PROBLEM SET

Do as many of the problems as you can. Some of the problems are not so easy, feel free to come and see me if you want to discuss any of them or some hints - I'm mainly interested in you really thinking about them.

Problem 1: Let G be a random d-regular graph with n vertices ($d \geq 3$ is fixed). Show that the probability that G is an expander tends to 1 as n tends to infinity.

Problem 2: Let G_n be a sequence of Erdos Renyi $G(n, d/n)$ random graphs. Let X_n be the number of components of size 2. Show that

$$\frac{1}{n} X_n \to c_d$$

in probability as $n \to \infty$ and find c_d.

Problem 3: Let G be a random d-regular graph with n vertices ($d \geq 3$ is fixed). Consider an SI infection model where infected vertices infect their susceptible neighbours at rate 1 and thereafter always remain infected (think an incurable nonfatal disease). Starting from a single infected vertex, let T_n be the time until the whole graph is infected. Show that there exists $0 < c \leq c' < \infty$ such that

$$\mathbb{P}[c \leq \frac{1}{\log n} T_n \leq c'] \to 1.$$

Extra challenge: show that $\frac{1}{\log n} T_n$ converges in probability.

Problem 4: Let G be a random graph n vertices chosen according to the configuration model with a random IID degree sequence d_v where $d_v \geq 3$ and $\mathbb{P}[d_v = k] \asymp k^{-s}$ for $2 < s < 3$, a scale free graph. Show that the mixing time of a random walk satisfies

$$\mathbb{P}[c \leq \frac{1}{\log n} t_{mix} \leq c'] \to 1.$$

for some constants $0 < c \leq c' < \infty$.

Problem 5: Let G be a random d-regular graph with n vertices ($d \geq 3$ is fixed). Let Z be the partition function of the Ising model with inverse temperature β and zero external field. Find

$$\lim \frac{1}{n} \log \mathbb{E} Z.$$

Let σ be a configuration chosen from the Ising distribution. If $U, U' \in V$ are uniformly chosen vertices for which β are $\sigma_U, \sigma_{U'}$ asymptotically independent as $n \to \infty$.

Problem 6: For $d \geq 3$ show that on the infinite d-regular tree the reconstruction problem is solvable for the hardcore model with fugacity λ for large enough λ.

Problem 7: Let Z be the number of independent sets on a random d-regular graph. In class we showed that showed that Z^2 was the partition function for spin system that is a pair of independent sets. Show that the BP equations for this spin system have a unique fixed point.