Reconstruction Threshold

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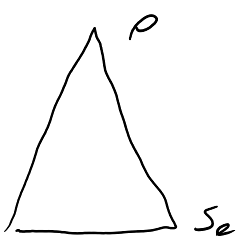
- A mixture of Gibbs measures is a Gibbs measure. A Gibbs measure is <u>extremal</u> or <u>pure</u> if it cannot be written as a non-trivial convex combination of Gibbs measures.

- <u>Reconstruction Problem</u>

Given a Gibbs measure μ. and $\sigma \sim \mu$.



Let $S_e = \{u : d(u, \rho)) = e\}$

$$S_{e+} = \bigcup_{e' \geq e} S_{e'}$$

Let $n_e(x) = n_e(\sigma)(x) = \mathbb{P}[\sigma_\rho = x \mid \sigma_{S_e}]$

* note this is a random variable

Since $n_e(x) = \mathbb{P}[\sigma_\rho = x \mid \sigma_{S_{e+}}]$

it is a backwards martingale so

$$n_e(x) \xrightarrow{a.s.} n(x).$$

We say the <u>reconstruction problem</u> is non-solvable if $\eta(x) = \mu(\sigma_\rho = x)$ a.s.

and <u>solvable</u> if $\exists x$ s.t. $\mathbb{P}[\eta(x) \neq \mu(\sigma_\rho = x)] > 0$.

<u>Lemma</u>: A Gibbs measure is extremal iff the reconstruction problem is non-solvable.

<u>Proof</u>: Let $m^{(0)}_{u \to v}(x) = I(\sigma_u = x)$

and $m^{(\ell)}_{u \to v} = BP(\{m^{(\ell-1)}_{w \to u}\}_{\substack{w \sim u \\ w \neq v}})$.

and $m_{u \to v} = \lim_{\ell \to \infty} m^{(\ell)}_{u \to v}$.

We have that $\eta^{(\ell)} = BP(\{m^{\ell-1}_{v \to \rho}\}_{v \sim \rho})$.

- $\{m$ is deterministic$\} \Leftrightarrow \{\eta$ deterministic$\}$ $\Leftrightarrow \{$reconstruction is non-solvable$\}$

- If m is not deterministic then μ is a non-trivial mixture of Gibbs measures.

- If μ is a non-trivial mixture of Gibbs measures, m cannot be deterministic as it would be a mixture of μ_m.

Ising Model

- μ_+ / μ_- are extremal since if

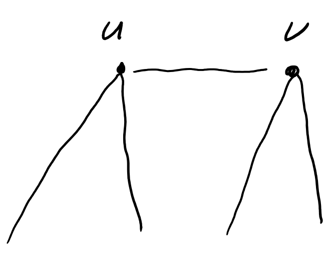
  μ_+ mixture of μ_1 & μ_2,

  couple $\sigma_+ \geq \sigma_1$, $\sigma_+ \geq \sigma_2$

  and $\mathbb{E}\,\sigma_+(u) = \mathbb{E}\, p_1 \sigma_1(u) + p_2 \sigma_2(u)$

  so $\mathbb{E}\,\sigma_+(u) = \mathbb{E}\,\sigma_i(u) \implies \sigma_+(u) = \sigma_i(u)$ a.s.

- Free Ising Measure



$$\mathbb{P}[\sigma_u = x, \sigma_v = y] \propto e^{\beta x y}\, m_{u \to v}(x)\, m_{v \to u}(y)$$

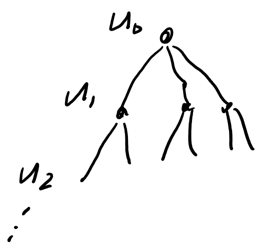
$$\propto e^{\beta x y}$$

So $\mathbb{P}[\sigma_u = x, \sigma_v = y] = \dfrac{e^{\beta x y}}{2(e^{\beta} + e^{-\beta})}$

$$= \frac{1}{2}\left[\frac{1}{2} + \frac{1}{2}\tanh \beta x y\right]$$

and $\mathbb{P}[\sigma_u = \sigma_y] = \frac{1}{2} + \frac{1}{2}\tanh\beta$.

- <u>Broadcast Model</u>:



Along each ray in the tree

$\sigma_{u_0}, \sigma_{u_1}, \ldots$

is a Markov Chain with transition matrix

$$M = \begin{bmatrix} \frac{1}{2} + \frac{1}{2}\theta & \frac{1}{2} - \frac{1}{2}\theta \\ \frac{1}{2} - \frac{1}{2}\theta & \frac{1}{2} + \frac{1}{2}\theta \end{bmatrix}, \qquad \theta = \tanh\beta$$

and eigenvalues $\{1, \theta\}$.

- Transition corresponds to same with prob θ, random with prob $1-\theta$.

$\mathrm{Cov}(\sigma_{u_0}, \sigma_{u_j}) = \theta^j$

<u>Kester - Stigum Bound</u>

Kesten - Stigum Bound

Reconstruction is solvable if $d\theta^2 > 1$ (true for any broadcast model).

Proof:

Let $Y_\ell = (d\theta)^{-\ell} \sum_{u \in S_\ell} \sigma_u$

$$\mathrm{Var}\, Y_\ell = (d\theta)^{-\ell} \sum_{u \in S_\ell} \sum_{v \in S_\ell} \mathrm{Cov}(\sigma_u, \sigma_v)$$

$$= (d\theta)^{-\ell} \sum_{u \in S_\ell} 1 + \sum_{j=0}^{\ell} \theta^{2j} \cdot (d-1) \cdot d^{\ell-1}$$

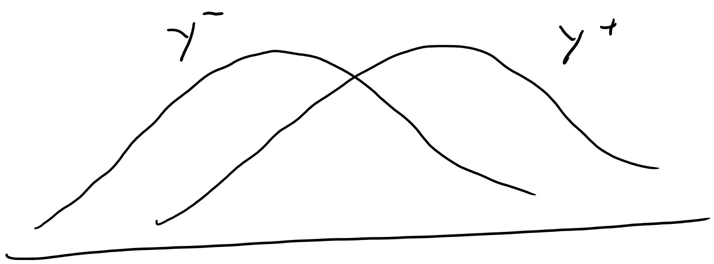
$$\asymp (d\theta)^{-\ell} \sum_{u \in S_\ell} C \cdot d^{\ell} \cdot \theta^{2\ell}$$

$$\asymp C.$$

And $\mathrm{Cov}(Y_\ell, \sigma_\rho) = (d\theta)^{-\ell} \cdot \sum_{u \in S_\ell} \mathrm{Cov}(\sigma_\rho, \sigma_u)$

$$= (d\theta)^{-\ell} \cdot d^{\ell} \cdot \theta^{\ell} = 1.$$

Can show that $Y_\ell \to Y$ a.s., $\mathrm{Cov}(Y, \sigma_\rho) = 1$,
$\mathrm{Var}(Y) = C$ so



$$\mathrm{Cov}(\tanh(Y), \sigma_\rho) > 0$$

and guess $\sigma = 1$ with prob

$$\frac{1}{2} + \frac{1}{2}\tanh(Y)$$

is correct with probability

$$\mathbb{E}\Big(\tfrac{1}{2} + \tfrac{1}{2}\tanh(Y)\Big)\cdot\Big(\tfrac{1}{2}(1+\sigma)\Big) + \Big(\tfrac{1}{2} - \tfrac{1}{2}\tanh(Y)\Big)\cdot\Big(\tfrac{1}{2}(1-\sigma)\Big)$$

$$= \frac{1}{2} + \frac{1}{2}\mathbb{E}\tanh(Y)\cdot\sigma$$

$$= \frac{1}{2} + \frac{1}{2}\mathrm{Cov}(\tanh(Y), \sigma) > \frac{1}{2}.$$

Non-reconstruction

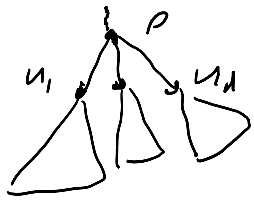
Clearly uniqueness $\Rightarrow$ non-reconstruction

so $d\theta \le 1 \Rightarrow$ non-reconstruction.

- What if $\frac{1}{d} < \theta \le 1/\sqrt{d}$?

Let $m_u^{(n)} = m_{u \to u^+}^{(n)}(+)$ $\quad u^+$ parent of u.

Let $m_u = m_{u \to u'}$



- Let $x_n = \mathbb{E}[m_\rho^{(n)} \mid \sigma_\rho = +] - \frac{1}{2}$

- $$m_\rho^{(n+1)} = \frac{\prod_{i=1}^{d} \left(1 + 2\theta(m_{u_i}^{(n)} - \frac{1}{2})\right)}{\prod_{i=1}^{d} \left(1 + 2\theta(m_{u_i}^{(n)} - \frac{1}{2})\right) + \prod_{i=1}^{d} \left(1 - 2\theta(m_{u_i}^{(n)} - \frac{1}{2})\right)}$$

$$= \frac{Z_+}{Z_+ + Z_-}.$$

Write $\mathbb{E}_+[\cdot]$ for $\mathbb{E}[\cdot \mid \sigma_\rho = +]$.

$$\mathbb{E}_+[m_{u_i}^{(n)} - \tfrac{1}{2}] = (\tfrac{1}{2} + \theta)\,\mathbb{E}[m_{u_i}^{(n)} - \tfrac{1}{2} \mid \sigma_{u_i} = +]$$
$$+ (\tfrac{1}{2} - \theta)\,\mathbb{E}[m_{u_i}^{(n)} - \tfrac{1}{2} \mid \sigma_{u_i} = -]$$
$$= \theta\,\mathbb{E}[m_{u_i}^{(n)} - \tfrac{1}{2} \mid \sigma_{u_i} = +] = \theta x_n$$

$$\mathbb{E}[Z_\pm \mid \sigma_\rho = +] = (1 \pm 2\theta^2 x_n)^d$$

by conditional independence.

$$\mathbb{E}\left[m_\rho^{(n)} - \tfrac{1}{2}\right]^2 = \mathbb{E}\left[\left(m_\rho^{(n)} - \tfrac{1}{2}\right)^2 \mid \sigma_\rho = +\right]$$

$$x_n + \tfrac{1}{2} = \mathbb{E}[m_\rho^{(n)} \mid \sigma_\rho = +]$$

$$= \sum_A \mathbb{P}[\sigma_\rho = + \mid \sigma_{S_n} = A] \cdot \mathbb{P}[\sigma_{S_n} = A \mid \sigma_\rho = +]$$

$$= \sum_A \mathbb{P}[\sigma_\rho = + \mid \sigma_{S_n} = A] \cdot \frac{\mathbb{P}[\sigma_\rho = + \mid \sigma_{S_n} = A] \cdot \mathbb{P}[\sigma_{S_n} = A]}{\mathbb{P}[\sigma_\rho = +]}$$

$$= 2 \sum_A (m^{(n)}(A))^2 \cdot \mathbb{P}[\sigma_{S_n} = A]$$

$$= 2\, \mathbb{E}(m^{(n)})^2$$

Also $\mathbb{E}(m^{(n)} - \frac{1}{2})^2 = \mathbb{E}(m^{(n)})^2 - \mathbb{E}\, m^{(n)} + \frac{1}{4}$

$$= \mathbb{E}(m^{(n)})^2 - \frac{1}{4}$$

so $\mathbb{E}(m^n - \frac{1}{2}) = \frac{1}{2}(x_n + \frac{1}{2}) - \frac{1}{4} = \frac{1}{2} x_n$

Since $m^{(n)} - \frac{1}{2} \mid \sigma_\rho = + \overset{d}{=} -(m^{(n)} - \frac{1}{2}) \mid \sigma_\rho = -$,

$$\mathbb{E}_+\left[(m_\rho^{(n)} - \tfrac{1}{2})^2\right] = \mathbb{E}_-\left[(m_\rho^{(n)} - \tfrac{1}{2})^2\right]$$

$$= \mathbb{E}\left[(m_\rho^{(n)} - \tfrac{1}{2})^2\right] = \frac{x_n}{2}.$$

$$\mathbb{E}_+\left[(m_{u_i}^{(n)} - \tfrac{1}{2})^2\right] = \frac{x_n}{2}$$

Now $\mathbb{E}[Z_+^2 \mid \sigma_\rho = +] = \left(\mathbb{E}_+(1 + 2\theta(m_n - \tfrac{1}{2}))^2\right)^d$

$$= \left(1 + 4\theta\, \mathbb{E}_+[m_{u_i}^{(n)} - \tfrac{1}{2}] + 4\theta^2\, \mathbb{E}_+(m_n - \tfrac{1}{2})^2\right)^d$$

$$= (1 + 6\theta^2 x_n)^d$$

$$= (1 + 6\theta^2 x_n)^d$$

$$\mathbb{E}_+[Z_+ Z_-] = \left(\mathbb{E}_+\left(1 + 2\theta\,(m_{u_i}^{(n)} - \tfrac{1}{2})\right)\left(1 - 2\theta\,(m_{u_i}^{(n)} - \tfrac{1}{2})\right)\right)^d$$

$$= (1 - 2\theta^2 x_n)^d$$

$$\mathbb{E}_+[Z_-^2] = \left(\mathbb{E}_+\left(1 - 2\theta(m_{u_i}^{(n)} - \tfrac{1}{2})\right)^2\right)$$

$$= (1 - 2\theta^2 x_n)^d$$

Now $\dfrac{1}{s+r} = \dfrac{1}{s} - \dfrac{r}{s^2} + \dfrac{r^2}{s^2}\,\dfrac{1}{s+r}$

So $m_\rho^{(n+1)} - \dfrac{1}{2} = \dfrac{Z_+}{Z_+ + Z_-} - \dfrac{1}{2}$

$$= \frac{Z_+}{2} - \frac{Z_+(Z_+ + Z_- - 2)}{4} + \frac{(Z_+ + Z_- - 2)^2}{4} \cdot \frac{Z_+}{Z_+ + Z_-} - \frac{1}{2}$$

$$\leq \frac{Z_+}{2} - \frac{Z_+(Z_+ + Z_- - 2)}{4} + \frac{(Z_+ + Z_- - 2)^2}{4} - \frac{1}{2}$$

$$= \frac{Z_+}{2} + \frac{(Z_- - 2)(Z_+ + Z_- - 2)}{4} - \frac{1}{2}$$

$$= \frac{Z_+ Z_- + (Z_- - 2)^2}{4} - \frac{1}{2}$$

$$x_{n+1} \leq \mathbb{E}_+\left[\frac{Z_+ Z_- + (Z_- - 2)^2}{4} - \frac{1}{2}\right]$$

$$= \frac{1}{2} - \frac{1}{2}(1 - 2\theta^2 x_n)^d$$

$$\leq \frac{1}{2} - \frac{1}{2}(1 - 2\theta^2 d x_n)$$

$$= d\theta^2 x_n$$

So if $d\theta^2 < 1$ then $x_n \to 0$

and $\mathbb{E}(m_n - \frac{1}{2})^2 \to 0$ so $m_n \to \frac{1}{2}$.

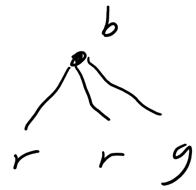
Hence we have non-reconstruction and the Kesten-Stigum bound is tight.

Not always tight:

For q-colourings

$$M = \begin{pmatrix} 0 & & {}^1/_{q-1} \\ & \ddots & \\ {}^1/_{q-1} & & 0 \end{pmatrix}, \qquad \theta = -\frac{1}{q-1}$$

reconstruction if $d \geq (q-1)^2$.



<u>Exact reconstruction</u>

If all $q-1$ colours except blue appear in the children of u, then u must be blue.

Let $f(\ell, k)$ be the succes probability for the coupon collector problem with ℓ coupons and k trials.

Then $f(\ell, (1-\varepsilon)\ell \log \ell) \to 0$

$f(\ell, (1+\varepsilon)\ell \log \ell) \to 1$ as $\ell \to \infty$.

Let $d = (1+\varepsilon) q \log q$.

and $p_n = \mathbb{P}[\, m_\rho^{(n)} = b \mid \sigma_\rho = b \,]$.

Then $p_{n+1} \geq \mathbb{E}[\, f(q-1, \mathrm{Bin}(d, p_n)) \,]$

and if $p_n \geq 1 - \frac{\varepsilon}{2}$ then

w.h.p. $\mathrm{Bin}(d, 1-\frac{\varepsilon}{2}) \geq (1+\frac{\varepsilon}{3}) q \log q$

so $\mathbb{E}[\, f(q-1, \mathrm{Bin}(d, 1-\frac{\varepsilon}{2})) \,] \geq 1 - \frac{\varepsilon}{2}$.

for large enough q.

But $d \approx q \log q$ is much smaller than the KS-bound of $d = (q-1)^2$.