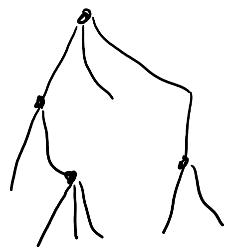
Local convergence and Branching Processes

Tuesday, February 6, 2018 3:25 PM

Galton – Watson Branching Process

Offspring distribution X, $\mathbb{E}X = \mu$.



Each vertex has independent offspring with law X.

Z_n = # in level n,

$$Z_{n+1} = \sum_{i=1}^{Z_n} X_{i,n}$$

$$\mathbb{E} Z_{n+1} = \mu \, \mathbb{E} Z_n = \mu^{n+1}.$$

If $\mu < 1$ then $\mathbb{E} Z_n \to 0$.

Does it ever die out

- $\mathbb{P}[\text{Survive}] = p > 0$ iff $\mu > 1$, where p solves $p = \mathbb{E}\, p^X$.

Local neighbourhood

$\mathrm{Bin}(n-1, d/n)$

$\mathrm{Bin}(n-3, \frac{d}{n})$

$\mathrm{Bin}(n-8, d/n)$

. . .

Locally close to Branching Process

with Poisson(d) offspring dist.

- <u>Local Weak Limit</u>

Metric on rooted graphs

We say $(G,v) \simeq (G',v')$

if $\exists$ graph homomorphism

$\varphi : V \to V'$ bijection, $\varphi(v) = v'$

$(w,v) \in E$ iff $(\varphi(w), \varphi(v)) \in E'$.

I will write $(G,v) \simeq_r$ if $B_r(G,v) \cong B_r(G',v')$.

$$d\big((G,v),(G,v')\big) = 2^{-\max\{r : (G,v) \simeq_r (G,v)\}}$$

Let I_n uniformly chosen vertex of G_n

$(G_n, I_n) \xrightarrow{lwc} (G, I)$ weak convergence w.r.t. local metric.

- If $\mathbb{P}[(G_n, I_n) \simeq_r (g,v)] \to \mathbb{P}[(G,I) \simeq_r (g,v)]$ $\forall (g,v)$.

- If G bounded degree then space is relatively compact. Enough that $|B_r(G_n, I_n)|$ is tight.

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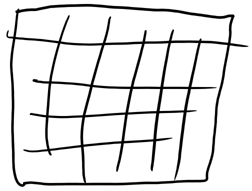
Ex: $n \times n$ grid

 $\xrightarrow{\text{L.W.C}}$ $\mathbb{Z}^2$

$$G(n, \tfrac{d}{n}) \xrightarrow{\text{lwc}} GWC(\text{Pois}(d))$$

What about random d-regular?

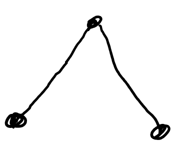
Limit is infinite d-regular tree

Ex  depth n binary tree $\longrightarrow$ Canopy tree

What about random graph with degree sequence with law Y.

$q_k = \mathbb{P}[Y = k]$,

- Use configuration model:

 Root distribution Y

$\mathbb{P}[\text{child degree } k]$

$$= \frac{\#\text{vertices degree } k \cdot k}{\#\text{edges}}$$

$\#$ edges

$$\approx \frac{p_k \cdot n \cdot k}{\mathbb{E} Y \cdot n} = \frac{p_k \cdot k}{\mathbb{E} Y}.$$ sized biased distribution Y^*.

Limit is a modified GWBP with root distribution $Y^* - 1$.

<u>Fact:</u> If Y is Poisson then, $Y^* - 1 \overset{d}{=} Y$.

- This is for the configuration model, reveal multiple local neighbourhoods to get back to the simple graph.