Cardy's Formula

Monday, May 1, 2017 4:38 PM

A map $\varphi : U \to \mathbb{C}$, $U \subset \mathbb{C}$ is conformal if φ is complex differentiable and $\varphi' \neq 0$.

- Preserves angles

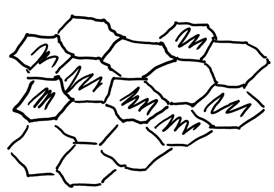
<u>Riemann Mapping Theorem</u>

If $U, U' \subsetneq \mathbb{C}$, simply connected open domains, then $\exists\ \varphi : U \to U'$ conformal map.

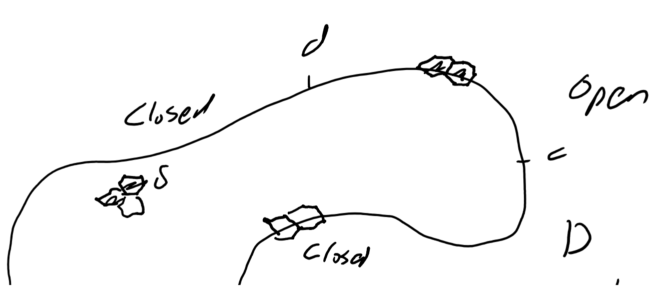
<u>Critical Crossing probabilities</u>

Hexagonal Lattice $\mathbb{H}$

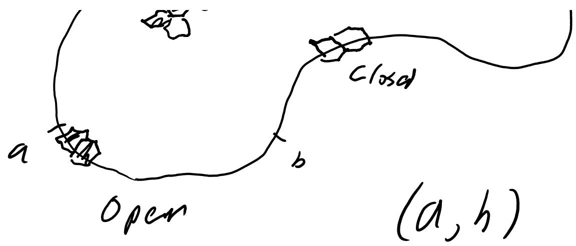
- Site percolation
  - $p = \frac{1}{2}$ critical



Rescale $\delta \mathbb{H}$



D domain, simply connected



(a,b)

D domain, simply connected
$a,b,c,d \in \partial D$ in
anti-clockwise order

— $C(D, a,b,c,d, \delta) = \mathbb{P}[$ open crossing from (a,b) to (c,d) $]$

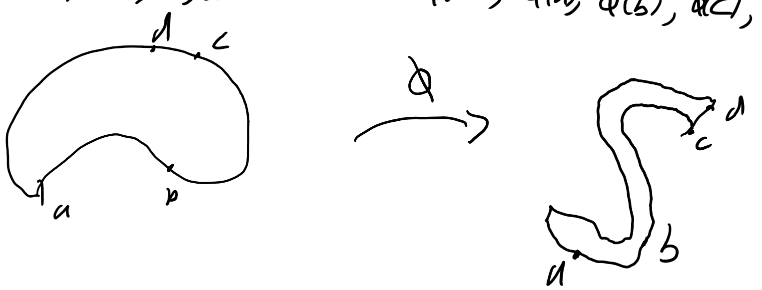
Smirnov's Theorem (Cardy's Formula)

The limit

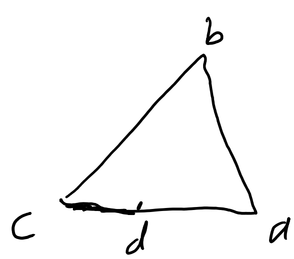
$$\lim_{\delta \to 0} \mathbb{P}[C(D,a,b,c,d,\delta)] = P_{D,a,b,c,d} \text{ exists,}$$

is conformally invariant so if $\phi : D \to \mathbb{C}$
is a conformal map

$$P_{D,a,b,c,d} = P_{\phi(D),\, \phi(a),\, \phi(b),\, \phi(c),\, \phi(d)}.$$



If $D = T$ is equilateral triangle, side length 1,
with corners a, b, c



$$P_{T,a,b,c,d} = |d-c|$$

We say

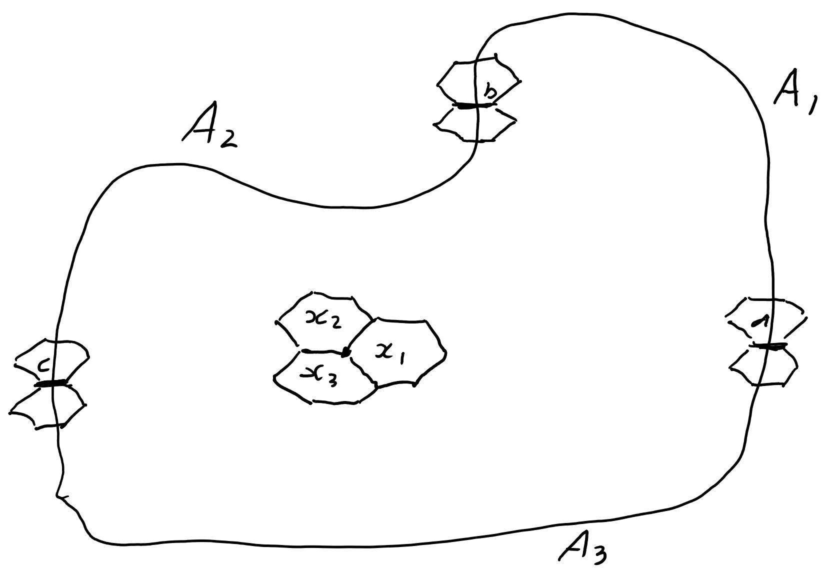
$x \in D$
$A \subset \partial D$

$x \longleftrightarrow A$ if open path linking x to A.

$x \longleftrightarrow_c A$ if closed " " " " "

Colour - Switching Lemma

Let D domain, $a, b, c \in \partial_\delta D$
edges joining boundary rectangles.



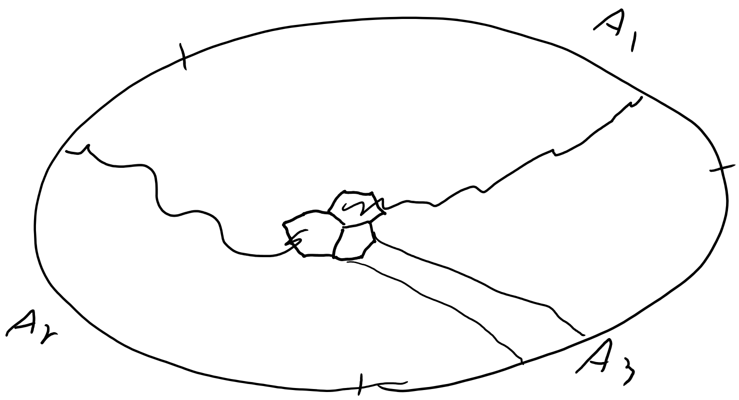
x_1, x_2, x_3 hexagons in $D \setminus \partial D$
meeting at a point

$B_j = \{ x_j \longleftrightarrow A_j \}$ $\quad R_j = \{ x_j \longleftrightarrow_c A_j \}$

Then $\mathbb{P}[B_1 \circ B_2 \circ R_3] = \mathbb{P}[B_1 \circ R_2 \circ B_3] = \mathbb{P}[R_1 \circ B_2 \circ B_3]$

(where $X \circ Y$ means events happen "disjointly")

- Note also $= \mathbb{P}[R_1, R_2, B_3] = \ldots$ since $p = 1/2$.



Proof: WLOG enough to prove

$$\mathbb{P}[B_1 \circ R_2 \circ R_3] = \mathbb{P}[B_1 \circ R_2 \circ B_3]$$

$$\Longleftrightarrow \mathbb{P}[B_1 \circ R_2 \circ R_3 \mid B_1 \wedge R_2] = \mathbb{P}[B_1 \circ R_2 \circ R_3 \mid B_1 \wedge R_2]$$

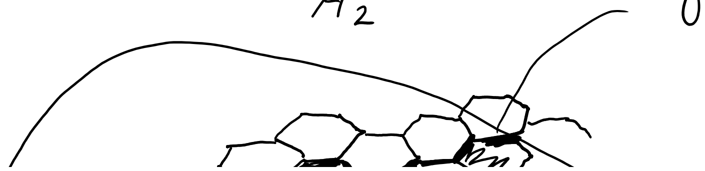
On the event $B_1 \wedge R_2$, x_1 is open

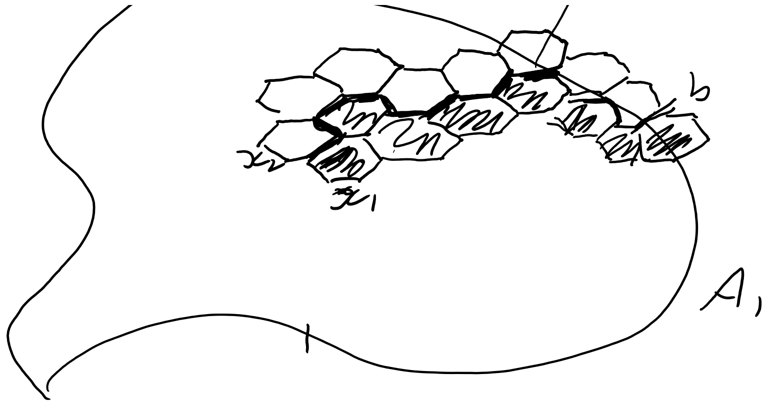
x_2 is closed.

Temporarily set edges in A_1 open, A_2 closed

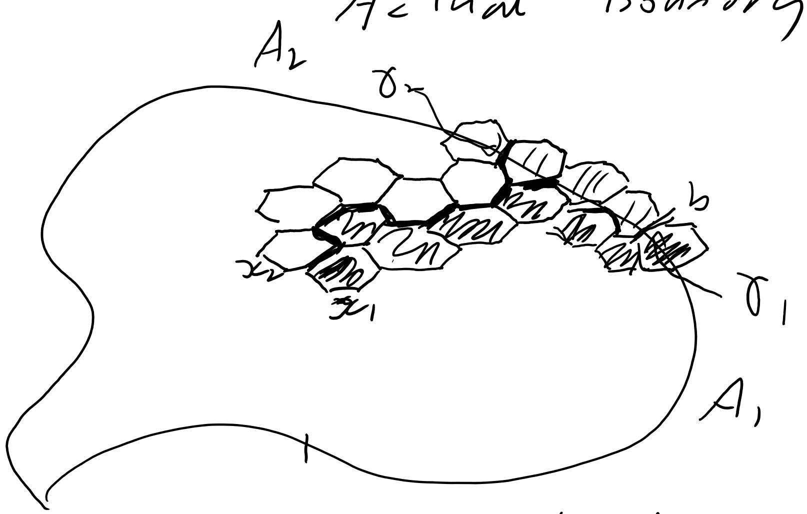
Explore the interface γ starting from

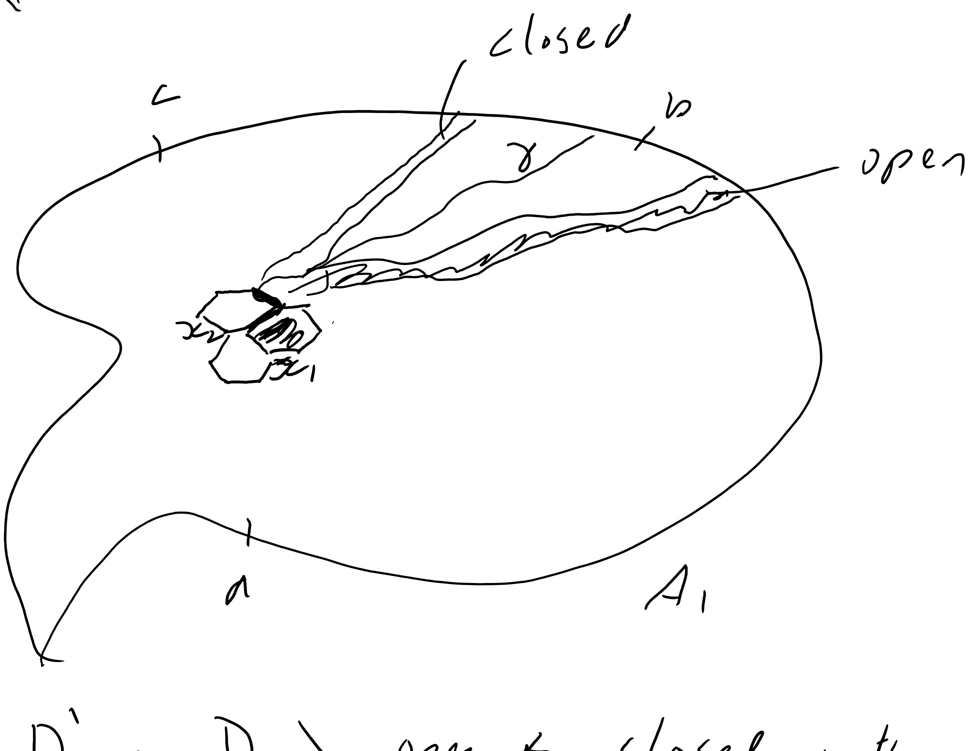
z - edge between x_1, x_2





Actual Boundry





$D' = D \setminus$ open & closed paths closest to γ

$$\mathbb{P}[B_1 \circ R_2 \circ B_3 \mid B_1 \cap R_2, D']$$

$$= \mathbb{P}[x_3 \longleftrightarrow A_3 \text{ in } D']$$

$$\parallel$$

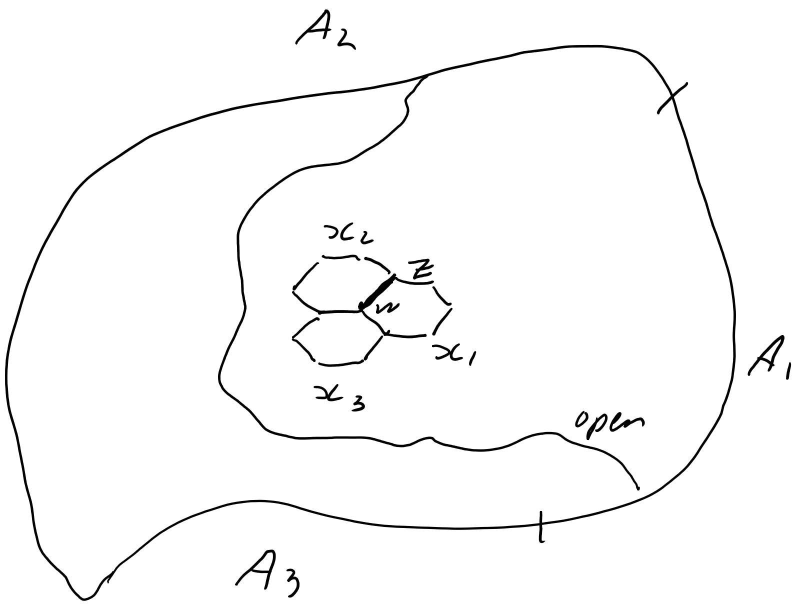
$$\mathbb{P}[B_1 \circ R_2 \circ R_3 \mid B_1 \cap R_2, D']$$

$$= \mathbb{P}[x_3 \longleftrightarrow_c A_3 \text{ in } D']$$

$\square$

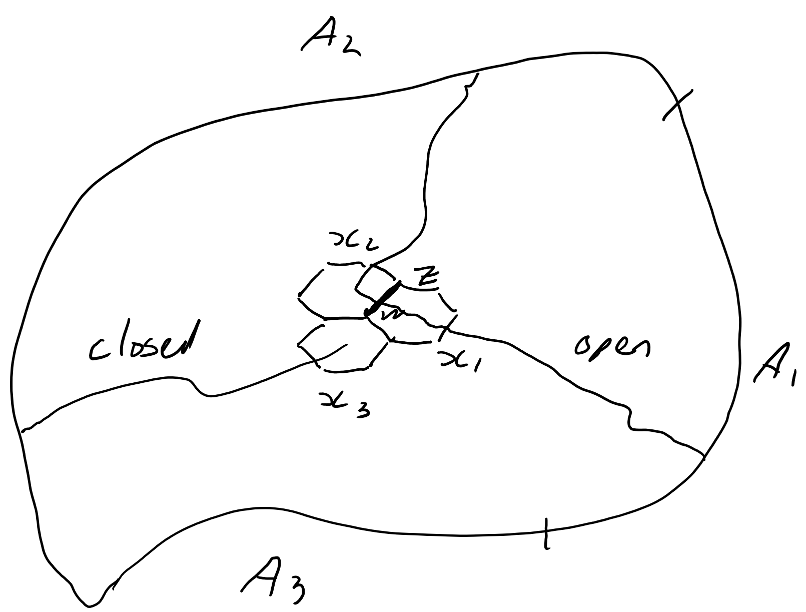
Creating a harmonic function.

Let w be point at the centre of x_1, x_2, x_3



$S^3(w) = \{$simple open path A_1 to A_2 separating w from A_3

<u>Event $S^3(z) \setminus S^3(w)$</u> $= B_1 \circ B_2 \circ R_3$

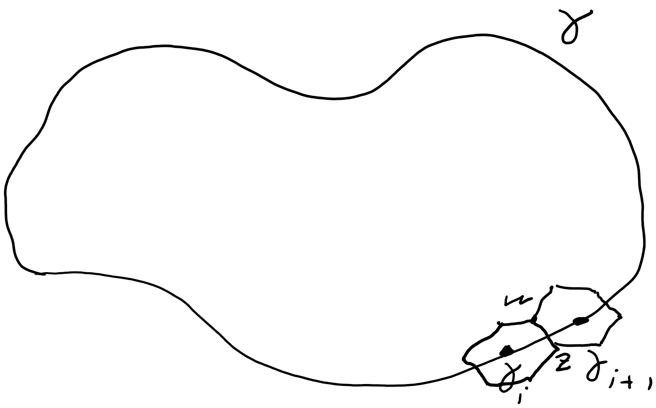


$B_1 \circ B_2 \circ R_3 \subseteq S^3(z) \setminus S^3(w)$ Clear

- $S^3(z) \setminus S^3(w) \subseteq B_1 \circ B_2$ since path separating z & A_3 must go through zw since it does not go separate w.

- x_3 closed
- component of x_3 must touch A_3

Contour Integral



$\gamma = \gamma_0, \dots, \gamma_e$ $\quad \gamma_0 = \gamma_e$ - centers of hexagons.

$[w, z]$ primal edge

$w = \rho \cdot \frac{\gamma_{i+1} - \gamma_i}{2} + \frac{\gamma_{i+1} + \gamma_i}{2}$ $\quad \rho = i/\sqrt{3}$

$w \in \text{int}\,\gamma.$

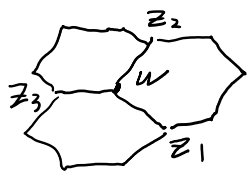
$$\oint_\gamma \varphi := \sum_i (\gamma_{i+1} - \gamma_i)\, \varphi\left(\rho \frac{\gamma_{i+1} - \gamma_i}{2} + \frac{\gamma_{i+1} + \gamma_i}{2}\right)$$

$$= \sum_{\substack{w \sim z \\ w \in \text{int}(\gamma) \\ z \notin \text{int}(\gamma)}} \rho^{-1} (w - z)\, \varphi(w)$$

Set $H_j(z) = \mathbb{P}[S^j(z)]$

$$P_j(z,w) = \mathbb{P}[S^j(z) \setminus S^j(w)]$$
$$= \mathbb{P}[S^j(z)] - \mathbb{P}[S^j(z) \cap S^j(w)]$$

Colour switch $\Rightarrow$



$$P_j(z_i, w) = P_{j+1}(z_{i+1}, w)$$
$$= P_{j+2}(z_{i+2}, w) :$$

indicies mod 3

$$P_j(z,w) - P_j(w,z) = H_j(z) - H_j(w)$$

$$\tau = e^{\frac{2\pi i}{3}}$$

$$H = \tau H_1 + \tau^2 H_2 + \tau^3 H_3$$
$$F = H_1 + H_2 + H_3$$

Goal: Use Morera's Theorem to show that H, F converge to holomorphic functions.

Morera: If $\forall \gamma$, $\oint_\gamma \phi = 0$ then ϕ is holomorphic.

Lemma: If γ has length L,

$$\left|\oint H\right|, \left|\oint F\right| \leq L\, \mathbb{P}[x \leftrightarrow B_n(x)] \to 0$$

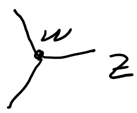
where $n = \delta^{-1} \operatorname{rad}(D)$.

$r(D)$ inf max $d(z, a_i)$

where $n = \dots$

$$\mathrm{rad}(D) = \inf_{z \in D} \max_{j \in \{1,2,3\}} d(z, A_j)$$

Proof:



$$\sum_{z \sim w} (w - z)\, \phi(w) = (1 + \tau + \tau^2)(w - z)\, \phi(w)$$
$$= 0$$

$I \in \mathrm{Int}\, \gamma$,

$$0 = \sum_{z \sim w \in I} (w - z)\, \phi(w) = \sum_{\substack{z \sim w \\ z, w \in I}} (w - z)\, \phi(w)$$
$$+ \sum_{w \in I,\, z \notin I} (w - z)\, \phi(w)$$

$$= \frac{1}{2} \sum_{z, w \in I} (w - z)\big(\phi(w) - \phi(z)\big) + \rho \oint_\gamma \phi$$

If $\phi = H_j$ then

$$\rho \oint_\gamma \phi = \frac{1}{2} \sum_{w, z \in I} (w - z)\big(P_j(z, w) - P_j(w, z)\big)$$

$$= \sum_{w, z \in I} (w - z)\, P_j(z, w)$$

$$= \sum_{w \in I} \sum_{z \in I} (w - z)\, P_j(z, w) \quad (*)$$

$$- \sum_{I \ni w \sim z \notin I} (w - z)\, P_j(z, w)$$

$\dots \frac{3}{2} H \dots H - \frac{3}{2} \tau^j H_j$

$I \ni w \sim z \notin I$

Take $\varphi = F = \sum_{i=1}^{3} H_i$ or $\varphi = H = \sum_{j=1}^{3} \tau^j H_j$

Then (#) cancels

Case 1 $\varphi = F$

$$\sum_{z \sim w} (w - z) \sum_{j=1}^{3} P_j(z, w) = (w - z_3) \sum_{j,k} \tau^k P_j(z_k, w)$$

$$= (1 + \tau + \tau^2) P_1(z_k, w) = 0$$

Case 2 $\varphi = H$

$$\sum_{z \sim w} (w - z) \sum_{j=1}^{3} \tau^j P_j(z, w) = (w - z_3) \sum_{j,k} \tau^{j+k} P_j(z_k, w)$$

$$= (1 + \tau + \tau^2) P_1(z_k, w) = 0$$

So $\rho \oint F = \sum_{I \ni w \sim z \in I^c} (z - w) \sum_j P_j(z, w)$

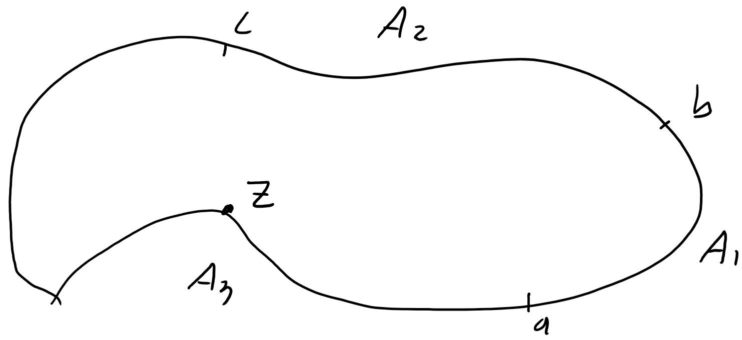
And $P_j(z, w) \Rightarrow$ connection from around w to each A_j

so $P_j(z, w) \le \mathrm{rad}(D)$.

Let $(H_{\delta_i}, F_{\delta_i}) \to (h, f)$ be a subsequential limit.

– Exists by Arzelà - Ascoli (+ regularity of H_j).

$\oint_\gamma H \,,\ \oint_\gamma F \to 0$ so h, f holomorphic.



If $z \in A_3$ then $H_3(z) = \mathbb{P}[\,z$ separated from A_1 by open path$\,]$

$= 0$

$$H_1(z) + H_2(z) = \mathbb{P}[\,(z,a) \longleftrightarrow (b,c)\,] + \mathbb{P}[\,(c,z) \longleftrightarrow (a,b)\,]$$

$$= \mathbb{P}[\,(z,a) \longleftrightarrow (b,c)\,] + \mathbb{P}[\,(c,z) \overset{c}{\longleftrightarrow} (a,b)\,]$$ since $p = \frac{1}{2}$

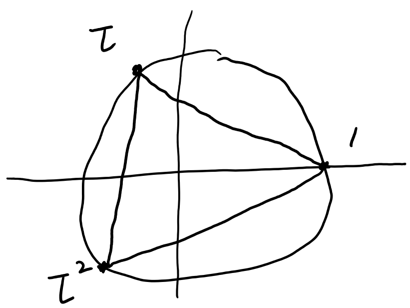
$= 1$ since complementary events

So $F = H_1 + H_2 + H_3 = 1$ on A_3.

$\Rightarrow f$ is holomorphic $f|_{\partial D} \equiv 1$.

$\Rightarrow f \equiv 1$.

Let T triangle



Since $H_1 + H_2 + H_3 = 1$

$h(z) = \tau H_1 + \tau^2 H_2 + \tau^3 H_3 \in T.$

convex combination of the corners.

Riemann Mapping Theorem

$\Rightarrow \exists\ \psi : T \to D$ bijective conformal map

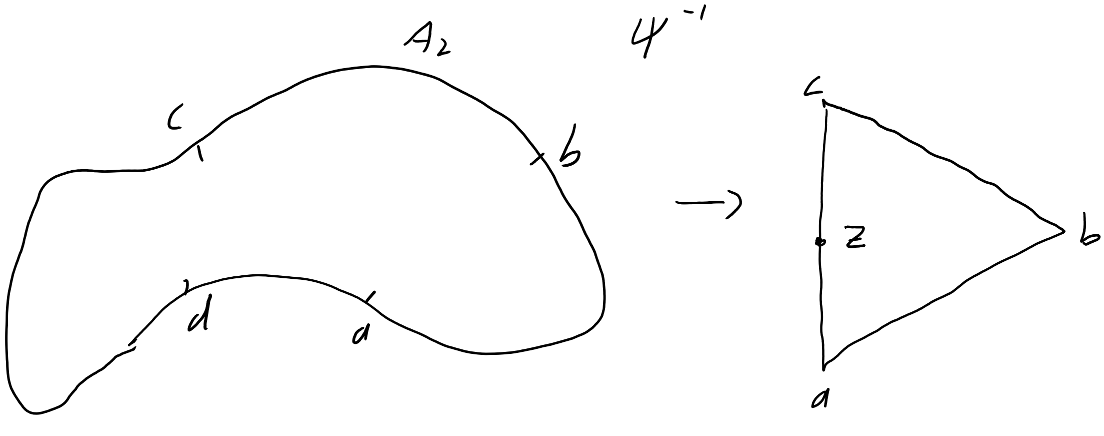
$\psi(1) = b, \quad \psi(\tau) = c, \quad \psi(\tau^2) = a.$

Then $g = h \circ \psi$ maps T to T.

- $g(\tau^j) = \tau^j$.
- g maps (τ^j, τ^{j+1}) to itself

$\Rightarrow g$ is the identity

Limiting map $h(z) = \psi^{-1}(z)$



For $d \in A_3 = (a, c)$

$$\mathbb{P}[(a,b) \longleftrightarrow (c,d)] = H_2(d)$$

$$\psi^{-1}(d) = \tau H_1(d) + \tau^2 H_2(d)$$

$$= \tau \quad + \tau(\tau - 1) H_2(d)$$

Conformally invariant!

$$z = \frac{|a-z|}{|a-c|}\tau + \frac{|c-z|}{|a-c|}\tau^2$$

So $\quad H_2(d) = \dfrac{|c-z|}{|c-a|}$.