Card shuffling

Tuesday, October 24, 2017 5:26 PM

Random walk on a group

G_n IID group elements

$X_n = G_n \dots G_0 \cdot X_0 = G_n X_{n-1}$.

π is uniform distribution.

Lazy RW on Hypercube $G = \{0,1\}^n$, $\mathbb{Z}_2^n$

- Pick $i \in \{1, \dots, n\}$
- $X_t(i) = X_{t-1}(i) + Z_n$ $\mathbb{P}[Z_t = 1] = \mathbb{P}[Z_t = 0] = \frac{1}{2}$.

Co-ordinates are random after update.

$$d_{TV}(X_t, \pi) \leq \mathbb{P}[\text{all co-ordinates updated by time } t]$$

<u>Coupon collector problem</u>

Sample $U_1, \dots, U_k$ IID with dist $U\{1, \dots, n\}$.

Let T be the first time such that all $j \in \{1, \dots, n\}$ have appeared. Then

$$\lim_{n \to \infty} \frac{T_n}{n \log n} = 1. \quad \text{in probability.}$$

Let S_k = time until k distinct U_i chosen.

Then $S_k - S_{k-1} \sim \mathrm{Geom}\left(\frac{n+1-k}{n}\right)$ independent.

$$\mathbb{E}\, S_k - S_{k-1} = \frac{n}{n+1-k} \qquad \boxed{\mathbb{E}\,\mathrm{Geom}(p) = \frac{1}{p}}$$

So $\mathbb{E}\, T = \mathbb{E}\, S_n = \sum_{k=1}^{n} \frac{n}{n+1-k} = n \sum_{k=1}^{n} \frac{1}{n} \approx n \log n.$

$$\mathrm{Var}\, T = \sum \mathrm{Var}(S_k - S_{k-1}) = \frac{(k-1)/n}{(n+1-k)^2/n^2}$$

$$\boxed{\mathrm{Var}\,\mathrm{Geom}(p) = \frac{1-p}{p^2}}$$

Now $\frac{(k-1)n}{(n+1-k)^2} \leq \frac{n^2}{(n+1-k)^2}$ so

$$\mathrm{Var}\, T \leq n^2 \sum_k \frac{1}{(n+1-k)^2} = n^2 \sum_{k=1}^{n} \frac{1}{k^2} = O(n^2).$$

So by Chebyshev's Inequality,

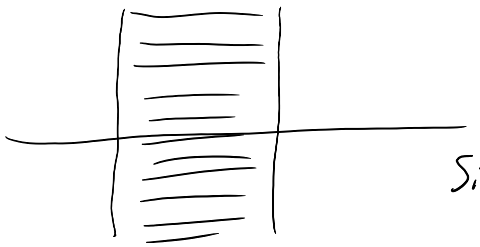
$$\mathbb{P}[\,|T - \mathbb{E}\,T| \geq \varepsilon n \log n\,] \leq \frac{C n^2}{\varepsilon^2 n^2 \log^2 n} \to 0.$$

So $\frac{T}{n \log n} \xrightarrow{P} 1.$

So $d_{TV}(X_{(1+\varepsilon) n \log n}, \pi) \to 0.$

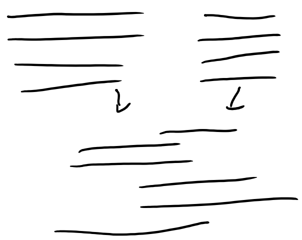
<u>Lower Bound:</u> Homework.

Riffle Shuffle Gilbert - Shannon - Reid



- Split into two decks
Size $Bin(n, 1/2)$

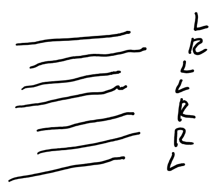
- Interlace drop with probability



$$\frac{M_L}{M_L + M_R}$$

M_L is # cards remaining on the left.

New permutation



If g correspond to one such split into N_L + $N_R = n - N_L$ deck and a choice $(L, L, R, R, \ldots, L, R)$ of interleaving. The interleavings have equal probability $\binom{n}{N_L}^{-1}$ since

$$\mathbb{P}[\,(L, L, R, R, \ldots) \mid N_L\,]$$

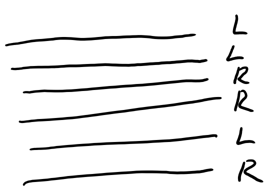
$$= \frac{N_L}{n} \cdot \frac{N_L - 1}{n-1} \cdot \frac{N_R}{n-2} \cdot \frac{N_R - 1}{n-3} \cdot \ldots \frac{1}{1}$$

$$= \frac{N_L! \cdot N_R!}{n!} = \binom{n}{N_L}^{-1}.$$

- $\mathbb{P}[\,g\,] = \mathbb{P}[\,\mathrm{Bin}(n, \tfrac{1}{2}) = N_L\,] \cdot \binom{n}{N_L}^{-1}$

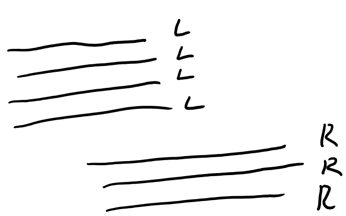
$$= \binom{n}{N_L} \left(\tfrac{1}{2}\right)^{N_L} \left(\tfrac{1}{2}\right)^{n - N_L} \cdot \binom{n}{N_L}^{-1} = 2^{-n}$$

Inverse Map



Assign each card L or R.

Split into L & R decks preserving the order



Place L deck on top.

For the same sequence $(L, L, R, \ldots)$ this operation is g^{-1}! Easier to analyse.

Claim: Random walk & inverse walk have the same d_{TV}.

$$X_t = G_t G_{t-1} \cdots G_1$$
$$Y_t = H_t \cdots H_1$$

$$\mathbb{P}[X_t = x] = \sum_{g_1,\dots,g_t} \mathbb{P}[G_1 = g_1, \dots, G = g \quad I(g_t \cdots g_1 = x)$$

$$= \sum_{g_1,\dots,g_t} \mathbb{P}[G_1 = g_t, \dots, G = g_1] \, I(g_t \cdots g_1 = x)$$

$$= \sum_{g_1,\dots,g_t} \mathbb{P}[H_1 = g_t^{-1}, \dots, H_n = g_1^{-1}) \, I(g_1^{-1} \cdots g_t^{-1} = x^{-1})$$

$$= \mathbb{P}[Y_t = x^{-1}]$$

$$d_{TV}(X_t, \pi) = \frac{1}{2} \sum_x \left| \mathbb{P}[X_t = x] - \frac{1}{|G|} \right|$$

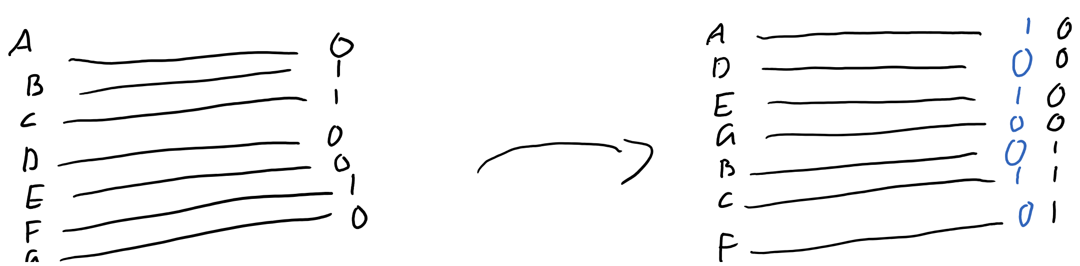
$$= \frac{1}{2} \sum_x \left| \mathbb{P}[Y_t = x^{-1}] - \frac{1}{|G|} \right|$$

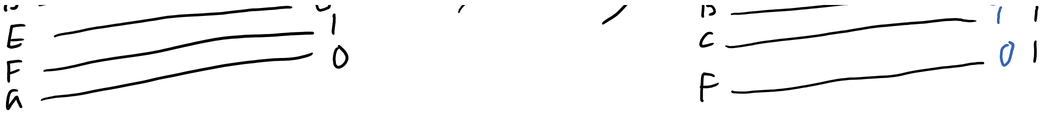
$$= \frac{1}{2} \sum_y \left| \mathbb{P}[Y_t = y] - \frac{1}{|G|} \right|$$

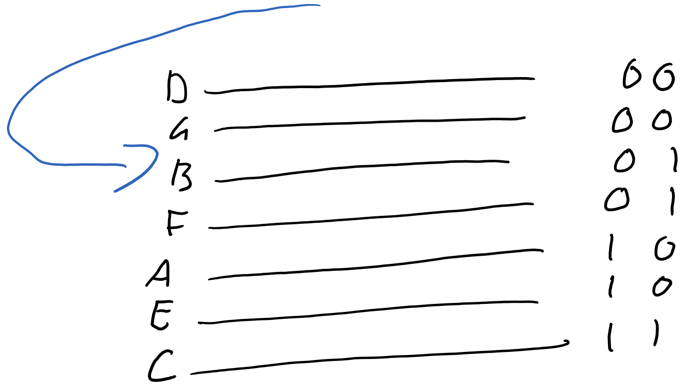
$$= d_{TV}(Y_t, \pi)$$

Analysis of Inverse Chain

Switch R/L with 1/0.







In binary order

Will be mixed once each card has a unique string.

$$\mathbb{P}[A, B \text{ same string}] = 2^{-t}$$

Union bound over $\binom{n}{2}$ pairs,

$$d_{TV}(X_t, \pi) \le \mathbb{P}[\text{all string unique}]$$
$$\le \binom{n}{2} 2^{-t}$$
$$\to 0 \quad \text{if} \quad t = (2+\varepsilon)\log_2 n.$$

Right bound $t \approx \frac{3}{2}\log_2 n$.

For 52 cards:

t	1	…	4	5	6	7	8
$d_{TV}(X_t, \pi)$	1.000		1.000	0.924	0.611	0.334	0.167 …