Infection models

Tuesday, February 27, 2018 10:07 PM

- SIS and SIR

- Susceptible, infected, suceptible/removed

  - Each infected infects a susceptible at rate λ (or each neighbour w.p. p).

  - A susceptible becomes suceptible/removed at rate 1.

- Well mixed population (complete graph).
  - infection rate λ/n

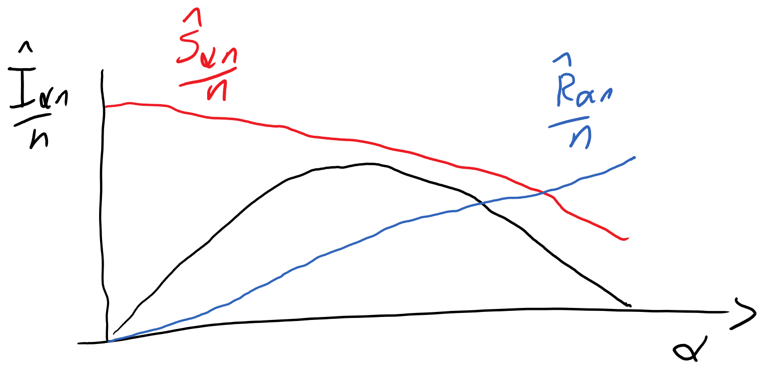
- What happens?

- Write S_t, I_t, R_t # at time t.

  $\hat{S}_n, \hat{I}_n, \hat{R}_n$ - # after n-th change

  $\mathbb{P}[\hat{I}_n = \hat{I}_n + 1 \mid \hat{S}_n] = \dfrac{\lambda \hat{S}_n}{\lambda \hat{S}_n + n}$

  $\geq \frac{1}{2}$ when $\hat{S}_n \geq \frac{1}{\lambda} \cdot n$



$$\hat{I}_k + 2\hat{R}_k = k$$

$$\hat{S}_k + \hat{I}_k + \hat{R}_k = n, \qquad \hat{S}_k = n - \hat{I}_k - (k - I_k)/2$$

$$\hat{S}_k = n + \frac{k}{2} - \frac{3}{2}\hat{I}_k$$

$$\mathbb{E}\left(\hat{I}_{k+1} - \hat{I}_k \mid \hat{I}_k\right] = \frac{\lambda \hat{S}_k}{\lambda \hat{S}_k + n} - \frac{n}{\lambda \hat{S}_k + n}$$

$$f(\alpha) = \hat{I}_{\alpha n}/n, \qquad \hat{S}_{\alpha n}/n \approx \left(n + \frac{\alpha}{2}\right) - \frac{3}{2} f(\alpha)$$

$$f'(\alpha) = \frac{\lambda\left(1 + \frac{\alpha}{2}\right) - \frac{3}{2} f(\alpha) - 1}{\lambda\left(1 + \frac{\alpha}{2}\right) - \frac{3}{2} f(\alpha) + 1}$$

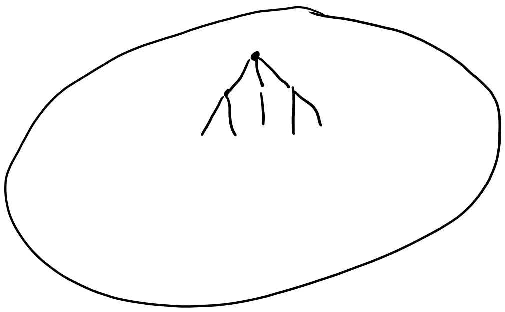
Total time $O(\log n)$.

<u>On a random graph:</u>

In fect each neighbour w.p. p.

- Bond percolation on G

- keep each edge w.p. p
- remove " " " $1-p$.



- Set of infected vertices is the same distribution as bond percolation cluster.

  - For edge u,v - present if 1st vertex infected infects other vertex.

- In a random graph with degree distribution $d_v \sim \nu$ does infection infect a linear fraction?

  - Consider the B.P., offspring dist ν^*.
    - after percolation $\mathbb{E}\nu_p^* = p\,\mathbb{E}d_v^*$

- Yes iff $\dfrac{\mathbb{E}\,D(D-1)p}{\mathbb{E}\,D} > 1$.

- What about infection rate λ?

Let $S_{n,t}$, $I_{n,t}$ be the

number of degree k vertices at time t.

- Let $A_t = \sum k I_{k,t}$, $B_t = \sum k S_{k,t}$
- At rate $\lambda A_t \cdot \frac{k S_{k,t}}{A_t + B_t - 1}$, $S_{k,t} \to S_{k,t} - 1$
  $I_{k-1,t} \to I_{k-1,t} + 1$.
- At rate $\sum I_{k,t}$, $I_{k,t}$ reduced by 1.
  - sample k edges from $A_t + B_t$ edges and remove them, E.G.
    $I_{j,t} \to I_{j,t} - 1$ + $I_{j-1,t} \to I_{j-1,t} + 1$.

$$\frac{1}{\Delta} \mathbb{E}\left[A_{t+\Delta} - A_t \mid \mathcal{F}_t\right] = \frac{\lambda A_t \sum k(k-1) S_{k,t}}{A_t + B_t - 1} - \lambda A_t$$

$$- \sum I_{k,t} \cdot k$$

$$- \sum I_{k,t} \cdot k \cdot \frac{A_t}{A_t + B_t}$$

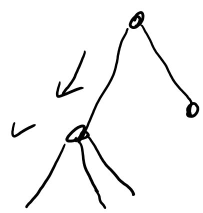
For small time when $B_t \approx n \mathbb{E} d_v$

$$\frac{1}{\Delta} \mathbb{E}\left[A_{t+\Delta} - A_t \mid \mathcal{F}_t\right] \approx A_t \left[\lambda \frac{\mathbb{E} d_v (d_v - 1)}{\mathbb{E} d_v} - \lambda - 1 \right]$$

Positive growth if $\lambda > \dfrac{1}{\mathbb{E} d_v^* - 1}$

Alternative Formulation

On a tree



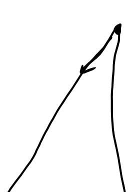
If v has d children each child has probability $\dfrac{\lambda}{1+\lambda}$ of being infected before v is removed.

Branching rate $\dfrac{\lambda}{1+\lambda} \mathbb{E} d_v^* > 1$

$$\iff \lambda > \frac{1}{\mathbb{E} d_v^* - 1}.$$

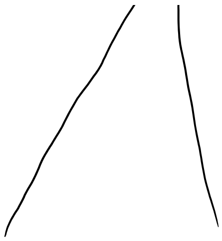
Contact Process / SIS

- Infected become susceptible again.
  - Stochasticalled dominates SIR.
- Start with infinite d-regular tree.



Upper bound

Branching random walk $\exists \epsilon$



Branching random walk $\mathcal{X}_t$

- particles branch to neighbouring site at rate λ
- die at rate 1.

$$\frac{1}{\Delta}\mathbb{E}[A_{t+\Delta t} - A_t \mid \mathcal{F}_t] = -A_t + d\lambda A_t$$
$$= (d\lambda - 1) A_t$$

Critical values is $\lambda = 1/d$.

Let $W_t = \sum_{x \in \mathcal{X}_t} \alpha^{-d(x,\rho)}$

$$\frac{1}{\Delta}\mathbb{E}[W_{\Delta+t} - W_t \mid \mathcal{F}_t] \leq W_t\left[-1 + \lambda(\alpha + (d-1)\alpha^{-1})\right]$$

$\mathbb{E}W_t = e^{\xi t}$ where $\xi = \left[-1 + \lambda(\alpha + (d-1)\alpha^{-1})\right]$

Set $\alpha = \sqrt{d}$

$$-1 + 2\sqrt{d}\,\lambda$$

So $\mathbb{E}W_t \to 0$ if $\lambda < \frac{1}{2\sqrt{d}}$

For $\frac{1}{d} < \lambda < \frac{1}{2\sqrt{d}}$, $\mathbb{E}A_t \to \infty$, $\mathbb{E}W_t \to 0$.

Second Moment Method

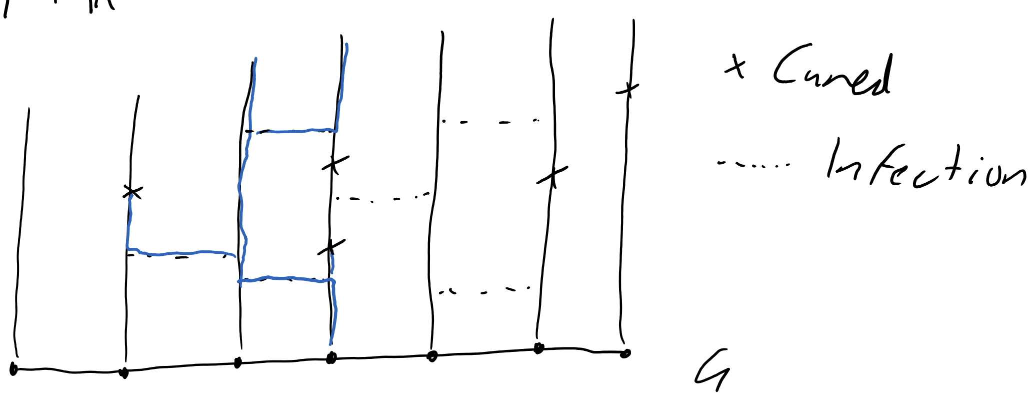
$$\mathbb{E}W_t^2 = \mathbb{E}W_t + \mathbb{E}\sum_{x \in \mathcal{X}_t}\sum_{x' \in \mathcal{X}_t \setminus x} \alpha^{-d(x,\rho) - d(x',\rho)}$$

$\mathbb{E}W_t \quad e^{t}$

$$= \mathbb{E} W_t + \int_0^t e^{2\gamma(t-s)} e^{\gamma t} \cdot \lambda \, ds$$

$$= O(e^{2\gamma t}) = O(\mathbb{E} W_t^2)$$

Contact Process is percolation on $G \times \mathbb{R}$



(**) $\mathbb{P}[v \text{ infected}]$ Increasing in λ.

<u>Theorem:</u> Pemantle '92

- Two thresholds

(a) $\lambda_c \asymp 1/d$ Infection survives

(b) $\lambda_r \asymp c/\sqrt{d}$ Root infected infinitely often.

Question: Which determines the behaviour on random regular graph?

Proof of Theorem:

- Upper bound by BRW.

- If $u \in T$, $d(u,\rho) = k$ with parent v, let X_u be the event
  - infection on (u,v) for $t \in [k-1, k]$
  - No removal at u for $t \in [k-1, k+1]$

$\mathbb{P}[X_u] = (1 - e^{-\lambda}) \cdot e^{-2} \approx \lambda e^{-2}$ small λ.

If $\rho, u_1, \dots, u_k = u$ is path from ρ to u,
if $X_{u_1}, \dots, X_{u_n}$ hold then u infected at time k.

Infinite component of $X_u = 1 \Rightarrow$ Infection survives

$d \lambda e^{-2} \geq 1$

- Check infection up the tree

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Claim: (a) If $\lambda < \lambda_c$ then $\mathbb{E} \sum_u X_t(u) \le e^{-cb}$

(b) If $\lambda > \lambda_c$ then $\mathbb{E} \sum_u X_t(u) \ge e^{cb}$.

Proof of (b): On d-ary tree.

Let $Z_t = \sum_u X_t(u)$. Let $A = \{\text{Survives}\}$. $\mathbb{P}[A] = p$.

- $\mathbb{P}[Z_{k+1} = 0 \mid Z_k \le L] \ge e^{-cL}$.

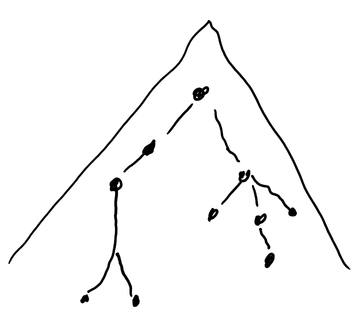
- $\sum_k \mathbb{P}[Z_k \le L] \le e^{cL} \sum_k \mathbb{P}[Z_k \le L, Z_{k+1} = 0]$

$$\le e^{cL}$$

so $\mathbb{P}[Z_k \le L] \to 0$.

$\mathbb{E} Z_k \ge (p - \mathbb{P}[Z_k \le L]) \cdot L \to L$

so $\mathbb{E} Z_k \to \infty$.



Let $I \subset T_d$, $|I| < \infty$ and

$F(I) = \{v : d(v, I) = 1, T_v \cap I = \emptyset\}$.

- u is an I-child of v if $v \in I$ is the first element of I on path from u to ρ. So

$$|F(I)| \leq \sum_{v \in I} (d-1) - \sum_{u \in I} I(u \text{ I-child of } v)$$

$$= (d-2)|I|.$$

Choose T so that $\mathbb{E}\, Z_{T-1} \gg 1$.

After time T set of expected size $R = \lambda e^{-2} \mathbb{E}\, Z_{T-1}$ infected with distinct subtrees.

$$\mathbb{E}\ Z_{kT} \geq R^k.$$

<u>On the graph:</u>

<u>Claim</u>: For all $d \geq 3$, $\exists$ $\delta > 0$ s.t. $\forall r$ $\exists \varepsilon$ s.t.

on a random regular graph, w.h.p.

$\forall I \subseteq V$, $|I| \leq \varepsilon n$,

$A = \#\{v : \exists u \in I, d(u,v) = r\}$, $|A| \geq \delta d(d-1)^{r-1} |I|$.

<u>Theorem</u>: Lalley - Su '15.

On random d-regular, let T = time infection ends.

- $\lambda < \lambda_c$ $\mathbb{P}[T > C \log n \mid X_0 = I_v] \to 0$.
- $\lambda > \lambda_c$ $\mathbb{P}[T > e^{cn} \mid |X_0| = 1] \geq \delta > 0$.

<u>Proof</u>: (a) $\lambda < \lambda_c$, $\mathbb{E}\ Z_k \leq \frac{1}{2}$.

Percolation is subadditive so

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$$\mathbb{E}[Z_{t+k} \mid X_t = B] \le \sum_{u \in B} \mathbb{E}[Z_{t+k} \mid X_t = u]$$

$$\le \tfrac{1}{2}|B| = \tfrac{1}{2}|X_t|.$$

So $\mathbb{E}\, Z_{mk} \le 2^{-m} n.$

(b) $\lambda > \lambda_c$: We can choose k s.t.

$$\mathbb{E}_T[Z_k \mid X_0 = I_\rho] \ge 2k^3.$$

$$\mathbb{E}_T\Big[\sum_{u : d(u,\rho) \ge k^2} X_t(u) \mid X_0 = I_\rho\Big] = o(1)$$

so $\exists\ 0 \le r \le k^2$,

$$\mathbb{E}_T\Big[\sum_{u : d(u,\rho) = r} X_t(u) \mid X_0 = I_\rho\Big] \ge k \ge \frac{2}{\delta}.$$

So $\mathbb{P}[X_t(u) = 1 \mid X_0 = I_\rho] \ge \frac{2}{\delta}\left(d(d-1)^{r-1}\right)^{-1}$.

for $d(\rho, u) = r$

With A as in the claim,

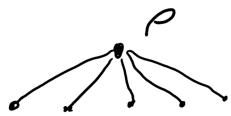
$$\mathbb{E}\Big(\sum_{u \in A(I)} X_{t+k} \mid X_t = I\Big) \ge |A(I)| \frac{2}{\delta}\left(d(d-1)^{r-1}\right)^{-1}$$

$$\ge 2|I|.$$

So if $\mathbb{P}[Z_{t+k} \ge \varepsilon n \mid Z_t \ge \varepsilon n] \ge 1 - e^{-c\varepsilon n}$.

Non-homogeneous graphs

Star graph



k - children

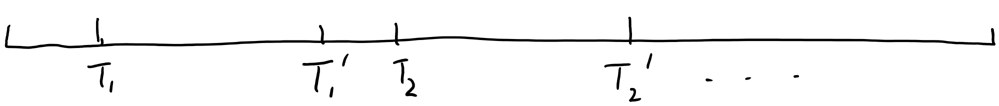
<u>Lemma</u>: For any $\delta > 0$, $\exists c > 0$ s.t.
if at least δk children infected at time 0,
$\mathbb{P}[T > e^{ck}] \geq 1 - e^{-ck}$.

- Fix $\varepsilon > 0$ small, assume $3\varepsilon k$ infected at time 0.

$$\mathbb{P}[\min_{0 \le t \le 1} W_1 \geq \varepsilon k] \le e^{-c\varepsilon k}$$

since each initially infected gets no removal w.p. $1/e > 1/3$.

Let $S = \int_0^1 X_t(\rho)\, d\rho$.



Sequence of uninfected times

$$T_i - T_{i-1}' \preccurlyeq Exp(\varepsilon k)$$

- $\mathbb{P}\left[\sum_{i=1}^{k\varepsilon/3} Exp(\varepsilon k) \geq \frac{1}{2}\right] \le e^{-ck}$

- $\mathbb{P}[\underbrace{\#\text{recoveries of } \rho}_{Pois(1)} \geq k\varepsilon/3] \le e^{-ck}$

So $\mathbb{P}[S \geq \frac{1}{2}] \geq 1 - e^{-ck}$.

- Number of vertices infected

$\sim Bin(1 - 3\varepsilon k, (1 - e^{-1/2}) \cdot e^{-1})$

$\geq \frac{k}{10}$ w.p. $1 - e^{-ck}$.

So $\mathbb{P}[W_1 \geq 3\varepsilon k] \geq 1 - e^{-ck}$.

BP tree with offspring dist

$\mathbb{P}[d_v = k] \sim k^{-\alpha}$.

<u>Lemma</u> On BP tree $\lambda_c = 0$.

<u>Proof</u>:

Suppose we can find

$v_1, v_2, \ldots$ with

- $d_{v_j} \geq 2^j$
- $d(v_j, v_{j+1}) \leq \alpha j$.

- B_j event v_j remains infected in the sense that $\varepsilon 2^j$ neighbours infected to time $\exp(c 2^j)$.

to time $\exp(c 2^j)$.

$\mathbb{P}[B_j] \geq 1 - \exp[-c 2^j]$

- A_j event v_{j+1} infected within time e^{cj} of v_j infected

$\mathbb{P}[A_j \mid B_j] \geq 1 - e^{-cj}$

Then $\mathbb{P}[\bigcap_{j \geq 1} (A_j \cap B_j)] > 0$ for all $\lambda > 0$.

- On the random graph.

  maximum degree $\sim n^{1/\alpha}$

  Infection time of at least $\exp(n^{1/\alpha})$.

- Actually w.h.p. infection lasts time

  - Fix large L.
  - Let V_L = vertices of degree $\geq L$.
  - $v, u \in L$ we write $v \sim_L u$ if $d(v,u) \leq \sqrt{L}$

Claim: Graph G_L is a good expander.

Theorem: For all $\lambda > 0$, $\exists c_\lambda > 0$, infection survives for time e^{cn} w.h.p.