Discrete Fourier Analysis

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Let $(\Omega, \mathcal{F}, \mu)$ discrete prob space

$L^2(\mu)$ set of L^2 functions / Random variables

- $(u_0, u_1, \dots, u_{|\Omega|-1})$ is a standard basis for $L^2(\mu)$ if
  - $u_0 \equiv 1$
  - $\mathbb{E} u_i u_j = \delta_{ij}$. orthonormal

Then if $f = \sum a_i u_i$, $g = \sum b_i u_i$

- $\mathbb{E} f = a_0$
- $\mathbb{E} f^2 = \sum_{i \geq 0} a_i^2$
- $Var(f) = \sum_{i \geq 1} a_i^2$
- $Cov(f, g) = \sum_{i \geq 1} a_i b_i$

- Simplest example $\Omega = \{-1, 1\}_\theta$, $\mu_\theta(\{x\}) = \frac{1 + \theta x}{2}$.

  $\mathbb{E} X = \theta$, $Var\, X = 1 - \theta^2$, $u_0 = 1$, $u_1 = \frac{X - \theta}{\sqrt{1 - \theta^2}}$

<u>Product Space</u>: $L^2(\mu_1 \times \mu_2)$ on $\Omega_1 \times \Omega_2$,

- $\mu_1 \times \mu_2 (A \times B) = \mu_1(A) \mu_2(B)$.

- Define tensor product of $f \in L^2(\mu_1)$, $g \in L^2(\mu_2)$

  $(f \otimes g)(x, y) = f(x) g(y)$.

If $L^2(\mu_i)$ has basis $u_0^i, \dots, u_{m_i}^i$

$L^2(\times_{i=1}^k \mu_i)$ has basis $\{u_{j_1}^1 \otimes u_{j_2}^2 \otimes \dots \otimes u_{j_k}^k\}_{j_1, \dots, j_k}$

Std basis for $\{-1,1\}^n_\theta$ is

$$U_S(x_1, \dots, x_n) = \prod_{i \in S} \frac{x_i - \theta}{\sqrt{1-\theta^2}}$$

Fourier $\hat{f}(S) = \mathbb{E}[f U_S]$ discrete Fourier transform

Write $f_S = \hat{f}(S) U_S$, $\qquad f = \sum_S \hat{f}(S) U_S = \sum_S f_S$.

$$\mathrm{Var}\, f = \sum_{S \neq \emptyset} \hat{f}(S)^2$$

Influence of i on f: $I_i(f) = \mathbb{E}[\mathrm{Var}[f \mid \{X_j\}_{j \neq i}]]$

If $f : \{\pm 1\}^n \to \pm 1$ then $I_i(f) = (1-\theta^2) \mathbb{P}[f(X) \neq f(X^{\oplus i})]$
Flip i.

Lemma: $I_i(f) = \sum_{S \ni i} \hat{f}(S)^2$

$$I_i(f) = \mathbb{E}[\mathrm{Var}[f \mid X_{-i}]] = \mathbb{E}[\mathrm{Var}[\sum_{S \ni i} f_S + \sum_{S \not\ni i} f_S \mid X_{-i}]]$$

$$= \mathbb{E}\left(\mathbb{E}\left[\left(\sum_{S \ni i} f_S\right)^2 \mid X_{-i}\right]\right) = \mathbb{E}\left(\sum_{S \ni i} f_S\right)^2 = \sum_{S \ni i} \hat{f}(S)^2$$

Corol: $\sum I_i(f) = \sum_{|S|} |S| \hat{f}(S)^2 \geq \sum_{S \neq \emptyset} \hat{f}(S)^2 = \mathrm{Var}\, f$.

Noise Operators: A linear operator $T : L^2(\mu) \to L^2(\mu)$ is a noise operator if

1) $f \geq 0 \Rightarrow Tf \geq 0$.
2) $T(1) = 1$ constant functions
3) $\mathbb{E}[(Tf)^2] = \mathbb{E}[f^2]$.
4) $\mathbb{E}[(Tf) \cdot g] = \mathbb{E}[f \cdot (Tg)]$ like reversibility/self-adjoint

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Ex: Markov transition, M reversible w.r.t. μ,

$$(T_M f)(x) = \sum_y M(x,y) f(y) = \mathbb{E}[f(X_1) \mid X_0 = x].$$

Special case: Bonami-Beckner operator (resampling).

$T_\eta f = \eta f + (1-\eta)\mathbb{E} f$. Markov chain

$$X_{i+1} = \begin{cases} X_i & \text{w.p. } \eta \\ \text{indep } \mu & \text{w.p. } 1-\eta. \end{cases}$$

$T_\eta u_0 = u_0$, $T_\eta u_1 = \eta u_1$

Tensoring: If T_i are noise operators on $L^2(\mu_i)$

$T = \bigotimes_{i=1}^n T_i$ defined by

$$T(u_1 \otimes \dots \otimes u_m) = (T_1 u_1) \otimes \dots \otimes (T_m u_m)$$

is a noise operator.

For Bonami-Beckner on $\{-1,+1\}^n_\theta$

$$T_\eta f = \sum_S \eta^{|S|} \hat{f}(S) u_S.$$

Hyper Contraction:

$T: L^p \to L^p$ is a contraction if $\|Tf\|_p \le \|f\|_p$

All Markov operators are contractions.

T is a (p,q) hyper-contraction if $1 < p < q$ &

$$\|Tf\|_q \le \|f\|_p$$

Lemma: If T_i are (p,q) hyper-contractions then

So is $\bigotimes_{i=1} T_i$.

<u>Proof</u>: By Fubini / generalized Minkowski.

For Bonami - Bechner if $\eta^2 \le \frac{p-1}{q-1}$

- T_η is (p,q) - hyper contractive. on $\{-1,+1\}_0$

Enough to check by scaling $f(x) = 1 + ax$, $a \in (0,1)$

$$\forall a: \left(\tfrac{1}{2}(1+a)^p + \tfrac{1}{2}(1-a)^p\right)^{1/p} \ge \left(\tfrac{1}{2}(1+\eta a)^q + \tfrac{1}{2}(1-\eta a)^q\right)^{1/q}$$

<u>Other results</u>: On $\{-1,1\}_\theta$, $\alpha = \min_x \mu_\theta(x) = \frac{1-|\theta|}{2}$

If $\eta^2 \le \frac{\alpha^2}{3}(p-1)$ then

T_η is $(p,2)$ hyper contractive.

By tensoring same for $T: \{-1,+1\}_\theta^n$.

Set $\Delta_i f = \sum_{S \ni i} \hat{f}(S) u_S$, $\|\Delta_i f\|_2^2 = I_i(f)$.

<u>Theorem</u> (Talagrand) There exists universal constant C such that,

On $\{+1,-1\}_\theta^n$, $\alpha = \frac{1-|\theta|}{2}$,

$$Var(f) \le C \log(1/\alpha) \cdot \sum_i \frac{\|\Delta_i f\|_2^2}{\log(\|\Delta_i f\|_2 / \|\Delta_i f\|_1)}$$

<u>Theorem</u> Kahn - Kalai - Linial KKL

There exists universal C s.t., $f: \{+1,-1\}_\theta^n \to \{-1,1\}$.

$$\max_i I_i(f) \ge C \frac{1}{\log(1/\alpha)} \frac{Var f}{n} \cdot \log n$$

i

Proof:

$$Var[f \mid X_{-i}] = (1-\theta^2)\, I[f(X) \neq f(X^{\oplus i})].$$

$$\therefore \|\Delta_i f\|_2^2 = I_i(f) = \sum_{S \ni i} \hat{f}(S)^2$$

$$= \mathbb{E}[Var[f \mid X_{-i}]] = (1-\theta^2)\, \mathbb{P}[f(X) \neq f(X^{\oplus i})]$$

We also have that when $\theta = 0$,

$$f(X) - f(X^{\oplus i}) = \sum_S \hat{f}(S)\Big(\prod_{j \in S} X_j - \prod_{j \in S} X^{\oplus i}\Big)$$

$$= \sum_{S \ni i} \hat{f}(S)\, 2 \prod_{j \in S} X_j = 2\Delta_i f.$$

so $|\Delta_i f| = I(f(X) \neq f(X^{\oplus i}))$

$$\|\Delta_i f\|_1 = \mathbb{P}[f(X) \neq f(X^{\oplus i})] = \|\Delta_i f\|_2^2 = I_i(f)$$

Also $\|\Delta_i f\|_1 = \|\Delta_i f\|_2^2$ for $\theta \neq 0$ as well, see below

So either $\exists i$ s.t. $I_i(f) \geq \frac{\log n}{n}$, then we are done or,

$$\|\Delta_i f\|_2 / \|\Delta_i f\|_1 \geq \sqrt{n/\log n}$$

and so

$$\log(\|\Delta_i f\|_2 / \|\Delta_i f\|_1) \geq (\tfrac{1}{2} + o(1)) \log n$$

Then $$var(f) \leq C \log(1/\alpha) \frac{n \max I_i(f)}{\frac{1}{2}\log n}$$

For $\theta \neq 0$

$f(X) - f(X^{\oplus i}) \in \{-2, 2\}$ on $\{f(X) \neq f(X^{\oplus i})\}$
and 0 o.w.

$$\frac{2X_i}{\sqrt{1-\theta^2}} \sum_{S \ni i} \hat{f}(S)\, U_{S \setminus i} \in \{-2, 2\} \text{ on } \{f(X) \neq f(X^{\oplus i})\}$$

so $$\sum_{S \ni i} \hat{f}(S)\, U_{S \setminus i} \in \pm\sqrt{1-\theta^2}$$

Now $\Delta_i f = \frac{X_i - \theta}{\sqrt{1-\theta^2}} \sum_{S \ni i} \hat{f}(S) U_{S \setminus i}$

So $|\Delta_i f| = \left|\frac{X_i - \theta}{\sqrt{1-\theta^2}}\right| \left|\sum_{S \ni i} \hat{f}(S) U_{S \setminus i}\right|$

$= |X_i - \theta| \, I(f(X) \neq f(X^{\oplus i}))$

So $\|\Delta_i f\| = \left(\frac{(1+\theta)}{2}(1-\theta) + \frac{1-\theta}{2} \cdot (1+\theta)\right) \mathbb{P}[f(X) \neq f(X^{\oplus i})]$

$= (1-\theta^2) \mathbb{P}[f(X) \neq f(X^{\oplus i})] = \|\Delta_i f\|_2^2$.

Proof of Talagrand:

Claim Enough to prove for all $g \neq 0$, $\mathbb{E} g = 0$,

$$\sum_S \frac{\hat{g}(S)^2}{|S|} \leq C \log(1/\alpha) \cdot \frac{\|g\|_2^2}{\log(\|g\|_2 / \|g\|_1)}$$

Take $g_i = \Delta_i f$.

$$\sum_i \frac{\hat{g}_i(S)^2}{|S|} = \sum_i \sum_S \frac{\hat{f}(S)^2 I(i \in S)}{|S|} = \sum_S \hat{f}(S)^2 = \mathrm{Var}(f)$$

$$\leq C \log(1/\alpha) \sum_i \frac{\|\Delta_i f\|_2^2}{\log(\|\Delta_i f\|_2 / \|\Delta_i f\|_1)}$$

Proof of Claim:

Take $\eta^2 = \frac{\alpha^2}{6}$, $\|T_\eta g\|_2 \leq \|g\|_{3/2}$

$T_\eta g = \sum \eta^{|S|} \hat{g}(S) U_S$

So

$$\|g\|_{3/2}^2 \geq \|T_\eta g\|_2^2 = \sum \eta^{2|S|} \hat{g}(S)^2 \geq \sum_{|S|=k} \eta^{2k} \hat{g}(S)^2.$$

$$\|g\|_{3/2}^2 \geq \|T_\eta g\|_2^2 = \sum \eta^{2|S|} \hat{g}(S)^2 \geq \sum_{|S|=k} \eta^{2k} \hat{g}(S)^2.$$

$$\sum_{|S|=k} \hat{g}(S)^2 \leq \left(\frac{6}{\alpha^2}\right)^k \|g\|_{3/2}^2$$

Fix m,

$$\sum_{|S|} \frac{\hat{g}(S)^2}{|S|} = \sum_{|S| \geq m} \frac{\hat{g}(S)^2}{m} + \sum_{k \leq m} \frac{\left(\frac{6}{\alpha^2}\right)^k}{k} \|g\|_{3/2}^2$$

$$\leq \frac{2}{m}\left[\left(\frac{6}{\alpha^2}\right)^m \|g\|_{3/2}^2 + \|g\|_2^2\right]$$

Choose largest m such that

$$\left(\frac{6}{\alpha^2}\right)^m \|g\|_{3/2}^2 \leq \|g\|^2$$

$$\Rightarrow \quad m+1 \geq \frac{2\log(\|g\|_2 / \|g\|_{3/2})}{\log(6/\alpha^2)}$$

By Holder

$$\|g\|_1 \cdot \|g\|_2 \cdot \|g\|_2 \geq \|g^3\|_{\frac{1}{\frac{1}{1}+\frac{1}{2}+\frac{1}{2}}} = \|g^3\|_{1/2} = \|g_{3/2}\|_2^3$$

so $$\frac{\|g\|_1}{\|g\|_2} \geq \left(\frac{\|g\|_{3/2}}{\|g\|_2}\right)^3$$

so $$m+1 \geq \frac{\frac{1}{3}\log(\|g\|_2/\|g\|_1)}{\log(6) + \log(1/\alpha)}$$

$$\sum \frac{\hat{g}(S)^2}{|S|} \leq \frac{C\log(1/\alpha)}{\log(\|g\|_2/\|g\|_1)} \cdot \|g\|_2^2 \quad \checkmark$$

Monotone Graph Properties have sharp thresholds

- Monotone
- Non-constant
- Invariant (in distribution) under permutations of vertices.

If A is monotone, $p \in (1-\delta, \delta)$
$\mathbb{P}_p[A] > \varepsilon$ then $\mathbb{P}_{p+\frac{c \log n}{\log(1/\delta)}}[A] \geq 1-\varepsilon$

Proof: $\mathbb{P}_p[A]$ is increasing and by Russo's Formula

$$\frac{d}{dp}\mathbb{P}_p[A] = \sum_i I_i(1_A) = n\, I_i(1_A) \geq \frac{1}{\log(1/\delta)} \cdot \log(n) \operatorname{Var}[1_A]$$
$$= \frac{1}{\log(1/\delta)} \log(n) \cdot \mathbb{P}_p[A](1-\mathbb{P}_p[A]).$$

First Passage Percolation

Cost on each edge $X_e = \begin{cases} a & \text{w.p. } 1/2 \\ b & \text{w.p. } 1/2 \end{cases}$ $a, b > 0$.

$$Y_n = \min_\gamma \sum_{e \in \gamma} X_e \qquad \gamma \text{ path } (0,0) \text{ to } (0,n)$$

Theorem: Benjamini - Kalai - Schramm

$$\operatorname{Var}(Y_n) \leq C\, n/\log n$$

Proof:

Let X_e' be independent copy & Y_n^e has X_e' replacing X_e.

$$I_e(Y_n) = \mathbb{E}[\operatorname{Var}[\quad Y_n \quad | X_{-e}]]$$
$$= \tfrac{1}{2}\mathbb{E}[\mathbb{E}(Y_n - Y_n^e)^2 \mid X_{-e}]]$$
$$= \tfrac{1}{2}\mathbb{E}[(Y_n - Y_n^e)^2]$$
$$= \mathbb{E}[(Y_n - Y_n^e)^2 I(Y_n < Y_n^e)]$$

$$= (a-b)^2\, \mathbb{P}[Y_n < Y_n^e]$$
$$\le (a-b)^2\, \mathbb{P}[e \in \gamma].$$

$$\mathrm{Var}(Y_n) \le \sum_e I_e(Y_n) \le (a-b)^2\, \mathbb{E}|\gamma| = Cn$$

We use Talagrand to get a better bound.

$\Delta_e Y \le C$, $\mathbb{P}[\Delta_e Y \neq 0] \le 2\mathbb{P}[e \in \gamma]$.

No known method to prove $\mathbb{P}[e \in \gamma] \le n^{-c}$.

Instead BKS use an averaging trick.

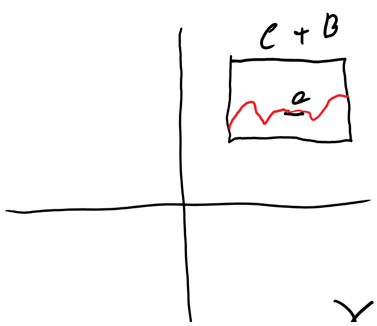
Let $Y_x = \min_{\gamma : x \to x+(0,n)} \sum X_e$

Set $Z = \frac{1}{|B|} \sum_{x \in B} Y_x$ where $B = \{x : \|x\|_\infty \le n^\delta\}$

$$|Y_x - Y| \le Cn^\delta$$

so $|Z - Y| \le Cn^\delta \Rightarrow \mathrm{Var}(Z) \le 2\mathrm{Var}(Y) + Cn^{2\delta}$.

$$\|\Delta_e Z\|_1 \le \frac{1}{|B|} \sum_{x \in B} \|\Delta_e Y_x\|_1 = \frac{1}{B} \sum_{x \in B} \|\Delta_{e-x} Y\|_1$$
$$\le \frac{C}{B} \sum_{x \in B} \mathbb{P}[e - x \in \gamma]$$
$$\le \frac{C}{|B|} \mathbb{E}[\gamma \cap e+B]$$
$$\le \frac{C' n^\delta}{n^{2\delta}} \le C' n^{-\delta}.$$



Since $\gamma \cap e+B$, the time that

γ spends in box side length $2n^\delta$

Since $\gamma \cap e + B$, the time that γ spends in box side length $2n^\delta$ is at most Cn^δ since it is the shortest path.

Also

$$\| \Delta_e Z \|_2^2 = \| \frac{1}{|B|} \sum_{x \in B} \Delta_e Y_x \|_2^2$$

$$\leq \frac{1}{|B|} \sum_{x \in B} \| \Delta_e Y_x \|_2^2 \quad \text{Cauchy-Schwartz}$$

$$\leq \frac{C}{|B|} \sum_{x \in B} \mathbb{P}[e - x \in \gamma] =: A_e^2 \leq C' n^{-\delta}$$

$$\mathrm{Var}(Z) \leq C \frac{\sum \| \Delta_e Z \|_2^2}{\log(\|\Delta_e\|_2 / \|\Delta_e\|_1)}$$

$$\leq C \frac{\sum A_e^2}{\log(A_e / \|\Delta_e\|_1)} \quad \text{increasing in } \|\Delta_e Z\|_2$$

$$\leq C \frac{\sum A_e^2}{\log(A_e / c A_e^2)} \quad \text{increasing in } \|\Delta_e Z\|_1$$

$$\leq \frac{C}{\delta \log n} \sum A_e^2$$

$$= \frac{C}{\delta \log n} \sum_e \sum_x \frac{1}{|B|} \mathbb{P}[e - x \in \gamma]$$

$$= \frac{C}{\delta \log n} \mathbb{E}|\gamma| \quad \text{Counting each } e, |B| \text{ times.}$$

$$\leq \frac{C' n}{\log n}.$$

∎

Friedgut's Theorem

On $\{-1,+1\}^n_0$ let $f: \{-1,+1\}^n \to \{-1,+1\}$ and

(A) $\sum_i I_i(f) = \sum_S \hat{f}(S)^2 |S| \le b$

Then for any $\varepsilon > 0$, there exists g depending on only $\exp(C(b+1)/\varepsilon)$ co-ordinates such that

$\mathbb{P}[f \neq g] \le \varepsilon.$

where C is a universal constant. Note, no dependence on n.

Can be generalized to $\{-1,1\}^n_\theta$.

Note that (A) means not much weight on large S.

Proof:

Since $f(X) - f(X^{\oplus i}) = 2\Delta_i f$, $\Delta_i f \in \{-1, 0, 1\}$

so $|\Delta_i f|_q = |\Delta_i f|_2^{2/q}$

Let $J = \{i : I_i(f) > \gamma\}$ and

$h = \sum_{S : S \cap J \neq \emptyset} \hat{f}(S) u_S.$

Then $\mathrm{sign}(f-h)$ depends only on J^c, $|J^c| \le \frac{b}{\gamma}$.

$\mathbb{P}[f \neq \mathrm{sign}(f-h)] \le \mathbb{P}[|h| \ge 1]$

$\le \|h\|_2^2$

So we need $\|h\|_2^2 < \varepsilon$.

If T_η is $(\frac{3}{2}, 2)$-hyper contractive,

$\sum \|T_\eta \Delta_i f\|_2^2 = \sum \sum \eta^{2|S|} \hat{f}(S)^2$

$i \in J$ $\qquad$ $i \in J \quad S \ni i$

$$= \sum_S |J \cap S|\, \eta^{2|S|} \hat{f}(S) \quad (\#)$$

$$\leq \sum_{i \in J} \|\Delta_i f\|_{4/3}^2$$

$$= \sum_{i \in J} \|\Delta_i f\|_2^{8/3}$$

$$\leq \gamma^{2/3} \sum_{i \in J} \|\Delta_i f\|_2^2$$

$$\leq \gamma^{2/3} b. \quad (\#\#)$$

Comparing (#) & (##),

So $$\sum_S \hat{f}(S)^2\, I\left(|S \cap J| > \frac{2\gamma^{2/3} b\, \eta^{-2|S|}}{\varepsilon}\right) \leq \frac{\varepsilon}{2}$$

and by theorem hypothesis (A)

$$\sum_S \hat{f}(S)^2\, I\left(|S| > \frac{2b}{\varepsilon}\right) \leq \frac{\varepsilon}{2}$$

Combining we have

$$\sum_S \hat{f}(S)^2\, I\left(|S \cap J| > \frac{2\gamma^{2/3} b\, \eta^{-4b/\varepsilon}}{\varepsilon}\right) \leq \varepsilon.$$

If $\gamma < \left(\frac{2b\eta^{-4b/\varepsilon}}{\varepsilon}\right)^{-3/2}$ then this is

$$\|h\|_2^2 = \sum_S \hat{f}(S)\, I(|S \cap J| \geq 1) \leq \varepsilon$$

and $|J^c| \leq b\left(\frac{2b\eta^{-4b/\varepsilon}}{\varepsilon}\right)^{3/2}$ suffices.