Coupling

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Basic Question

If $p < q$ is

$\mathbb{P}[\,Bin(n,p) \geq k] \leq \mathbb{P}[\,Bin(n,q) \geq k]$

$\sum_{i=k}^{n} \binom{n}{i} p^i (1-p)^{n-i} \qquad \sum_{i=k}^{n} \binom{n}{i} q^i (1-q)^{n-i}$

Coupling: Compare or equate X & Y,

find joint distribution (X', Y') with

$X \overset{d}{=} X'$, $Y \overset{d}{=} Y'$.

Ex: Let $X \sim Ber(p)$ $Y \sim Ber(q)$

$U \sim U[0,1]$ $X' = I(U \leq p)$, $Y' = I(U \leq p)$

If $p < q$ then $X' \leq Y'$.

$U_1, \ldots, U_n$ IID $U[0,1]$

$Bin(n,p) \overset{d}{=} \sum_{i=1}^{n} I(U_i \leq p) \leq \sum_{i=1}^{n} I(U_i \leq q)$

$\overset{d}{=} Bin(n,q)$

Percolation $\theta_p = \mathbb{P}[\,0 \in \text{infinite component}]$

θ_p is increasing

Set $\xi_e(p) = I(U_e \leq p)$ indicator of edge e open.

For each p, $\xi_e(p)$ is percolation with prob p and so $X(p) = I(0 \in I.C.)$ is increasing.

$\Rightarrow \theta_p$ increasing.

Critical percolation

$$p_c = \inf\{p : \theta_p > 0\}$$

Later: Kesten's Theorem

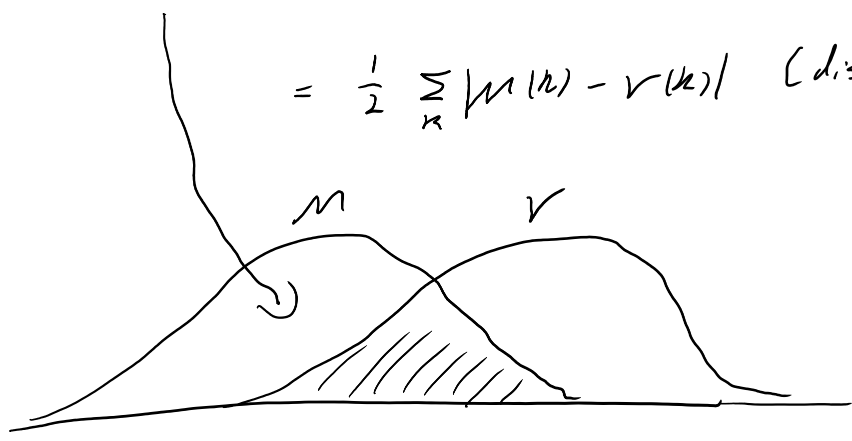
Q: Is $\theta_{p_c} = 0$ on $\mathbb{Z}^d$? Yes for $d=2$, $d \geq 4$.

$d=3$ open!

Total Variation Distance

$$d_{TV}(\mu, \nu) = \sup_A |\mu(A) - \nu(A)|$$

$$= \frac{1}{2} \sum_x |\mu(x) - \nu(x)| \quad \text{(discrete)}$$



Optimal coupling: $X \sim \mu$, $Y \sim \nu$

If $X' \sim \mu$, $Y' \sim \nu$ then

$$\mathbb{P}[X' \neq Y'] \geq d_{TV}(\mu, \nu)$$

Pf: $\mathbb{P}[X' \neq Y'] \geq \sup_A \mathbb{P}[X' \in A, Y' \notin A]$

$$\geq \sup_A \mathbb{P}[X' \in A] - \mathbb{P}[Y' \in A]$$

$\exists$ X', Y' such that $\mathbb{P}[X' = Y'] = d_{TV}(\mu, \nu)$.

Let $$\lambda_1(k) = \frac{\min(\mu(k), \nu(k))}{1 - d_{TV}(\mu, \nu)}$$

$$\lambda_2(k) = \frac{\max(\mu(k) - \nu(k), 0)}{d_{TV}}$$

$$\lambda_3(k) = \frac{\max(\nu(k) - \mu(k), 0)}{d_{TV}}$$

$W_i \sim \lambda_i$, $U = Ber(d_{TV}(\mu, \nu))$ independent

- If $U = 0$ set $X' = Y' = W_1$

If $U = 1$ set $X' = W_2$, $Y' = W_3$

$$\mathbb{P}[X' = Y'] = \mathbb{P}[U = 0] = 1 - d_{TV}(\mu, \nu)$$

$$\mathbb{P}[X' = k] = \mathbb{P}[U = 0, W_1 = k]$$

$$+ \mathbb{P}[U = 1, W_2 = k]$$

$$= \min(\mu(k), \nu(k)) + \max(\mu(k) - \nu(k), 0)$$

$$= \mu(k)$$

For two measures on a Polish space

μ singular w.r.t ν

$\Leftrightarrow$ $d_{TV}(\mu, \nu) = 1$

$\Leftrightarrow$ $\forall$ coupling $X \sim \mu$, $Y \sim \nu$: $\mathbb{P}[X = Y] = 0$.

<u>Example</u>: Perturbed lattice.