \geq q$ $\|X\|_p \geq \|X\|_q.$

$X_n \xrightarrow{L^p} X$ if $\|X_n - X\|_p \to 0.$

$X_n \xrightarrow{a.s.} X$ Along a subsequence

$\Rightarrow$ $X_n \xrightarrow{P} X$

If $\|\limsup X_n\|_p < \infty$

$X_n \xrightarrow{L^p} X$ $\Rightarrow$

Lemma: If $X_n \xrightarrow{a.s.} X$ then $X_n \xrightarrow{P} X.$

Let $A_n = \{\sup_{m \geq n} |X_m - X| > \varepsilon\}$

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Let A_n ... $m \geq n$ ...

so $\mathbb{P}[A_n] \searrow \mathbb{P}[\bigcap_{n=1}^{\infty} A_n] = 0$

$\therefore \mathbb{P}[|X_n - X| > \varepsilon] \leq \mathbb{P}[A_n] \to 0.$

<u>Lemma</u>: If $X_n \xrightarrow{L^p} X$ then $X_n \xrightarrow{P} X$.

Since $\|X_n - X\|_p \geq \|X_n - X\|_1 \Rightarrow X_n \xrightarrow{L^1} X$

WLOG $p = 1$.

$$\mathbb{P}[|X_n - X| > \varepsilon] \leq \frac{\mathbb{E}|X_n - X|}{\varepsilon} = \frac{\|X_n - X\|_1}{\varepsilon} \to 0.$$

<u>Ex</u>: $X_n = n \operatorname{Ber}(1/n)$ independent

$\mathbb{P}[|X_n| > \varepsilon] \leq \frac{1}{n} \to 0$, so $X_n \xrightarrow{P} 0$.

$\|X_n - 0\|_1 = \mathbb{E} \; X_n = n \cdot \frac{1}{n} = 1$, $X_n \not\xrightarrow{L^1} 0$.

Weak Law of Large Numbers

Let X_n be IID R.V., $\mathbb{E} X_n = \mu$, $\operatorname{Var} X_n = \sigma^2 < \infty$. Then $S_n = \sum_{i=1}^{n} X_i$,

$$\frac{1}{n} S_n \xrightarrow{P} \mu, \quad \frac{1}{n} S_n \xrightarrow{L^2} \mu.$$

<u>Proof</u>:

$\mathbb{E} \frac{1}{n} S_n = \mu$

$$\operatorname{Var} \tfrac{1}{n} S_n = \frac{1}{n^2} \operatorname{Var} S_n = \frac{\sigma^2}{n}$$

By Chebyshev's Inequality

$$\mathbb{P}\left[\left| \tfrac{1}{n} S_n - \mu \right| > \varepsilon \right] \le \frac{\sigma^2}{\varepsilon^2 n} \to 0.$$

$$\left\| \tfrac{1}{n} S_n - \mu \right\|_2 = \sqrt{\mathbb{E} | S_n - \mu |^2}$$

$$= \sigma / \sqrt{n} \to 0.$$

Note: Only used variance, would hold if X_n were uncorrelated,

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We say A_n happens infinitely often

$$A_n \text{ i.o.} := \limsup A_n = \lim_{m \to \infty} \bigcup_{n=m}^{\infty} A_n$$

$$= \{ \omega \text{ in infinitely many } A_n \}$$

Note that

$$X_n \xrightarrow{a.s.} X \iff \forall \varepsilon > 0 \ \mathbb{P}[\{ |X_n - X| > \varepsilon \} \text{ i.o.}] = 0$$

Borel-Cantelli Lemmas

a) If A_n sequence of sets

$$\sum_{i=1}^{n} \mathbb{P}[A_i] < \infty \implies \mathbb{P}[A_i \text{ i.o.}] = 0$$

b) If A_i independent then

$$\sum_{i=0}^{n} \mathbb{P}[A_i] = \infty \implies \mathbb{P}[A_i \text{ i.o.}] = 1$$

Proof: a) If $\sum_{i=0}^{\infty} \mathbb{P}[A_i] < \infty$, $\exists m$ s.t. $\sum_{i=m}^{\infty} \mathbb{P}[A_i] < \varepsilon$.

Then $\mathbb{P}[A_i \text{ i.o.}] \leq \mathbb{P}[\bigcup_{i \geq m} A_i]$

$$\leq \sum_{i=m}^{\infty} \mathbb{P}[A_i] < \varepsilon.$$

b) $\mathbb{P}[\bigcup_{i=m}^{\infty} A_i] = 1 - \mathbb{P}[\bigcap^{\infty} A_i^c]$

$$= 1 - \prod_{i=1}^{\infty} \mathbb{P}[A_i^c]$$

$$= 1 - \prod_{i=1}^{\infty} (1 - \mathbb{P}[A_i])$$

$$\geq 1 - \prod_{i=1}^{\infty} \exp(-\mathbb{P}[A_i])$$

since $e^{-x} \geq 1 - x$

$$= 1 - \exp\left(-\sum_{i=n}^{\infty} \mathbb{P}[A_i]\right) = 1.$$

So $\mathbb{P}[A_i \text{ i.o.}] = \mathbb{P}[\lim_m \bigcup_{i=m}^{\infty} A_i]$

$$= \lim_{m \to \infty} \mathbb{P}[\bigcup_{i=m}^{\infty} A_i] = 1.$$

Ex If $X_n \sim Ber(1/n)$, independent then $X_n \xrightarrow{P} 0$,

$$\mathbb{P}[\{X_n(\omega) \not\to 0\}] \geq \mathbb{P}[\{X_n = 1\} \text{ i.o.}] = 1.$$

Ex Independence assumption is needed

If $U \sim$ uniform on $[0,1]$ and $A_n = \{U < \frac{1}{n}\}$,

$\sum \mathbb{P}[A_i] = \sum 1/i = \infty$.

But $\limsup A_n = \lim_m A_m = \emptyset$.

Ex: If $Y_n = n\, I(A_n)$ then $Y_n \xrightarrow{a.s.} 0$ but $Y_n \not\xrightarrow{L^1} 0$.

since $\mathbb{E}\, Y_n = 1$.

Strong Law of large Numbers

If X_n IID, $\mathbb{E} X_n = \mu$ then

$\frac{1}{n} S_n \xrightarrow{a.s.} \mu$.

Proof:

Since $X_n = X_n^+ - X_n^-$ reduce to case $X_n \geq 0$.

(*) We will assume $\mathbb{E} X_n^2 = \sigma^2 < \infty$ (not needed).

Fix $\varepsilon > 0$. Let $A_n = \{ |\frac{1}{n} S_n - \mu| > \varepsilon \}$

By Chebyshev, $\mathbb{P}[A_n] \leq \frac{\sigma^2}{\varepsilon^2 n}$ not summable

But $\sum_{n=1}^{\infty} \mathbb{P}[A_{n^2}] = \sum_{n=1}^{\infty} \frac{\sigma^2}{\varepsilon^2 n^2} < \infty$

So $\frac{1}{n^2} S_{n^2} \xrightarrow{a.s.} \mu$.

For $m^2 < n < (m+1)^2$

$$\frac{m^2}{n} \cdot \frac{S_{m^2}}{m^2} < \frac{S_n}{n} < \frac{S_{(m+1)^2}}{(m+1)^2} \frac{(\lfloor\sqrt{n}\rfloor + 1)^2}{n}$$

$$\limsup \frac{S_n}{n} \le \limsup \frac{S_{m^2}}{m^2} \cdot \limsup \frac{(\lfloor\sqrt{n}\rfloor+1)^2}{n}$$

$$= \mu \quad \text{a.s.}$$

Similarly $\liminf \frac{S_n}{n} = \mu$. a.s.

Let $U_1, \dots, U_n$ IID Unif $[0,1]$

$$V_n = \prod_{i=1}^n U_i, \qquad \mathbb{E} V_n = \prod_{i=1}^n \mathbb{E} U_i = 2^{-n}$$

What is typical value of V_n?

$$V_n = \exp\left(\sum_{i=1}^n \log U_i \right)$$

$$\mathbb{P}[-\log U_i > x] = \mathbb{P}[U_i < e^{-x}] \qquad x > 0$$
$$= e^{-x}$$

so $-\log U_i \sim \text{Exp}(1)$.

By SLLN $\frac{1}{n} \log V_n \to 1$ a.s.

So $\mathbb{P}\left[e^{-(1+\varepsilon)n} \le V_n \le e^{-(1-\varepsilon)n} \right] \to 1.$