*Integration and Expectation

Tuesday, September 5, 2017 1:49 PM

[KS Sec 3.1, D Sec 1.4]

Goal: Define $\int_S f \, d\mu$ Lebesgue Integration for a measurable function f on $(S, \mathcal{S}, \mu)$

The expectation of a random variable X is

$$\mathbb{E} X := \int_\Omega X(\omega) \, d\mathbb{P}(\omega)$$

Discrete Random variables

$$\mathbb{P}[\{\omega\}] = p(\omega), \quad \sum_{\omega \in \Omega} p(\omega) = 1, \quad \mathbb{P}[A] = \sum_{\omega \in A} p(\omega).$$

Then $\mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega) p(\omega).$

i) Linear $\mathbb{E}[cX + dY] = c\,\mathbb{E}[X] + d\,\mathbb{E}[Y]$

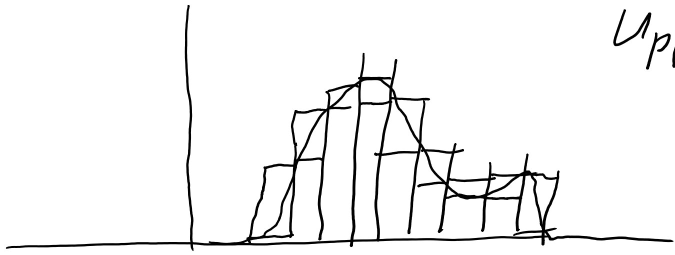
ii) If $0 \le X \le M$ then $0 \le \mathbb{E} X \le M \mathbb{P}[\Omega]$

iii) $\int X \ge \int Y$ if $X \ge Y \ge 0$

iv) $\int a \, I(\Omega) = a.$

v) $\left| \int X \, d\mu \right| \le \int |X| \, d\mu$

Riemann Integration



Upper and lower sums converge

$$\mathbb{P}[U \in Q] = \int_0^1 I(Q)(x)\,dx$$

should be 0 but Riemann integral not defined

Lebesgue Integration on $(S, \mathcal{S}, \mu)$

* Assume $\mu(S) < \infty$.

4 Steps:

a) $f = \sum a_i I(A_i)$ simple function

b) f non-negative & bounded

c) f non-negative

d) Any f.

Simple Functions

$$X = \sum a_i I(A_i)$$

A_i partition of S.

Then $\int_S X \, d\mu := \sum a_i \mu(A_i)$

(iv) holds.

Linear

If $Y = \sum b_i I(B_i)$,

$$aX + bY = \sum_{i,j} (c\, a_i + d\, b_j) I(A_i \cap B_j)$$

$$\int [aX + bY] = \sum_{i,j} (c a_i + d b_j) \mu(A_i \cap B_j)$$

$$= c \sum_i a_i \underbrace{\sum_j \mu(A_i \cap B_j)}_{=\mu(A_i)}$$

$$+ d \sum_j b_j \sum_i \mu(A_i \cap B_j)$$

$$= c \int X \, d\mu + d \int X \, d\mu$$

Bounded Functions $X_n \le M$.

If X_n simple and $X_n \uparrow X$ then

$$\int X \, d\mu := \lim_n \int X_n \, d\mu.$$

Three things to check

a) Limit exists

b) Some sequence X_n exists

c) Limit does not depend on X_n

- If $X_n = 2^{-n} \lfloor 2^n X_n \rfloor$, $X_n \uparrow X$ + X_n simple

- Since $X_{n+1} - X_n \geq 0$,

$$\int X_{n+1} d\mu = \int X_n d\mu + \int X_{n+1} - X_n \, d\mu \geq \int X_n \, d\mu$$

So $\int X_n \, d\mu$ is a bounded increasing sequence.

- Suppose also $X_n' \uparrow X$.

Let $C_n = \{X - X_n > \varepsilon\}$, $C_n' = \{X - X_n' > \varepsilon\}$.

$\mu(C_n) \searrow 0$, $\mu(C_n') \searrow 0$.

For $(C_n \cup C_n')^c$, $|X_n - X_n'| < \varepsilon$, so

$$\left|\int X_n - \int X_n' d\mu\right| \leq \int |X_n - X_n'| \, d\mu$$

$$\leq \int \varepsilon + M I(C_n \cup C_n') \, d\mu$$

$$\leq \varepsilon \mu(S) + M \underbrace{\mu(C_n \cup C_n')}_{\to 0}$$

So $\lim \int X_n - \int X_n' \leq \varepsilon \implies \int X_n = \int X_n'$

c) <u>Nonnegative Functions</u>

$x \wedge y := \min\{x, y\}$, $x \vee y := \max\{x, y\}$.

$$\int X := \lim_n \int X \wedge n \, d\mu$$

May be infinite.

__Exercise__: Check satisfies properties.

d) General Functions

Write $X^+ = X \vee 0$, $X^- = (-X) \vee 0$

so $X = X^+ - X^-$

Define $\int X \, d\mu := \int X^+ \, d\mu - \int X^- \, d\mu$

Not defined if $\mathbb{E} X^+ = \mathbb{E} X^- = \infty$

Integral on a subset:

$$\int_A X \, d\mu := \int X \cdot I(A) \, d\mu$$

__Expectation__: $\mathbb{E} X = \int X \, d\mathbb{P}$

__Convergence__

- $f_n \to f$ almost everywhere (a.e.) if $\mu(\{ w : f_n(w) \not\to f(w) \}) = 0$.

– Does $f_n \to f_n$ a.e. $\Rightarrow \int f_n \, d\mu \to \int f \, d\mu$?

No: $f_n(\omega) = n \, I(0 \le \omega \le \frac{1}{n})$

$\int f_n \, d\mu = 1.$

But $f_n \to 0$ a.e. $\int 0 \, d\mu = 0.$

Or $h(\omega)$ compactly supported,
$f_n(\omega) = h(\omega - n)$

Dominated Convergence Theorem [KS Thm 3.26]

If $f_n \to f$ a.e. and for some φ,

- $\sup_n |f_n| \le \varphi$ a.e.
- $\int \varphi \, d\mu < \infty$

Then $\int f_n \, d\mu \to \int f \, d\mu$

Proof: For random variables i.e. $\mu(\Omega) = 1.$

Let $\varepsilon > 0.$

Let $A_n = \{ |f_n - f| > \varepsilon \}$

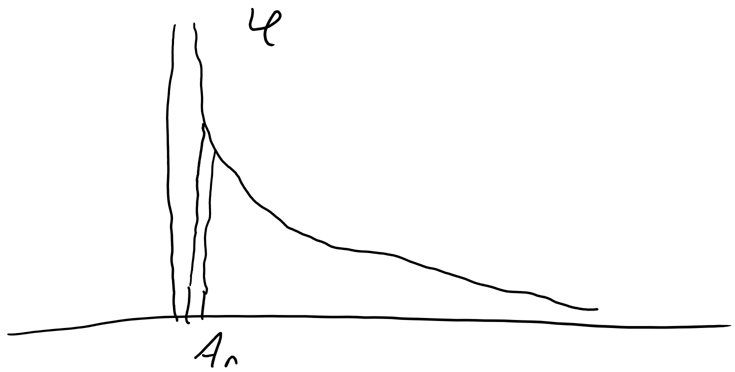
We have that $\mu(A_n) \to 0$

So $\left| \int f_n \, d\mu - \int f \, d\mu \right| = \left| \int f_n - f \, d\mu \right|$

$\le \int |f_n - f| \, d\mu$

$$\le \int \varepsilon + 2\varphi \cdot I(A_n)\, d\mu$$

$$= \varepsilon + 2\int_{A_n} \varphi\, d\mu$$



$$\lim_{n\to\infty} \int_{A_n} \varphi\, d\mu = \lim \int_{A_n} \varphi \wedge M\, d\mu + \int_{A_n} \varphi - \varphi\wedge M\, d\mu$$

$$\le \lim_{n\to\infty} M\underbrace{\mu(A_n)}_{=0} + \underbrace{\int \varphi\, d\mu - \int \varphi\wedge M\, d\mu}_{\xrightarrow{M\to\infty} 0}$$

$$\therefore \quad \int f_n\, d\mu \to \int f\, d\mu.$$

Monotone Convergence Theorem [KS Thm 3.27]

If $f_n \ge 0$, $f_n \uparrow f$ then $\int f_n\, d\mu \to \int f\, d\mu$.

Fatou's Lemma [KS Lem 3.28]

If $f_n \ge 0$ then

$$\liminf \int f_n\, d\mu \ge \int (\liminf f_n)\, d\mu.$$

Expectations with densities:

If a R.V. X has density $f(x)$,

$$\mathbb{E}X = \int x f(x) dx, \quad \mathbb{E}[g(X)] = \int g(x) \cdot f(x) dx$$

Ex: $X \sim Exp(1)$,

$$\mathbb{E}X = \int_0^\infty x e^{-x} dx = 1.$$

$$\mathbb{E}X^2 = \int_0^\infty x^2 e^{-x} dx = 2.$$

Gaussian:

$$\mathbb{E}X = \int x \frac{e^{-x^2}}{2} \cdot \frac{1}{\sqrt{2\pi}} dx = 0$$

$$\mathbb{E}X^2 = \int x \cdot \left(x e^{-x^2/2} \cdot \frac{1}{\sqrt{2\pi}}\right) dx$$

$$= \int e^{-x^2/2} \cdot \frac{1}{\sqrt{2\pi}} dx = \mathbb{P}[\Omega] = 1.$$

Moments:

$\mathbb{E}X^{k}$ called k-th moment.

Variance defined as

$$Var X = \mathbb{E}(X - \mathbb{E}X)^2$$

$$= \mathbb{E}X^2 - 2\mathbb{E}[X \cdot \mathbb{E}X] + (\mathbb{E}X)^2$$

$$= \mathbb{E}X^2 - (\mathbb{E}X)^2$$