*Introduction

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Treat Discrete & Cts in unified way

Ex: Toss series of coins

$$X = \sum_{i=1}^{\infty} 2^{-i} I(\text{coin } i \text{ Heads})$$

discrete

$X \sim U[0,1]$

0110001... Binary

$$Y = 2 \sum_{i=1}^{\infty} 3^{-i} I(\text{coin } i \text{ Heads})$$

Cantor Set

Y is singular continuous

Dimension $\frac{\log 2}{\log 3} \approx 0.63$

Probability Space $(\Omega, \mathcal{F}, \mathbb{P})$ [KS Sec 1.1]

Ω is sample or outcome space

- Ex Roll 3 dice $\{1, \dots, 6\}^3$

Sequence of coin tosses $\{H, T\}^{\mathbb{N}}$

- Abstract Set

- $\mathcal{F}$ a set of subsets of Ω represents events:

Ex: $\{2$ on dice $1\}$

$\{$Tosses $1 + 2$ the same$\}$

A collection $\mathcal{F}$, subsets of Ω is an an algebra

a) $\Omega \in \mathcal{F}$

b) Complements: $C \in \mathcal{F} \Rightarrow \Omega \setminus C \in \mathcal{F}$

c) Finite Unions $C_1, C_2 \in \mathcal{F} \Rightarrow C_1 \cup C_2 \in \mathcal{F}$

An algebra is a σ-algebra if

countable unions: $C_1, \ldots \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{\infty} C_i \in \mathcal{F}$

[KS Lemma 1.1]

Properties: If $\mathcal{F}$ is a σ-algebra

a) $\emptyset \in \mathcal{F}$

b) $C_1, \ldots \in \mathcal{F} \Rightarrow \bigcap_{i=}^{\infty} C_i \in \mathcal{F}$

c) $C_1, C_2 \in \mathcal{F} \Rightarrow C_1 \setminus C_2 \in \mathcal{F}$

Pf: a) $\Omega^c = \emptyset \in \mathcal{F}$

b) $\bigcap_{i=1}^{\infty} C_i = \left(\bigcup_{i=1}^{\infty} C_i^c \right)^c \in \mathcal{F}$

c) $C_1 \setminus C_2 = C_1 \cap C_2^c \in \mathcal{F}$.

- We write $\sigma(\mathcal{A})$ smallest σ-algebra containing $\mathcal{A}$.

- Borel σ-algebra $\mathcal{B}(S)$ is the smallest σ-algebra of all open sets of S.

  → Write $\mathbb{B}$ for $\mathcal{B}(\mathbb{R})$

[KS Def 1.12]

A measure μ on $(\Omega, \mathcal{F})$ is a function

$$\mu : \mathcal{F} \to [0, \infty)$$

such that for $C_1, \ldots$ disjoint

$$\mu\left(\bigcup_{i=1}^{\infty} C_i \right) = \sum_{i=1}^{\infty} \mu(C_i) \quad \text{(Countable additivity)}$$

It is a probability measure if

$$\mu(\Omega) = 1.$$

Properties: i) $\mu(\emptyset) = 0$ Pf $\mu(A \cup \emptyset) = \mu(A) + \mu(\emptyset)$

<u>Properties</u>: i) $\mu(\emptyset) = 0$ P.f $\mu(A \cup \emptyset) = \mu(A) + \mu(\emptyset)$

ii) If $A \subset B \Rightarrow \mu(A) \le \mu(B)$

iii) <u>Subadditive</u> $\mathbb{P}\left(\bigcup_{i=1}^{\infty} C_i\right) \le \sum_{i=1}^{\infty} \mu(C_i)$

iv) If $C_i \searrow C$ then $\mu(C_i) \searrow \mu(C)$ <u>if $\mu(C_1) < \infty$.</u>

v) If $C_i \nearrow C$ then $\mu(C_i) \nearrow \mu(C)$

<u>Pf</u>: iv) $\mu(C_m) = \mu(C) + \sum_{i=m}^{\infty} \mu(C_i \setminus C_{i+1})$

A probability space is a tripled $(\Omega, \mathcal{F}, \mathbb{P})$ where $\mathcal{F}$ is a σ-algebra and $\mathbb{P}$ is a probability measure.

<u>Examples of measures</u>:

Countable discrete set:

$$\mu(A) = \sum_{w \in A} p(w), \qquad \Omega = \{1, \dots, 6\}, \; p(w) = \tfrac{1}{6}$$

<u>Caratheodory's Theorem</u>. [KS Sec 3.4]

If $\mathcal{A}$ is a <u>semi-algebra</u> and m is a sigma-additive function

$$m\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} m(A_i) \qquad A_i \text{ disjoint}$$

then $\exists$ measure μ on $\sigma(\mathcal{A})$ such that $\mu(A) = m(A)$

$\mathcal{A}$ is a <u>Semi-algebra</u> if

i) $\Omega \in \mathcal{A}$

ii) $C, C' \in \mathcal{A} \Rightarrow C \cap C' \in \mathcal{A}$

iii) If $C \subset C'$ then $\exists A_1, \dots, A_n \in \mathcal{A}$ disjoint, $C \cap A_i = \phi$, $C' = C \cup A_1 \cup \dots \cup A_n$

<u>Lebesgue measure</u> on $\mathbb{R}$.

- Define $\mu((a,b]) = b-a$.

Extension of μ to $\mathbb{B}$ by

<u>Properties</u> of Lebesgue measure

- Defined on $\mathbb{B}$
- $\mu((a,b)) = \mu([a,b]) = b-a$.
- $\mu(\{x\}) = 0$
- $\mu(A+x) = \mu(A)$ translation invariance.

Lebesgue measure on $[0,1]$.

$([0,1], \mathbb{B}([0,1]), \mu)$ is a probability space.

<u>Non-measurable Sets</u>

__Vitali set.__

Equivalence Class $x \sim y$ if $x - y \in \mathbb{Q}$

E.G. $\mathbb{Q}$, $\{\sqrt{2} + q : q \in \mathbb{Q}\}$, $\mathbb{Q} + y$

$A \subset [0,1]$ consists of one point in each equivalence class.

__Claim:__ A is not measurable

$$V = \bigcup_{q \in \mathbb{Q} \cap [-1,1]} A + q$$

$[0,1] \subset V \subset [-1,2]$

So $1 \leq \mu(V) \leq 3$

If $\mu(A) = 0$ then $\mu(V) = 0$.

If $\mu(A) > 0$ then $\mu(V) = \infty$. Contradiction.

__Banach - Tarski Paradox__

A ball in $\mathbb{R}^3$ can be decomposed into 5 pieces and re-arranged into 2 balls!