Brownian motion

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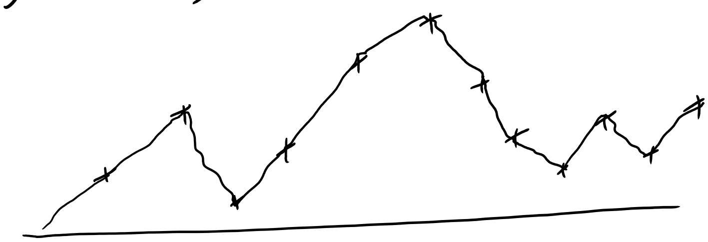
Let $S_n = \sum_{i=1}^{n} X_i$, X_i IID,

$\mathbb{E} X_i = 0$, $\text{Var } X_i = 1$ E.G. SRW.

By CLT, $S_n/\sqrt{n} \xrightarrow{d} N(0,1)$



$$Y_n(t) = S_{nt}/\sqrt{n}.$$

So $Y_n(1) \xrightarrow{d} N(0,1)$

Also $Y_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{tn} X_i \xrightarrow{d} N(0,t)$

This gives us convergence at a particular time. But we may want path properties E.G.

- Maximum of $Y_n(t)$, time of maximum
- Time $Y_n(t)$ is positive

$$\text{Cov}(Y_n(t), Y_n(s))$$

$$= \frac{1}{n} \text{Cov}\left(\sum_{i=1}^{tn} X_i, \sum_{j=1}^{sn} X_j\right)$$

$$= \frac{1}{n} \cdot (nt \wedge ns) = t \wedge s.$$

Take $t_1 < t_2 < \dots < t_k$

$$\left(Y_n(t_1), \dots, Y_n(t_k) \right) \xrightarrow{d} N_k(0, V)$$

$$V_{ij} = t_i \wedge t_j.$$

Brownian Motion

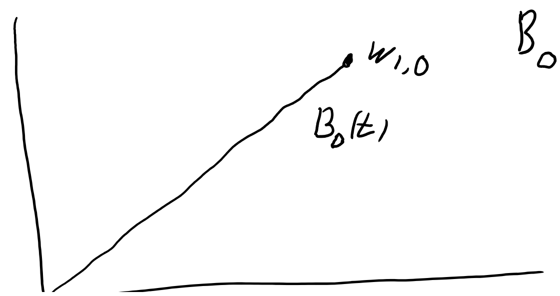
A random function $B: \Omega \to C[0,\infty)$.
such that:

- $B(t)$ is a Gaussian process.
- $B(0) = 0$, $\mathbb{E}\, B(t) = 0$, $\mathrm{Cov}(B(t), B(s)) = t \wedge s$.
- $B(t)$ is continuous almost surely.

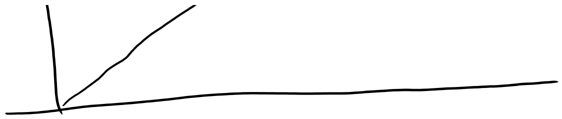
Construction: on $[0,1]$ via dyadic approximation

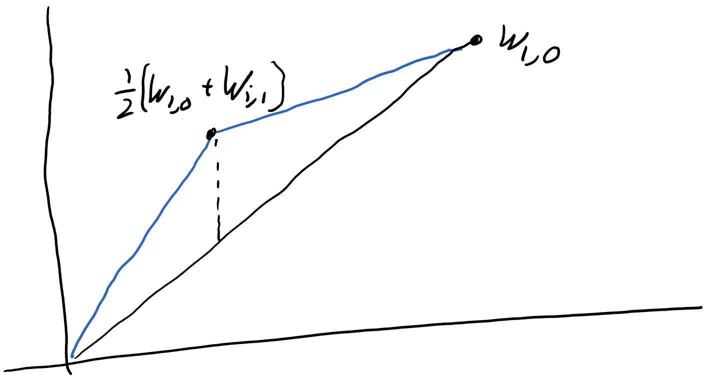
Let $W_{j,k}$ be IID $N(0,1)$

Set $B_0(t) = t\, W_{1,0}$



$B_0(t) \overset{d}{=} B(t).$





Set $B_1(\frac{1}{2}) = B_0(\frac{1}{2}) + \frac{1}{2} W_{1,1}$, $B_1(1) = B_0(1)$

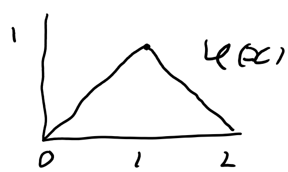
Linearly interpolate on $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$.

Claim: $\{B_1(k \cdot 2^{-1})\}_{k \in \{0,\dots,2\}} \overset{d}{=} \{B(k 2^{-1})\}_{k \in \{0,\dots,2\}}$

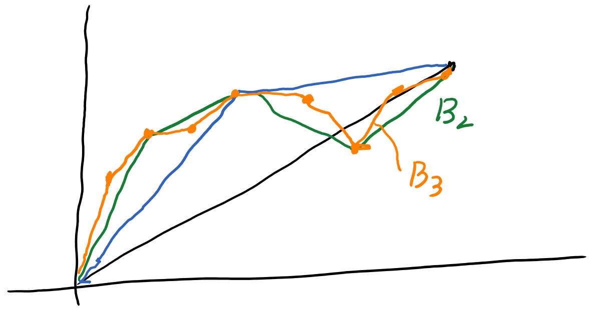
$$\mathrm{Var}(B_1(\tfrac{1}{2})) = \mathrm{Var}(\tfrac{1}{2} W_{1,0}) + \mathrm{Var}(\tfrac{1}{2} W_{1,1})$$
$$= \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{1}{2}$$

$$\mathrm{Cov}(B_1(\tfrac{1}{2}), B_1(1)) = \mathrm{Cov}(\tfrac{1}{2}(W_{1,0}+W_{1,1}), W_{1,0}) = \tfrac{1}{2}.$$

General Step: Let $\varphi(x) = \begin{cases} x & x \in [0,1] \\ 2-x & x \in [1,2] \\ 0 & \text{o.w.} \end{cases}$



Set $B_{k+1}(t) = B_k(t) + \sum_{j=1}^{2^k} 2^{-(k+2)/2} \cdot W_{j,k+1} \varphi(2^k x - j + 1)$



<u>Claim:</u> $\{B_k(j2^{-k})\}_{j\in\{0,\dots,2^k\}} \overset{d}{=} \{B(j2^{-k})\}_{j\in\{0,\dots,2^k\}}$

<u>Check</u>

$Cov(B_k(j2^{-k}), B_k(\ell 2^{-k})) = j2^{-k} \wedge \ell 2^{-k}$.

For j, ℓ even by induction.

Suppose $j = 2s_1+1$, $\ell = 2s_2+1$, $s_1 < s_2$

$$= Cov\left(\frac{B_{k-1}(s_1 2^{-(k-1)}) + B_{k-1}((s_1+1)2^{-(k-1)})}{2} + W_{s_1,k} 2^{-(k+1)/2}, \right.$$

$$\left. \frac{B_{k-1}(s_2 2^{-(k-1)}) + B_{k-1}((s_2+1)2^{-(k-1)})}{2} + W_{s_2,k} 2^{-(k+1)/2} \right)$$

$$= \frac{1}{4} 2^{-(k-1)}(s_1 + s_1 + (s_1+1) + (s_1+1))$$

$$= 2^{-k}(2s_1+1) = 2^{-k} j = 2^{-k} j \wedge 2^{-k}\ell.$$

$Var(B_k(j2^{-k}))$

$$= Var\left(\frac{B_{k-1}(s_1 2^{-(k-1)}) + B_{k-1}((s_1+1)2^{-(k-1)})}{2} + W_{s_1,k} \, 2^{-(k+1)/2} \right)$$

$$= \frac{1}{4}\left(3s_1 2^{-(k-1)} + (s_1+1)2^{-(k-1)}\right) + 2^{-(k+1)}$$

$$= 2^{-k}\left(2s_1 + \tfrac{1}{2} + \tfrac{1}{2}\right) = 2^{-k} j$$

Similarly if j or l odd.

<u>Claim</u> $B_k(t)$ converges uniformly a.s.

$\mathbb{P}[\|B_k(t) - B(t)\|_\infty > \varepsilon] \to 0.$

<u>Proof</u>: Let $D_k = \|B_k - B_{k-1}\|$

$$= 2^{-(k+1)/2} \cdot \max_{1 \le j \le 2^{k-1}} W_{j,k}$$

$$\mathbb{P}[N(0,1) > x] = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx \le e^{-x^2/2}. \quad x \geq 0.$$

So $\mathbb{P}[D_k > k\, 2^{-(k+1)/2}] = \sum_{j=1}^{2^{k-1}} \mathbb{P}[N(0,1) > k]$

$$\le 2^k e^{-k^2/2}$$

By Borel-Cantelli, since $\sum_k 2^k e^{-k^2/2} < \infty$,

$\mathbb{P}[\{D_k > k\, 2^{-(k+1)/2}\} \text{ i.o.}] = 0.$

Hence $\mathbb{P}[\sum D_k < \infty] = 1$ so $B_k \xrightarrow{L^\infty} B$.

$\Rightarrow B(t)$ is continuous.

<u>Convergence of Processes</u>

We say $X_n(t)$ converges in <u>finite dimensional distributions</u> if $\forall\, t_1 < t_2 < \ldots < t_k$

$$(X_n(t_1), \ldots, X_n(t_k)) \xrightarrow{d} (X(t_1), \ldots, X(t_k)).$$

Stronger Notion of Convergence

If $X_n(t), X(t) \in C[0,1]$ we say

$X_n(t) \to X(t)$ in the sup-norm topology if

for all bounded continuous functions

$f : (C[0,1], \|\cdot\|_\infty) \to \mathbb{R}$, (E.G. $f(X) = \max_{0 \le x \le 1} X(x)$).

$$\mathbb{E} f(X_n) \to \mathbb{E} f(X).$$

Functional Central Limit Theorem

<u>Theorem</u>: If $S_n = \sum_{i=1}^{n} X_i$, X_i IID, $\mathbb{E}X_i = 0$, $\mathrm{Var}\, X_i = 1$, $Y_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{nt} X_i$ then

$$Y_n(t) \xrightarrow{f.d.d.} Y(t).$$

<u>Homework</u>: If X_i are bounded then

$Y_n(t) \to Y(t)$ in the sup-norm.

Properties of Brownian Motion

a) Independent Increments

$B(t_1)$, $B(t_2) - B(t_1), \dots, B(t_k) - B(t_{k-1})$ are independent.

If $i<j$ then

$$\mathrm{Cov}(B(t_i) - B(t_{i-1}), B(t_j) - B(t_{j-1}))$$

$$= t_i \wedge t_j - t_i \wedge t_{j-1} - t_{i-1} \wedge t_j + t_{i-1} \wedge t_{j-1}$$

$$= t_i - t_i - t_{i-1} + t_{i-1} = 0.$$

Furthermore $W_s = B_{t+s} - B_t$ is a Brownian motion and is independent of $\mathcal{F}_t$.

Strong Markov Property says this also holds for S a stopping time.

b) $B(t)$ is a martingale

$$\mathbb{E}[B_t \mid \mathcal{F}_s] = \mathbb{E}[B_s + (B_t - B_s) \mid \mathcal{F}_s]$$

$$= B_s.$$

c) $B(t)$ is $\frac{1}{2}$-self similar (fractal)

If $Y(t) = s^{-1/2} B(ts)$

then $Y(t)$ is Brownian Motion.

$$\mathrm{Cov}(Y(t), Y(t')) = s^{-1} \mathrm{Cov}(B(st), B(st'))$$

$= s^{-1}(st \wedge st')$

$= t \wedge t'$

d) Non-differentiable

At one point

$$\frac{d}{dt}B(t) = \lim_{h \to 0} \frac{B(t+h)-B(t)}{h} \sim \frac{N(0,h)}{h} \sim N(0,\tfrac{1}{n})$$

$$\mathbb{P}[|N(0,1/n)| < M] = \mathbb{P}[|N(0,1)| < \tfrac{M}{\sqrt{n}}] \to 0.$$

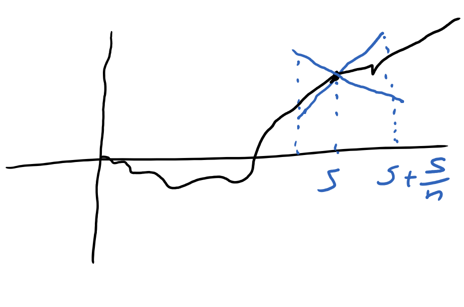
At all points

$$\mathbb{P}[\exists s \in [0,1], B(t) \text{ differentiable at } s] = 0.$$

Fix $C > 0$.

Let $A_n = \{\exists s \in [0,1] : |B(t) - B(s)| \le 2C|t-s|$ for $t \in [s - \frac{5}{n}, s + \frac{5}{n}]\}$

Must hold for all large n if $B(t)$ is differentiable at s with $|B'(s)| \le C$.



Let $Y_{k,n} = \max_{j \in \{0,1,2\}} \left\{ \left| B\left(\frac{k+j+1}{n}\right) - B\left(\frac{k+j}{n}\right) \right| \right\}.$

Let

Let

$$B_n = \{ \min_{0 \le k \le n-2} Y_{k,n} \le \frac{100C}{n} \}.$$

<u>Claim</u> $A_n \subseteq B_n$.

If s satisfies A_n and $\left(\frac{k+j}{n}, \frac{k+j+1}{n}\right) \subseteq \left[s - \frac{5}{n}, s + \frac{5}{n}\right]$

then by Δ inequality

$$\left|B\left(\tfrac{k+j+1}{n}\right) - B\left(\tfrac{k+j}{n}\right)\right| \le \left|B\left(\tfrac{k+j+1}{n}\right) - B(s)\right|$$
$$+ \left|B\left(\tfrac{k+j}{n}\right) - B(s)\right|$$
$$\le 2C \left|\tfrac{k+j+1}{n} - s\right| + 2C\left|\tfrac{k+j}{n} - s\right|$$
$$\le 2C\left(\tfrac{5}{n} + \tfrac{5}{n}\right) = \frac{20C}{n}.$$

<u>We can pick k such that $\left[\frac{k+j}{n}, \frac{k+3}{n}\right]$</u> $\subseteq \left[s - \frac{5}{n}, s + \frac{5}{n}\right]$ so $Y_{k,n} \le \frac{20C}{n}$.

$$\mathbb{P}\left[Y_{k,n} \le \frac{100C}{n}\right] = \mathbb{P}\left[|N(0, \tfrac{1}{n})| \le \frac{100C}{n}\right]^3$$
$$= \mathbb{P}\left[|N(0,1)| \le \frac{100C}{\sqrt{n}}\right]^3$$
$$\le \left(\frac{200C}{\sqrt{n}} \cdot \frac{1}{\sqrt{2\pi}}\right)^3 \quad \text{density} \le \frac{1}{\sqrt{2\pi}}$$
$$\le D n^{-3/2}.$$

$$\mathbb{P}[B_n] \le n \cdot D n^{-3/2} \le D/\sqrt{n}.$$

So $\mathbb{P}[A_n] \le \mathbb{P}[B_n] \le D/\sqrt{n} \to 0$.

But A_n is increasing in n so $\mathbb{P}[A_n] = 0$.

$\Rightarrow \mathbb{P}[B(t)$ is nowhere differentiable$] = 0.$

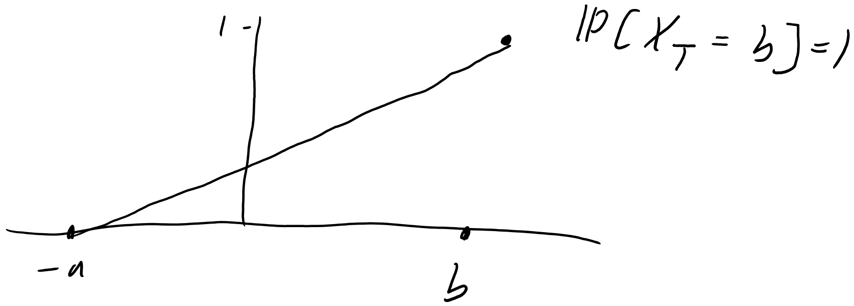
Hitting Probabilities:

Let T be the first hitting time of $-a$ or b.

$$\mathbb{E} X_T = \mathbb{E} X_0 = 0$$

$$= -a\,\mathbb{P}[X_T = -a] + b\,\mathbb{P}[X_T = b]$$

So $\mathbb{P}[X_T = b] = \dfrac{a}{b+a}$

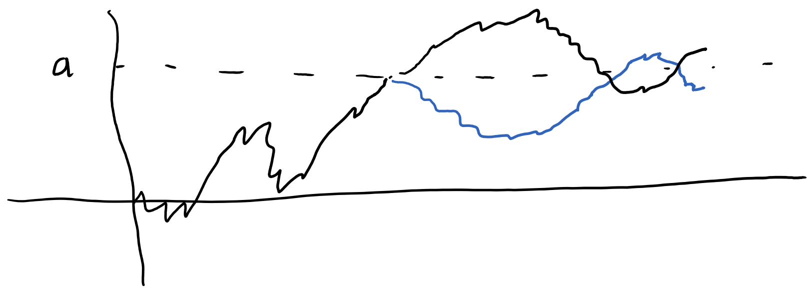


Reflection Principle

Let $M_t = \max_{0 \le s \le t} B(s)$.

Let T be the first hitting time of a.

Let $B^*(t) = \begin{cases} B(t) & t \le T \\ 2a - B(t) & t > T \end{cases}$



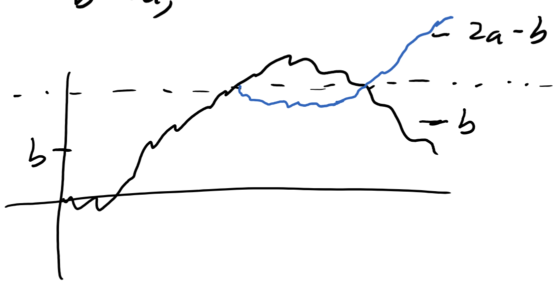
Since $-B(t) \overset{d}{=} B(t)$,

$$B(t+T)-a \overset{d}{=} -(B(t+T)-a) = a - B(t+T)$$

is B.M independent of $\mathcal{F}_T$.

So $B^*(t)$ is also B.M, let $M_t^* = \max_{s \le t} B^*(s)$,

for $b < a$,



$$\mathbb{P}[M(t) \geq a,\ B(t) \leq b] = \mathbb{P}[M^*(t) \geq a,\ B^*(t) \geq 2a-b]$$

$$= \mathbb{P}[B^*(t) \geq 2a-b]$$

$$= \mathbb{P}[B(t) \geq 2a-b]$$

Set $a=b$,

$$\mathbb{P}[M(t) \geq a] = \mathbb{P}[M(t) \geq a,\ B(t) \geq a] + \mathbb{P}[M(t) \geq a,\ B(t) \leq a]$$

$$= 2\,\mathbb{P}[B(t) \geq a]$$

So $M(t) \overset{d}{=} |B(t)|$.

Let $T_s = \inf\{t : B(t) = s\}$

Then T_{s_1}, $T_{s_2} - T_{s_1}$ are independent,

$T_1 \stackrel{d}{=} T_2 - T_1$ are independent.

$$\mathbb{P}[T_1 \leq t] = \mathbb{P}[M_t \geq 1] = \mathbb{P}[\,|N(0,t)| \geq 1]$$
$$= \mathbb{P}[\,|N(0,1)| \geq 1/\sqrt{t}\,]$$

so density of T_1 is

$$f_{T_1}(t) = \frac{d}{dt}\, 1 - \int_{-1/\sqrt{t}}^{1/\sqrt{t}} e^{-x^2/2}\, dx$$
$$= t^{-3/2} e^{-1/2t}$$

$$T_2 = \inf\{t : B(t) = 2\}$$
$$\stackrel{d}{=} \inf\{t : 2B(t/4) = 2\} \quad \text{Brownian Scaling}$$
$$= 4\inf\{t : B(t) = 1\}$$
$$= 4T_1.$$

So $X \sim T_1$, X, X' IID then

$X + X' \stackrel{d}{=} 4X$ so X is $\frac{1}{2}$-stable

$$T_1 \stackrel{d}{=} \sum_{x \in \Pi} x$$

where Π Poisson Process with intensity
$\lambda(x) = x^{-3/2}$.

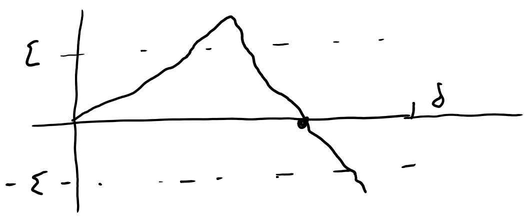
No isolated zero's

Let $Z = \{ t : B(t) = 0 \}$.

Theorem: Z has no isolated points almost surely.

Claim: $\inf\{ t > 0 : B(t) = 0 \} = 0$ a.s.

If $T_\varepsilon, T_{-\varepsilon} < \delta$ then $\exists\, 0 < t < \delta$ such that $B(t) = 0$



Since $\mathbb{P}[T_\varepsilon > \delta] = \mathbb{P}[T_{-\varepsilon} > \delta]$

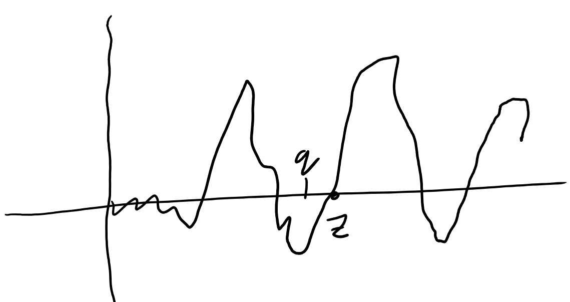
$$= \mathbb{P}[\varepsilon^2 T_1 > \delta]$$

$$= \mathbb{P}[T_1 > \delta/\varepsilon^2] \to 0$$

as $\varepsilon \to 0$.

$\Rightarrow \mathbb{P}[\inf\{ t > 0 : B(t) = 0 \}] = 1$.

Suppose $z \in Z$ is isolated. Then $\exists\, q \in \mathbb{Q}$, $q < z$ such that $[q, z) \cap Z = \emptyset$.



Let S_q be the stopping time $S_q = \inf\{ t \geq q : B(t) = 0 \}$.

Then $B(S_q) = 0$ and $\inf\{ t > S_q : B(t) = 0 \} > S_q$.

$\mathbb{P}[S_q$ isolated on the right$]$

$=$

$$= \mathbb{P}[\inf\{t > S_q : B(t) = 0\} > S_q]$$

$$= \mathbb{P}[\inf\{t > 0 : B(t) = 0\} > 0] = 0$$

since $\{B(t+S_q) - B(S_q)\}_{t \geq 0} \overset{d}{=} \{B(t)\}_{t \geq 0}$.

Taking a union bound over $q \in \mathbb{Q}$

$\mathbb{P}[\exists$ isolated point in $Z]$

$$= \mathbb{P}[\exists q \in \mathbb{Q} : S_q \text{ right isolated}]$$

$$\leq \sum_{q \in \mathbb{Q}} \mathbb{P}[S_q \text{ right isolated}] = 0.$$

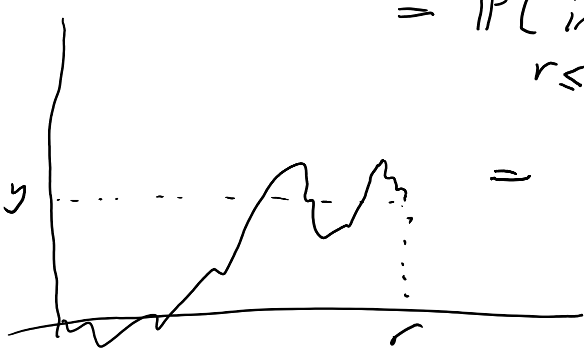
Last zero: let $R = \sup\{t \in [0,1] : B(t) = 0\}$.

$$\mathbb{P}[R \leq r] = \mathbb{E}[\,\mathbb{P}[R \leq r \mid \mathcal{F}_r]\,]$$

$$= \mathbb{E}[\,\mathbb{P}[\text{no zeros in } (r,1] \mid B(r)]\,]$$

$\mathbb{P}[\text{no zero's in } (r,1] \mid B(r) = y]$ for $y > 0$

$$= \mathbb{P}[\inf_{r \leq t \leq 1} B(t) - B(r) \geq -y \mid B(r) = y]$$



$$= \mathbb{P}[\inf_{0 \leq t \leq 1-r} B(t) \geq -y]$$

$$= \mathbb{P}[M_{1-r} \leq y]$$

$$= \mathbb{P}[|N(0, 1-r)| \leq |y|]$$

$$= 2\int_0^{|y|/\sqrt{1-r}} \frac{1}{\sqrt{2\pi(1-t)}} e^{-\frac{x^2}{2(1-t)}}\,dx$$

also by symmetry for negative y.

So $\mathbb{P}[R \le r] = \mathbb{E}\left[2\int_0^{|B_r|/\sqrt{1-r}} \frac{1}{\sqrt{2\pi(1-t)}} e^{-\frac{x^2}{2(1-t)}} dx\right]$

$$= \int_{-\infty}^{\infty} 2\int_{-|y|/\sqrt{1-r}}^{|y|/\sqrt{1-r}} \frac{1}{\sqrt{2\pi(1-t)}} e^{-\frac{x^2}{2(1-t)}} dx \frac{1}{\sqrt{2\pi t}} e^{-\frac{y^2}{2t}} dy$$

$$= \dots$$

$$= \frac{1}{2\pi} \arcsin(\sqrt{t}).$$

Question: Find $\mathbb{P}[Z \cap (a,b) = \emptyset]$?

Use scaling $= \mathbb{P}[Z \cap (\frac{a}{b}, 1) = \emptyset] = \frac{1}{2\pi} \arcsin\left(\sqrt{\frac{a}{b}}\right).$

Multi-dimensional Brownian Motion

$B(t) = (B_1(t), \dots, B_d(t))$ is d-dimensional B.M. with $B_i(t)$ independent B.M.

$B(t) \sim N(0, tI)$.

- Rotational invariance:

If R is a rotation matrix

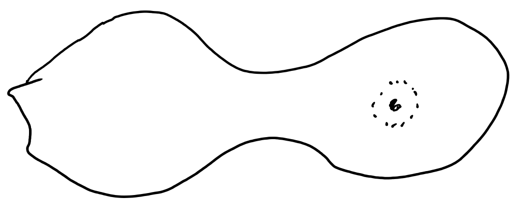
$\{R\, B(t)\}_{t\in\mathbb{R}} \overset{d}{=} \{B(t)\}_{t\in\mathbb{R}}.$

For each fixed t, $R\,B(t) \overset{d}{=} N(0, tI)$
since the density $f(x) = \frac{1}{(\sqrt{2\pi})^d} \cdot \exp(-\frac{1}{2}\sum x_i^2)$
is rotationally invariant.

So $R\,B(t_1)$, $R(B(t_2) - B(t_1))$, ...
independent, $N(0, (t_i - t_{i-1})I)$

Dirichlet Problem / PDE's.

Let D be an open set in $\mathbb{R}^d$ with smooth compact boundary, $g : \partial D \to \mathbb{R}$ smooth,



let $T = \inf\{t : B(t) \in \partial D\}$

Let $h(x) = \mathbb{E}[g(X_T) \mid X_0 = x] =: \mathbb{E}_x[g(X_T)]$

Then $\mathbb{E}_x[g(X_T) \mid \mathcal{F}_t]$ $\qquad$ $\mathbb{E}_x[\cdot] = \mathbb{E}[\cdot \mid X_0 = x]$

$= \mathbb{E}_x[g(X_T) \mid X_t] = \mathbb{E}_{X_t}[g(X_T)]$ for $t < T$.

So $h(X_{t \wedge T})$ is a martingale.

Let $B \subset D$ be a ball centred at x, S the first hitting time of ∂D.

Since S is a stopping time, by SMP

$$\mathbb{E}[\, g(X_T) \mid X_S, X_0 = x] = h(X_S)$$

so

$$h(x) = \mathbb{E}[\, g(X_T) \mid X_0 = x] = \mathbb{E}\, h(X_S).$$

By rotational symmetry X_S is uniform on ∂D so

$$h(x) = \mathbb{E}\, h(X_S) = \fint_{\partial D} h(u)\, du$$

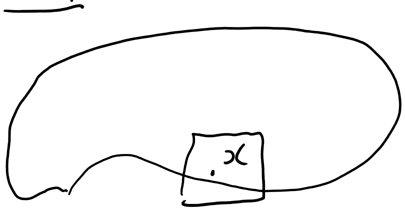
$\leftarrow$ average

So h satisfies the mean value property.

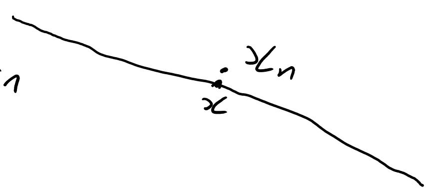
$$\Rightarrow \Delta h = \sum \frac{\partial^2}{\partial x_i^2} h = 0 \text{ and smooth on } D.$$

<u>Claim</u>: For $x_n \to x \in \partial D$, $h(x_n) \to g(x)$.

<u>Proof</u>:



Zoom in

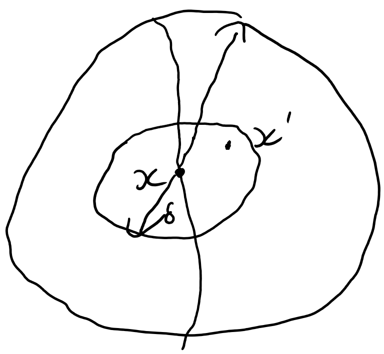


Enough to show $\forall \varepsilon > 0, \exists \delta$ such that if $|x' - x| < \delta$ then

$$\mathbb{P}[\,|g(X_T) - g(x)| > \varepsilon \mid X_0 = x'] < \varepsilon$$

since $\|g\|_\infty$ is bounded.

After rotation on small scales it is close to a hyperplanes,



$R\delta$, for $|x - x'| < \delta$,

If $M_t^{(i)} = \max_{0 \le s \le t} B(s)$, $I_t^{(i)} = \sup_{0 \le s \le t} -B(s)$

Pick $R \gg S \gg 1$,

$$\mathbb{P}[\, I_{S\delta^2} > \delta,\ \forall i,\ M^{(i)}_{S\delta^2}, I^{(i)}_{S\delta^2} < R\delta \,] \geq 1 - \varepsilon'$$

So $\mathbb{P}[\,|X_T - x| > R\delta\,] \leq 1 - \varepsilon'$.

<u>Transience</u>:

If h is harmonic, $\Delta h = 0$, then

$h(X_t)$ is a martingale since .

for a domain D, h solves the

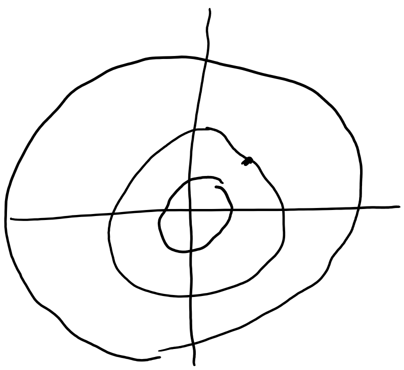
Dirichlet problem $\Delta g = 0$, $g(x) = h(x)$ on ∂D

so $\mathbb{E}[\, h(X_T) \mid X_0 = x] = h(x)$.

When $d=2$, let $h(x) = \log|x|$, $\Delta h = 0$.

- rotationally invariant.

Let $T_n = \inf\left\{ t \geq T_{n-1} : \dfrac{|B(T_n)|}{|B(T_{n-1})|} \in \{\tfrac{1}{2}, 2\} \right\}$.



Set $W_n = \begin{cases} 1 & \text{if } \dfrac{|B(T_n)|}{|B(T_{n-1})|} = 2 \\ -1 & \text{if } \dfrac{|B(T_n)|}{|B(T_{n-1})|} = \dfrac{1}{2} \end{cases}$

$$S_n = \sum_{i=1}^{n} W_i$$

So $|B(T_n)| = 2^{S_n}$.

By symmetry W_n is independent of $\mathcal{F}_{T_{n-1}}$.

$$\mathbb{E}[h(B_{T_n})] = h(B(T_{n-1})) = \log|B(T_{n-1})|$$

$$= (\log(2) + \log|B(T_{n-1})|)\, \mathbb{P}[W_n = 1]$$

$$+ (\log\tfrac{1}{2} + \log|B(T_{n-1})|)\, \mathbb{P}[W_n = -1]$$

$\Rightarrow \mathbb{P}[W_n = 1] = 1/2$ and S_n is SRW.

If $B(0)=(1,0)$ then

- $\mathbb{P}[\exists t: B(t)=0]$

$= \mathbb{P}[\exists M$ such that $\forall L$, S_n hits $-L$ before $M]$

$= \mathbb{P}[S_n \to -\infty] = 0.$

So $B(t)$ is transient.

Still, $\mathbb{P}[\exists t: B(t) \leq 2^{-k}]$

$= \mathbb{P}[\exists n: S_n \leq k] = 1.$

So $\mathbb{P}[0 \in \overline{\{B(t): t \geq 0\}}] = 1$

In fact $\overline{\{B(t): t \geq 0\}} = \mathbb{R}^2.$

We call $B(t)$ neighbourhood reccurent.

<u>When $d \geq 3$.</u>

Take $h(x) = |x|^{2-d}$, $\Delta h = 0.$

Define T_k, W_k same way, $\theta = 2^{2-d}$

$h(T_k) = \mathbb{E}\, h(T_{k+1})$

$= \theta\, h(T_k) \cdot \mathbb{P}[W_{k+1} = 1]$

$+ \quad \theta^{-1} \quad h(T_n)\, \mathbb{P}[W_{n+1} = -1].$

So $\mathbb{P}[W_{n+1} = 1] = \dfrac{1-\theta}{1-\theta^2} = \dfrac{1}{1+\theta} > \dfrac{1}{2}.$

Hence $S_n \to \infty$ a.s.

and

$\mathbb{P}[\inf |B(t)| \leq 2^{-k} \mid B(t) = (1,0)]$

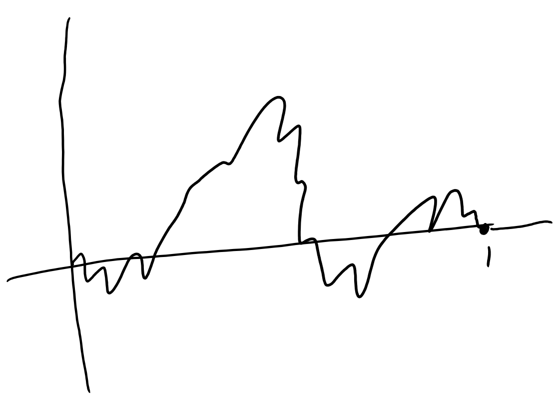
$= \mathbb{P}[\inf S_n \leq -k] \to 0$ as $k \to \infty$.

So $B(t)$ is not neighbourhood reccurrent.

Brownian Bridge

Brownian motion on $[0,1]$

Conditioned on $B(1) = 0$



Let $Y(t) = B(t) - t\,B(1)$.

$\mathrm{Cov}(Y(t), B(1)) = \mathrm{Cov}(B(t), B(1))$

$$- t \, Var(B(1))$$
$$= t - t = 0.$$

So $Y(t)$ is independent of $B(1)$,

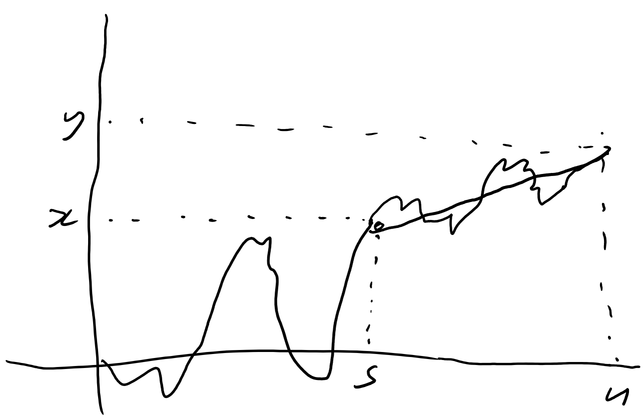
So $Y(t) \overset{d}{=} Y(t) \mid B(1) = 1 \overset{d}{=}$ Brownian Bridge.

If $0 < s < t < 1$,

$$Cov(Y(t), Y(s)) = Cov(B(t) - tB(1), B(s) - sB(1))$$
$$= t \wedge s - ts - st + ts$$
$$= t(1-s).$$

Brownian Bridge is the continous Gaussian process with $Cov(Y(s), Y(t)) = t(1-s)$.

If we condition on $B(s) = x$, $B(u) = y$, what is the distribution between?



$$X(t) := \frac{B(s + (u-s)t) - B(s) - t(B(u) - B(s))}{\sqrt{u-s}}$$

$\overset{d}{=}$ Brownian Bridge

Independent of $(B(s), B(u))$.

<u>Example</u> Empirical Process.

$X_1, \dots, X_n$ IID CDF $F(x)$.

Then $F_n(x) = \frac{1}{n} \#\{1 \le i \le n : X_i \le x\}$.

$\sim Bin(n, F(x))$.

If $X_i \sim Unif[0,1]$, then

$\sqrt{n}\,(F_n(x) - F(x)) \xrightarrow{f.d.d.} Y(t)$ Brownian Bridge.

Check covariance, for $0 < x < y < 1$,

$Cov(\sqrt{n}\, F_n(x), \sqrt{n}\, F_n(y))$

$$= \frac{1}{n} \sum_{i=1}^{n} Cov\left(I(X_i \le x), I(X_i \le y)\right)$$

$$= \mathbb{E}\, I(X_i \le x, X_i \le y) - \mathbb{P}[X_i \le x]\,\mathbb{P}[X_i \le y]$$

$$= x - xy = x(1-y).$$

If X_i have another distribution then

$\sqrt{n}\,(F_n(x) - F(x)) \xrightarrow{f.d.d.} Y(F(x))$.