Moments of Independent Sets

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Let $Z = \#$ IS of G.

$$\lim_n \frac{1}{n} \log \mathbb{E} Z = \max \Phi(h)$$

where h is the edge empirical measure of TIFP.

$$m = BP[m]$$
$$= \frac{(1-m)^{d-1}}{1+(1-m)^{d-1}}$$

- Decreasing in m so one fixed point
- If $m = \frac{\alpha \log d}{d}$, d large

$BP[m] \approx d^{-\alpha}$ so $m^* \approx \frac{\log d}{d}$.

$$\Phi(m) = \Phi_{vertex}(m) - \frac{d}{2} \Phi_{edge}(m)$$
$$= \log(1+(1-m)^{d-1}) - \frac{d}{2}\log(1-m^2)$$
$$\approx \frac{\log d}{d} + \frac{d}{2} \cdot \frac{\log^2 d}{d^2}$$
$$\approx \frac{\log^2 d}{2d}.$$

What about $\mathbb{E} Z^2$?

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- Spin system $\sigma = (\sigma^1, \sigma^2) \in (\mathcal{X}^2)^V$.
- Find fixed points:

$$m(+,+) = \frac{m(-,-)^{d-1}}{1 + m(-,-)^{d-1} + (m(-,-) + m(+,-))^{d-1} + (m(-,-) + m(-,+))^{d}}$$

$A = m(-,-)$, $B = m(-,-) + m(+,-)$,

$C = m(-,-) + m(-,+)$

Then

$$m(-,+) = \frac{B^{d-1}}{1 + A^{d-1} + B^{d-1} + C^{d-1}}$$

$$m(+,-) = \frac{C^{d-1}}{1 + A^{d-1} + B^{d-1} + C^{d-1}}$$

If $B > C \Rightarrow m(+,-) > m(-,+)$

and $m(-,+) > m(+,-)$

So $m(+,-) = m(-,+)$

Let $m := m(+,-) + m(+,+) = 1 - C = 1 - B$.

$m(+,+) = \gamma m^2.$

- Must have $m \approx \frac{\log d}{d}$, $m(+,+) \leq \frac{\varepsilon \log d}{d}$.

- Let $m = m^* + \varepsilon$, $m(+,+) = m^2 + \gamma$

$$\frac{m(+,+)}{m(+,-)} = \frac{m^2 + \gamma}{m - m^2 - \gamma} = \frac{1}{1-m} + \frac{\gamma}{m} +$$

$$= \frac{C^{d-1}}{B^{d-1}} = \frac{((1-m)^2 + \gamma)^{d-1}}{(1-m)^{d-1}}$$

$$= (1-m)^{d-1} \cdot \left(1 + \frac{\gamma}{1-m}\right)^{d-1}$$

So $\mathbb{E} Z^2 \lesssim n^{O(1)} (\mathbb{E} Z)^2$

actually

$$\frac{\mathbb{E} Z^2}{(\mathbb{E} Z)^2} \to C > 1.$$

- $\mathbb{P}[Z > \varepsilon \mathbb{E} Z] > \varepsilon.$

$\Leftrightarrow \mathbb{P}[\log Z - \log \mathbb{E} Z > \log \varepsilon] > \varepsilon$

• By Azuma - Hoeffding

$$\mathbb{P}\left[\,|\log Z - \mathbb{E}\log Z| > t\sqrt{n}\,\right] \le 2e^{-ct^2}$$

$$\therefore \quad |\mathbb{E}\log Z - \log \mathbb{E}Z| \le C\sqrt{n}.$$

So

$$\frac{1}{n}\log Z \to \Phi = \Phi_{\text{vertex}} - \frac{d}{2}\Phi_{\text{edge}}$$

$$= \log\left(1 + (1-m)^{d-1}\right) - \frac{d}{2}\log\left(1-m^2\right)$$

What about larger IS?

$\mathbb{E}$ # IS of size αn?

$$\exp\left(n\,\Phi_\alpha + o(n)\right)$$

where $\Phi_\alpha = \max\limits_{h:\,\bar{h}(1)=\alpha} \Phi(h)$

This is a constrained maximization problem

• Find h, γ such that

$$\Psi_\gamma(h) := \Phi(h) + \gamma\,\bar{h}(1)$$

is maximized at h, $\bar{h}(1) = \alpha$.

$\Phi(h) = \langle \log \Psi, \bar{h} \rangle + \dots$

in this case $\Psi = 1$,

so $\Phi_\gamma(h) = \langle \log \Psi \cdot e^{\gamma e_1}, \bar{h} \rangle + \dots$

so Φ_γ corresponds to the spin system

$$\mathbb{P}[\sigma] = \prod_{u \in V} e^{\gamma \sigma} \prod_{u \sim v} I(\sigma_u \sigma_v \neq 1)$$

$$= \lambda^{\Sigma \sigma_u} I(\sigma \in IS) \quad \text{Hardcore model.}$$

where $\lambda = e^\gamma$

Solve

$$m = \frac{\lambda (1-m)^{d-1}}{1 + \lambda (1-m)^{d-1}} \quad \text{one fixed point}$$

Density

$$\alpha_\lambda = \frac{\lambda (1-m_\lambda)^d}{1 + \lambda (1-m_\lambda)^d}$$

For d large, if $m_\lambda \approx \frac{s \log d}{d}$,

$$\frac{s \log d}{d} \approx \lambda \left(1 - \frac{s \log d}{d}\right)^{d-1}$$

$$\approx \lambda d^{-s} \quad \text{so} \quad \lambda \approx d^{s-1}.$$

$$\Phi_\alpha = \log\left(1 + \lambda (1-m)^d\right) - \frac{d}{2}\log\left(1 - m^2\right)$$

And

$$\Phi_\alpha = \Psi_\lambda - \alpha \log \lambda$$

$$= \log\left(1 + \lambda(1-m)^d\right) - \alpha \log \lambda - \frac{d}{2}\log\left(1-m^2\right)$$

$$\approx \alpha\left(1 - \log \lambda\right) + \frac{d}{2} m^2$$

$$\approx \alpha\left(1 - (s-1)\log d\right) + \frac{s^2 \log^2 d}{d}$$

$$\approx \frac{s(1-s)\log^2 d}{d} + \frac{s^2 \log^2 d}{2d}$$

$$\approx s(2-s)\frac{\log^2 d}{2d}$$

So positive when $s < 2$.

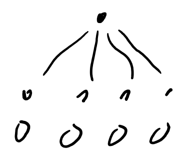
There must be some α_1 such that

$\Phi_{\alpha_1} = 0$, $\Phi_\alpha < 0$ for $\alpha > \alpha_1$.

Is α_1 size of max I.S..

- Suppose there exists an I.S. with size α_1
- Its local weak limit will be given by hardcore model with λ_α fugacity, B.P. fixed point $m_\alpha = BP_{\lambda_\alpha}[m_\alpha]$

  - $\mathbb{P}_{m,\lambda}[\sigma_v = 0, \sigma_{\partial v} \equiv 0] = m(0)^d \cdot \dfrac{1}{1+\lambda}$
    $> \delta.$



  - Constant fraction of sites which can be increased.

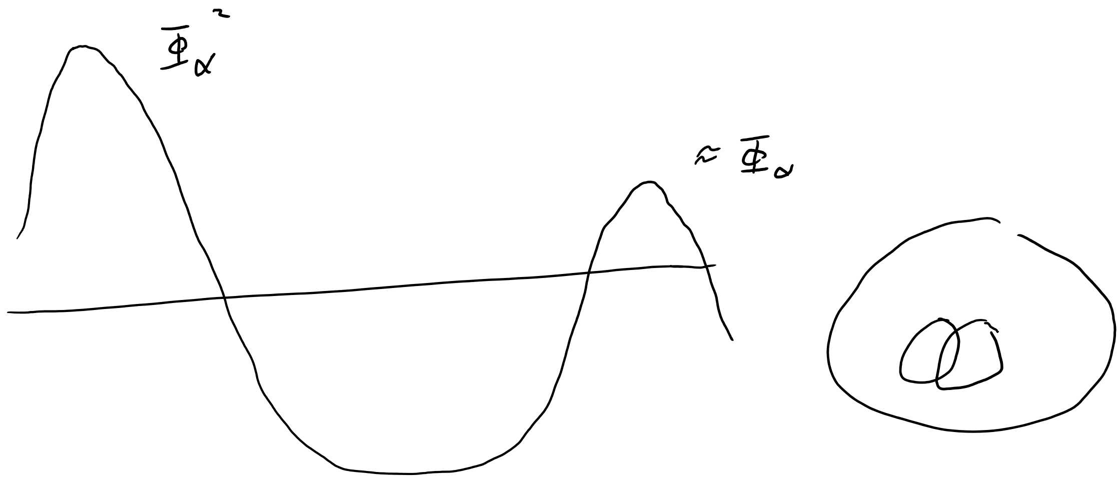
- $\mathbb{P}[\text{ max IS} = \alpha_1 n] \leq \mathbb{P}[\text{max IS} \geq (\alpha_1 + \delta')n]$
  $+ \mathbb{P}[\exists \text{ IS size} \geq \alpha_1 n \text{ without } \delta' n \text{ } 0\text{-neighbourhoods}]$
  $\leq e^{-cn}.$

Clustering

For $s > 1$, $\mathbb{E}\,\#$ pairs of IS size $\alpha = \dfrac{s \log d}{d}$, intersection $\gamma = \alpha$

$\sim \Phi^2$

intersection $\gamma = \alpha$



$\mathbb{E}$ # Pairs IS size αn, overlap γn

$$\approx \binom{n}{\alpha n}\binom{n(1-\alpha)}{(\alpha-\gamma)n}\binom{\alpha n}{\gamma n} \cdot \mathbb{P}[\text{ Both IS }]$$

$$\approx \exp\left(H(\alpha)\cdot n + H\left(\frac{\alpha-\gamma}{1-\alpha}\right) n(1-\alpha) \right)$$

$$\approx d^{\alpha n + (\alpha-\gamma)n} \cdot \prod_{i=1}^{d\gamma n}\left(1 - \frac{d(2\alpha-\gamma)n - i}{dn}\right)$$
$$\cdot \left(\prod_{i=1}^{d(\alpha-\gamma)n}\left(1 - \frac{d(\alpha-\gamma)n - i}{dn}\right)\right)^2$$

$$\approx d^{(2\alpha-\gamma)n} \cdot \exp\Big[-d\gamma n\left(2\alpha - \tfrac{3}{2}\gamma\right)$$
$$- dn(\alpha-\gamma)^2\Big]$$

setting $\gamma = t\alpha$,

$$\approx d^{(2-t)\alpha n} \exp\left[-d\alpha^2 n\left(\left(2t - \tfrac{3}{2}t^2\right) + (1-t)^2\right)\right]$$

$$= d^{(2-t)\alpha n} \exp\left[-d\alpha^2 n\left(1 - t^2/2\right)\right]$$

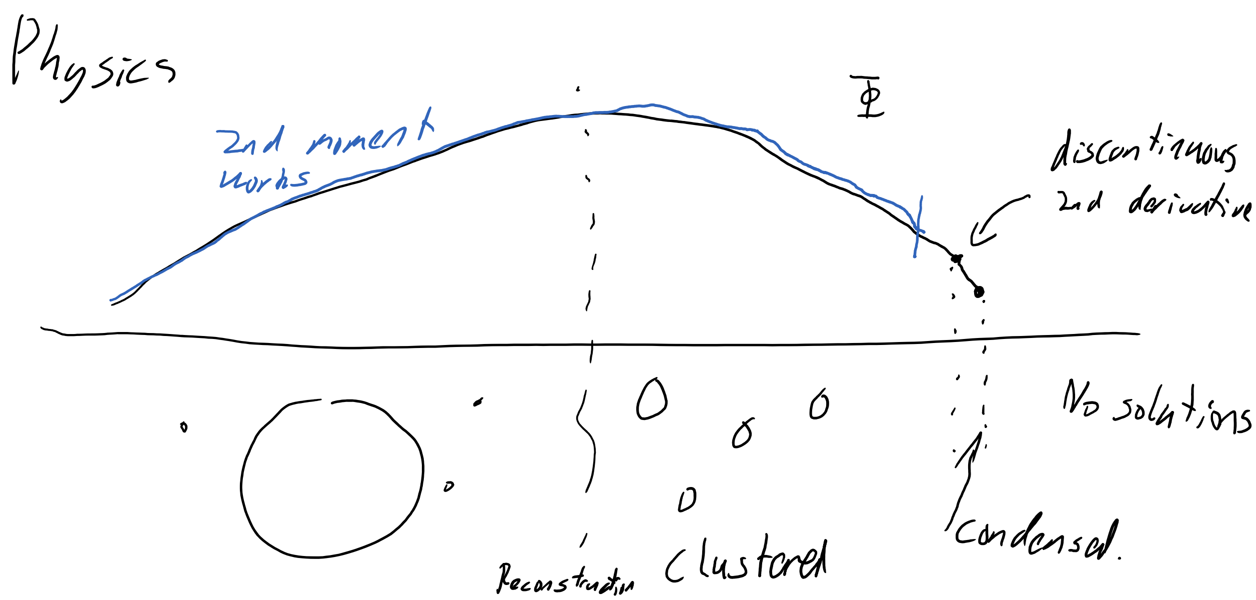
$$\approx d^{(2-t)\alpha n} \exp[-d\alpha^2 n(1-t^2/2)]$$

$$\approx d^{\alpha n(2-t-s(1-t^2/2))}$$

- #IS size α – 2nd moment method works for all $s < 2 - o_d(1)$.

- <u>Clustering:</u>
  - Solutions split into clusters.

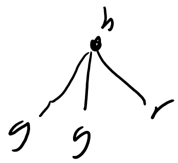
Physics



- What do they look like.
  - run BP, find fixed point

Ex: q-Colouring $\quad d \approx s q \log q \quad 1 < s < 2.$

Free measure is very frozen



Clusters are BP fixed pts.

- Spin system on directed edges.
  but with continuous spins

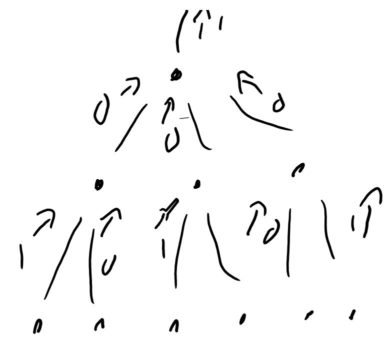
• Hardcore model

$$m_{u \to v} = \frac{\lambda \prod (1 - m_{u_i \to u})}{1 + \lambda \prod (1 - m_{u_i \to u})}$$

as $\lambda \to \infty$

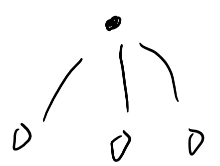
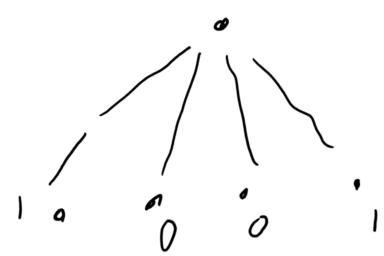
$$BP[\{m_{u_i \to u}\}] = \prod_i I(m_{u_i \to u} < 1).$$

This gives a discrete spin system



- It is rigid in the sense of a boundary condition has at most one continuation

- Getting back on IS.

 a 1  a 0

free variables
choose one as a 1

- <u>Weights:</u>

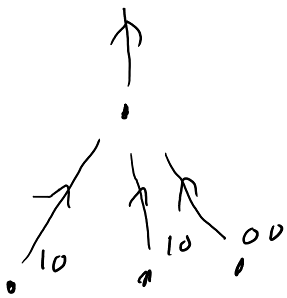
$$\mathbb{P}[r] = \frac{1}{Z}\,\theta^{\#1 + \frac{1}{2}\#?}\; I(r \in WP)$$

$$= \frac{1}{Z}\,\theta^{\#\{v:\ \text{outgoing } 1\}} \cdot \theta^{-\#\{e:\ r_e = (1,1)\}}\; I(r \in WP)$$

$$= \frac{1}{Z}\,\theta^{\#\{v:\ \text{outgoing } 1\}} \cdot \theta^{-\#\{e:\ r_e=(1,1)\}}\, I(r \in wp)$$

Possibilities

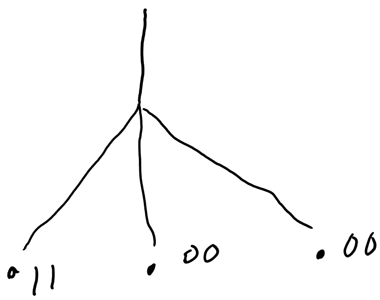
a)



$\underbrace{10,\ 10, \dots, 10}_{\geq 2}, 00, \dots 00$

root message
01 or 00.

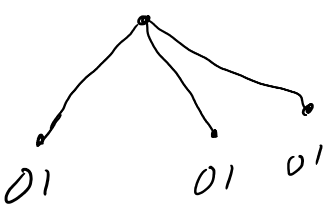
b)



root message 00

11, 00, ... 00

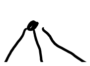
c)



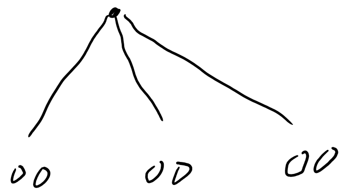
root message 10

01, ..., 01

d)



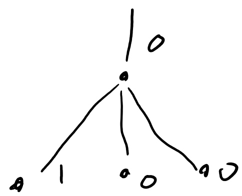
d)

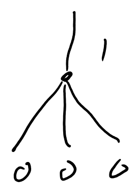


root message 11.

Find fixed point of the form

$q_{10} = q_{11} \qquad q_{01} = q_{00}$





$$q = \frac{\theta(1-q)^{d-1}}{1+(\theta-1)(1-q)^{d-1}} \,.$$

- Step 1: Solve for q_θ.

$$\alpha = \frac{\theta\,(1-q)^d}{1+(\theta-1)(1-q)^d}$$

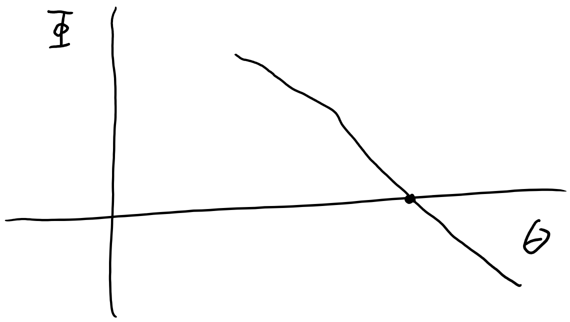
- Step 2:

$$\Phi_\theta = \Phi_{vert} - \frac{d}{2}\Phi_{edge} - \alpha \log\theta$$

$$= \log\left(1+(\theta-1)(1-q)^{d-1}\right)$$

$$-\frac{d}{2}\log\left(1-(1-\theta^{-1})q^2\right) - \alpha\log\theta$$

$\Phi \Big|$



- Set θ_0 as $\Phi_{\theta^*} = 0$.

- Size of MAX IS α_{θ^*}

  - This calculation gives the first moment.

Second Moment

- Pairs of I.S.

q_{11}

$$= \frac{\theta^2 q_{00}^{d-1}}{\theta^2 q_{00}^{d-1} + \underbrace{\theta\left((q_{00}+q_{10})^{d-1} - q_{00}^{d-1}\right)}_{q_{10}} + \theta\left((q_{00}+q_{01})^{d-1} - q_{00}^{d-1}\right) \leftarrow q_{01} + \underbrace{\left(1 - (q_{00}+q_{10})^{d-1} - (q_{00}+q_{01})^{d-1} + q_{00}^{d-1}\right)}_{q_{00}}}$$

- Two solutions 1) $\theta = \theta^*$

$$q(x,y) = q_*(x)\, q_*(y)$$

2) $\theta = \sqrt{\theta^*}$, $q(x,y) = q(x) \cdot \mathbb{1}(x=y)$.

— Lots of work to show that these are the only solutions.

Free energy away from Max IS.

A B.P. fixed point $\{r_{u \to v}\}_{(u,v) \in E}$

represents a cluster.

$$w(r) = \prod_{u \in V} \left(1 + \lambda \prod_{u' \in \partial u} 1 - r_{u' \to u}\right) \cdot \prod_{u \sim v} \left(1 - r_{u \to v} r_{v \to u}\right)$$

We want

$$Z = \sum w(r)$$

- A spin system with cts. values.

<u>Survey Propagation</u>: $r \in P(\{0,1\})$

$$\cong [0,1]$$

- S measure on r's

$$S \in P(P(\{0,1\})) \cong P([0,1])$$

$SP[\{S_{u' \to u}\}](A)$

$\frac{1}{\cdot} \left(\cdot \lambda \prod (1 - r_{u' \to u}) \quad \cdot \right)$

$$= \frac{1}{Z} \int I\left(\frac{\lambda \prod (1 - r_{u' \to u})}{1 + \lambda \prod (1 - r_{u' \to u})} \in A \right)$$

$$\cdot \left(1 + \lambda \prod (1 - r_{u' \to u}) \right) \prod ds_{u' \to u}(r_{u' \to u})$$

Solve for fixed point

$$SP[s] = s.$$

- Replica Symmetric Solution

$$\Phi_\lambda = \log \left(\int \left(1 + \lambda \prod (1 - r_j)\right) \prod_j s(dr_j) \right)$$

$$- \frac{d}{2} \log \left(\iint 1 - (1 - \lambda^{-1}) r r' \right) s(dr) s(dr'))$$

- Energy of a typical solution

$$\Psi_\lambda = \int \log\left(1 + \lambda \prod (1 - r_j)\right) \prod s(dr_j)$$

$$- \frac{d}{2} \iint \log \left(1 + (1 - \lambda^{-1}) r r'\right) s(dr) s(dr')$$

At some point $\Phi_\lambda > 0$ but

$$\Phi_\lambda < \Psi_\lambda.$$

Let $Z_\beta = \sum_r w(r)^\beta$.

$$SP_\beta[s](A) = \frac{1}{Z_\beta} \int I\left(\frac{\lambda \prod (1 - r_i)}{1 + \lambda \prod (1 - r_i)} \in A \right) \left(1 + \lambda \prod (1 - r_i)\right)^\beta$$

$\Pi \; s_i(dr_i)$