Moments

Tuesday, April 10, 2018 11:50 AM

Suppose we want to calculate the expected value of the partition function, $\mathbb{E} Z$ on a random d-regular graph

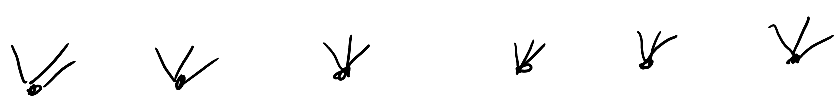
$$\mathbb{E} Z = \mathbb{E} \sum_{x \in \mathcal{X}^n} \prod \bar{\psi}(x_u) \prod_{u \sim v} \underline{\psi}(x_u, x_v)$$

Let $\underline{h} : \mathcal{X}^2 \times \mathcal{X}^V$, $\bar{h} : \mathcal{X} \times \mathcal{X}^V$

$$\underline{h}(y, y') = \#\{(u,v) \in E : \sigma_u = y, \sigma_v = y'\} / \binom{n}{2}$$

$$\bar{h}(y) = \#\{u \in V : \sigma_u = y\}$$

Expected # of configurations with $\underline{h} = h$.



Assign spins to vertices $\binom{n}{n\bar{h}} = \dfrac{n!}{\prod_{x \in \mathcal{X}} (n h(x))!}$.

Assign spins to half edges

$$\prod_x \binom{ndh(x)}{ndh(x, \cdot)} = \prod_x \frac{(ndh(x))!}{\prod_{x'} (ndh(x, x'))!}.$$

Prob of matching:

$$\prod_{x} M_{nd\,h(x,x)} \prod_{x \neq x'} \sqrt{M_{nd\,h(x,x')}} \Big/ M_{nd}.$$

Apply Stirling's Formula:

$\mathbb{E}$ # h-configs

$$= n^{O(1)} \exp\left(n\left[\frac{d}{2} H(h) - (d-1) H(\bar{h})\right]\right)$$

where $H(p) = -\sum p_x \log p_x$.

Weight:

$$\exp\left(n\left[\langle \log \bar{\psi}, \bar{h} \rangle + \frac{d}{2} \langle \log \psi, h \rangle\right]\right).$$

$$\Phi(h) = \frac{d}{2} H(h) - (d-1) H(\bar{h}) + \langle \log \bar{\psi}, \bar{h} \rangle + \frac{d}{2} \langle \log \psi, h \rangle$$

$$\mathbb{E} Z = \sum_{h} \mathbb{E} Z_h = \sum_{h} n^{O(1)} \exp(n \Phi(h))$$

So $\lim \frac{1}{n} \log \mathbb{E} Z = \max_{h} \Phi(h)$.

Theorem: If h is the global maximizer

of $\Phi(h)$ then there exists m such that

- $m = BP[m]$
- $h(x,x') = \psi(x,x')\, m(x)\, m(x') / Z_m$

$*$ Can be done via calculus, see Theorem 1.18 of Dembo, Montanari, Sun

<u>Alternative Proof:</u>

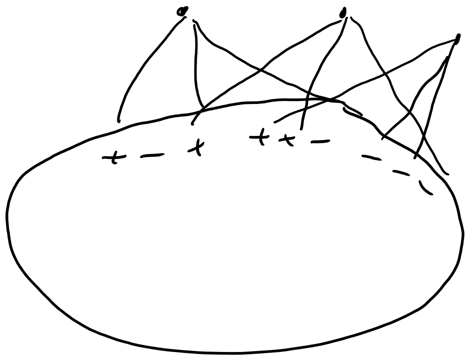
- $Z = M_{dn}^{-1} \sum_{(G,x)} w(G,x)$

  where $w(G,x) = \prod_u \bar{\psi}(x_u) \prod_{(u,v)\in E_G} \psi(x_u, x_v)$.
- Pick (G,σ) w.p. proportional to $w(G,x)$
- For $k \sim \delta n$ select k vertices u.a.r., remove any that are adjacent

$A = \{u_1, \dots, u_k\}$

- Remove edges in A, forget spins in A.

- resample (G', σ') given $(G(A^c), \sigma_{A^c})$.



Then $((G,\sigma,A), (G',\sigma',A))$ are exchangeable.

- Let $h_A(\sigma, x, x') = \frac{1}{d|A|} \# \{(u,u') : u \in A, \sigma_u = x, \sigma_{u'} = x'\}$.

w.h.p. $\bar{h}_A(\sigma) \approx \bar{h}(\sigma)$ by sampling.

So $\bar{h}_A(\sigma) \approx \bar{h}_A(\sigma')$.

What is the law of $h_A(\sigma')$?

k d-stars roots $B = (v_1, \ldots, v_k)$

with weights

$\prod \psi(x_{v_i}) \prod \psi(x \ldots x \ldots) \ldots D^*$

with weights

$$\prod_i \Psi(x_{u_i}) \prod_{u' \in \partial u_i} \Psi(x_{u_i}, x_{u'}). \sim \mathbb{P}^*$$

$$(*) = \mathbb{P}[\sigma'_{A \cup \partial A} \in \cdot \mid G_{A^c}, \sigma_{A^c}]$$

$$= \mathbb{P}^*[\sigma^*_{B \cup \partial B} \in \cdot \mid \frac{1}{dk} \sum_i \sum_{v \in \partial v_i} \delta_{\sigma^*_v} = \bar{h}_{\partial A}]$$

Let $\mathbb{P}^*_m \sim \prod_i \Psi(x_{v_i}) \prod_{v' \in \partial v_i} \Psi(x_{v_i}, x_{v'}) \, m(x_{v'})$

i.e. external field.

and m^* chosen so that for $v \in \partial A$.

$$\mathbb{P}^*_{m^*}[\sigma_v = x] = \bar{h}_{A^c}(x).$$

$$(*) = \mathbb{P}^*_{m^*}[\sigma_{B \cup \partial B} \mid \overbrace{\frac{1}{dk} \sum_i \sum_{v \in \partial v_i} \delta_{\sigma_v} = \bar{h}_{\partial A}}^{=: D}]$$

Now $\mathbb{P}^*_{m^*}[D] \asymp k^{-\frac{1}{2}(|\mathcal{X}|-1)}$ by local CLT.

Let $h_{m^*}(x, x') = \mathbb{P}^*_{m^*}[\sigma_{v_i} = x, \sigma_{v'} = x']$

for $v_i \in B$, $v' \in \partial B$.

$$\mathbb{P}^*_{m^*}[\,|\frac{1}{dk} \sum_i \sum_{v \in \partial v_i} \delta_{(\sigma_{v_i}, \sigma_v)} - h_{m^*}| > \varepsilon]$$

$$\lesssim e^{-c \varepsilon^2 k}$$

$\mathbb{P}^*_{m^*}[\,|\frac{1}{dk} \sum \sum$

$\approx e$

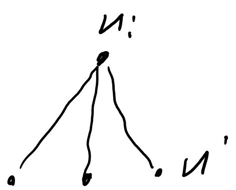
$$\mathbb{P}^*_{m^*}\Big[\Big|\frac{1}{dk}\sum_i\sum_{v\in\partial u_i}\delta_{(\sigma_{u_i},\sigma_v)} - h_{m^*}\Big| > \varepsilon \,\Big|\, D\Big]$$
$$\le e^{-c\varepsilon^2 k^2} k^{\frac{1}{2}(|\mathcal{X}|-1)}$$

So

$$\mathbb{P}\Big[\Big|\frac{1}{dk}\sum_i\sum_{u'\in\partial u_i}\delta_{(\sigma_{u_i},\sigma_{u'})} - h_{m^*}\Big| > \varepsilon \,\Big|\, G_{A^c},\sigma_{A^c}\Big]$$
$$\le e^{-c\varepsilon^2 k^2} k^{\frac{1}{2}(|\mathcal{X}|-1)}$$

Now w.h.p.

$$h_{m^*} \approx \frac{1}{dk}\sum_i\sum_{u'\in\partial u_i}\delta_{(\sigma_{u_i},\sigma_{u'})} \approx \frac{1}{dk}\sum_i\sum_{u'\in\partial u_i}\delta_{(\sigma_{u'},\sigma_{u_i})}$$



$$h_{m^*}(x,x') \propto \psi(x,x')\, m_{u_i\to u'}(x)\, m_{u'\to u_i}(x')$$
$$= \psi(x,x')\, BP(m^*)(x)\, m^*(x')$$

so $m^* = BP(m^*)$.

A (unique) such m^* exists. Why?

Let $X_i = \frac{1}{d}\sum_{v\in\partial u_i}\delta_{\sigma_v}$ random vector indexed by $\mathcal{X}$.

$X_i \sim \nu$

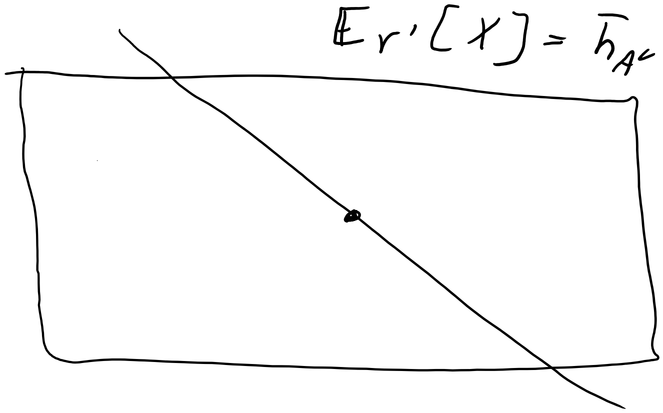
Consider the large deviation event

$$\mathbb{P}^*\Big[\frac{1}{k}\sum X_i \approx \bar{h}_{A^c}\Big] = \exp\big(-H(\nu^*|\nu)k + o(k)\big)$$

where $\nu^* = \arg\min_{\nu' : \mathbb{E}_{\nu'}[X_i] = \bar{h}_{A^c}} H(\nu'|\nu)$

Minimize $H(r'|r)$ on hyperplane

$\mathbb{E}_{r'}[X] = \bar{h}_{A^c}$



$H(r'|r)$ is strictly convex

Write $\varphi(\lambda) = \mathbb{E}\, e^{\lambda X}$, $\quad \dfrac{dr_\lambda}{dr} = e^{\lambda \cdot x}/\varphi(\lambda)$

Lagrange multiplier λ $\quad r^*$ minimizer of

$$H(r'|r) - \langle \lambda, E_{r'}[X_i] \rangle$$

$$= \sum_i r_i' \log \frac{r_i'}{r_i} - \sum_i \langle \lambda, x_i \rangle r_i'$$

$$= \sum_i r_i' \log \frac{r_i'}{r_i e^{\langle \lambda, x_i \rangle}/\varphi(\lambda)} - \log \varphi(\lambda)$$

$$= H(r' | r_\lambda) - \log(\varphi(\lambda)).$$

So $\mathbb{E}_{r_\lambda}[X] = \hat{h}_{A^c}$

$m(x) \propto e^{\lambda x}/d$

then

$$\frac{\partial \mathbb{P}^*_{m^*}[\sigma_{v_i \cup \partial v_i} \in \cdot\,]}{\partial \mathbb{P}^*[\sigma_{v_i \cup \partial v_i} \in \cdot\,]} \propto \prod_{v' \in \partial v_i} m^*(v')$$

$$\propto e^{\langle \frac{1}{d} \sum_{v'} \delta_{\sigma_{v'}}, \lambda \rangle}$$

$$= e^{\langle X, \lambda \rangle}$$

so $X \sim \nu_\lambda$.