Radio Channel Assignments

February 7, 2000

Contents

List of Figures

List of Tables

1 Introduction

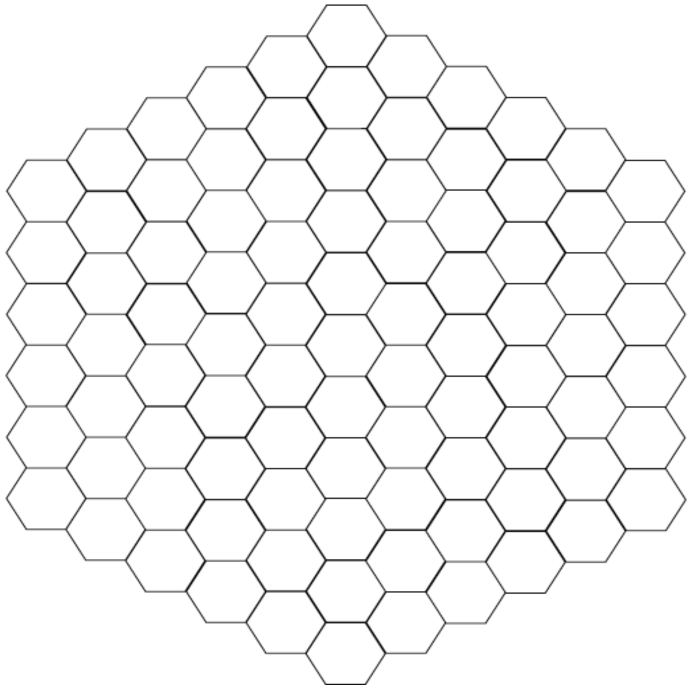

Figure 1: Planar grid of regular hexagons.

The problem of channel assignments has become increasingly relevant today due to the surge in wireless communications in all facets of life. With the ability to connect to the internet via cellular phone and with advanced technologies in wireless data transfer, the importance of an unhindered connection is more crucial than in the previous days of phone conversations and walkie-talkies. What was once an annoying static could now be a complete corruption of uncountable amounts of data. With this in mind, the ability to manage a network of radio frequencies without interference is clearly a necessity. Unfortunately, nature has limited the width of frequencies that machines can safely transmit (biologically), and so the management of these complex networks has come to include the problem of maximizing the number of possible frequencies while still allowing a large enough gap to prevent interference.

This of course would be an easy task (mathematically) if it were not for the ever-plaguing existence of friction. The strength of a signal slowly dies as the signal increases in distance from the source, and this allows frequencies to be used more than once over certain distances and makes wireless communication an easier task (realistically). Fortunate for the mathematician, this presents a new and interesting combinatorial problem: squeezing the maximum number of frequencies into a worldwide network with just enough space between transmitters so as to avoid any interference.

For this paper, we consider the generalized case of assigning frequencies to transmitters placed uniformly in a hexagonal grid. Each hexagon is given a positive integer value that corresponds to a given frequency, and the goal is to optimize the width of frequencies used for various situations. Hence, given initial constraints, we aim to cover any hexagonal region with positive integers so that the maximum integer used remains as small as possible. Since natural boundaries, changes in altitude, and even the curvature of the Earth aid in weakening radio signals (thus making distances of transmission smaller), we chose to stick to a 2-dimensional model(as in Figure 1) so as to create an effective upper bound for the real-life interpretation of this problem. This way we create a minimal upper bound for any region, given initial constraints on frequency distribution and signal strength.

The paper will be organized as follows: an introduction followed by a statement of the problem, definitions, the solutions, and a conclusion. The appendix contains proofs of theorems and lemmas we considered too lengthy for the body of the paper.

2 The Problem

Let s denote the length of a side of one of the hexagons in Figure 1. Then the distance from the center of one hexagon to the center of an adjacent hexagon is $s\sqrt{3}$.

For this paper, we concentrate on two levels of interference, which give us two respective constraints for the problem.

- If two transmitters are within a distance of $2s$ of each other, then their channels must differ by at least k_1.
- If two transmitters are within a distance of $4s$ of each other, then their channels must differ by at least k_2,

Under these constraints, what is the minimum width of the interval in the frequency spectrum that is needed to assign channels? We acheive this by a concept of a span. Span is the minimum of the largest channel assignments that satisfy the constraints. Lastly, it is not required to use every channel smaller than the span when determining the assignment.

3 Definitions

First, we will need to define several of the terms that we will be using in our paper. The following definitions are necessary for consistency throughout the paper.

Definition 1. *Let a* **region** *be a collection of hexagons, finite or otherwise.*

Definition 2. *Let u and v be hexagons in a region $\mathcal{X}$. Let* **D(u,v)** *be the minimum number of hexagons (including the beginning but not including the ending) that one must pass through in order to move from u to v in region $\mathcal{X}$. Let $D(u,u) = 0$.*

So, for example, the stipulation in the problem that any two different transmitters (in hexagons u and v respectively) are within distance $2s$ is equivalent to saying $D(u,v) \leq 1$. Similarly, it is worth noting that two hexagons are within distance $4s$ if and only if $D(u,v) \leq 2$.

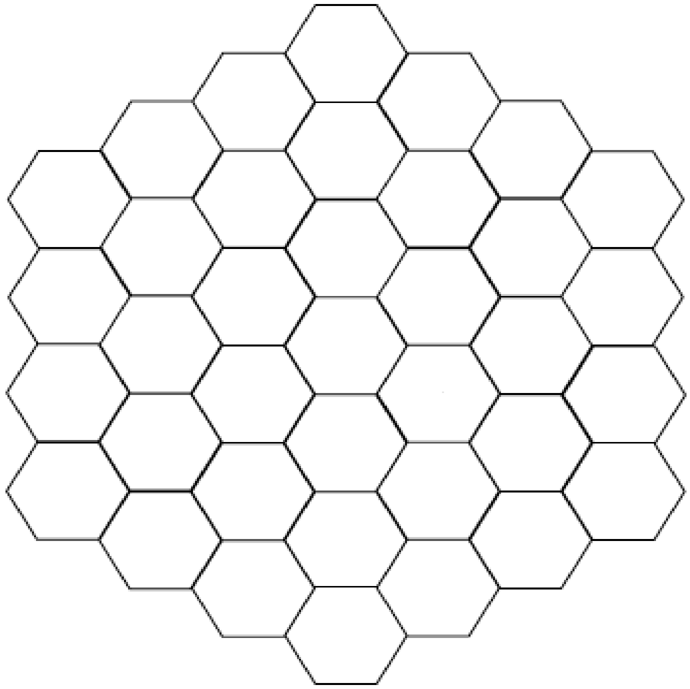

Figure 2: Region T

Definition 3. *Define T to be the the portion of a plane which is one hexagon, u, along with all hexagons, v such that $D(u, v) \leq 3$ (see Figure 2).*

Definition 4. *Define R to be an arbitrary planar hexagonal grid which contains T.*

Definition 5. *Let k_i be the minimum allowed difference in channels of two hexagons, u and v in a region R, with $D(u, v) = i$.*

For example, if $k_1 = 2$ and $k_2 = 1$, then any two transmitters in hexagons u and v which are adjacent to each other must have channels that differ by at least 2. If the two transmitters in hexagons u and v are two hexagons apart(i.e., $D(u, v) = 2$), then their channels must not be the same.

Definition 6. *Let C be a function of the hexagons in a region, R, to the positive integers. Given a set of constraints, call C a* **channel assignment** *to R under those constraints if C maps the hexagons to an allowed set of frequencies. Then, let $C(u)$ denote the frequency of the hexagon, u.*

Definition 7. *The width of the interval of the frequency spectrum in region R is the largest channel used. The minimum width over all channel assignments of a region R is called a span.*

Definition 8. *Let the function $S(l_1, l_2, \dots, l_n)$ of a region R be the span under the restrictions that $k_i = l_i$ for all i from 1 to n.*

Definition 9. *For a given $k_1 \geq 4$, define the set $\mathcal{N}_k = \{1, 2, 3, k+3, k+4, k+5, 2k+5, 2k+6, 2k+7\}$ as the channel assignment set. That is, for a region R, $C(R) \subseteq \mathcal{N}_k$.*

With these definitions, we are now able to provide our results.

4 Solution

In this section, we are concerned with planar regions that expand out in every direction infinitely, or that are finite. First we are going to need to prove some general results.

Lemma 1. *Let M be any positive integer. If $S(k_1, k_2, \ldots, k_n) = L$ then*

$$S(k_1, k_2, \ldots, k_{i-1}, k_i + M, k_{i+1}, \ldots k_n) \leq L + M(\lceil \frac{L}{k_i} \rceil - 1)$$

Proof. Let C_1 be an assignment of channels on the region R with span L and satisfying the given constraints. We will construct an assignment on the region which satisfies the new constraints, with the desired largest channel used. To do so, define a new channel arrangement, C_2 as follows:

$$C_2(u) = C_1(u) + M(\lceil \frac{C_1(u)}{k_i} \rceil - 1).$$

To see that the new set of constraints are satisfied, notice that

$$|C_2(u) - C_2(v)| \geq |C_1(u) - C_1(v)|$$

and so all the constraints for k_j, $j \neq i$ are still satisfied. Furthermore, if

$$|C_1(u) - C_1(v)| \geq k_i$$

then

$$|C_2(u) - C_2(v)| = |C_1(u) - C_1(v)| + M.$$

This is because if $|C_1(u) - C_1(v)| \geq k_i$ then $\lceil \frac{C_1(u)}{k_i} \rceil \neq \lceil \frac{C_1(v)}{k_i} \rceil$. This demonstrates that our constraint for the new value of k_i is now satisfied. Thus, the only channels that are used are of the form

$$C_1(u) + M(\lceil \frac{C_1(u)}{k_i} \rceil - 1) \leq L + M(\lceil \frac{L}{k_i} \rceil - 1)$$

Therefore, the channel assignment we have constructed is valid, and furthermore we have shown that

$$S(k_1, k_2 \ldots k_{i-1}, k_i + M, k_{i+1}, \ldots k_n) \leq L + M(\lceil \frac{L}{k_i} \rceil - 1)$$

as desired. □

Lemma 2. *On region any R containing T (see Figure 2), $S(4, 1) > 14$.*

The proof of this Lemma is given in Appendix A.

Lemma 3. *On region any R containing T, $S(2, 1) > 8$ and $S(3, 1) > 11$.*

Proof. If $S(3,1) = L \leq 11$ then by Lemma 1 we know that

$$S(4,1) \leq L + \lceil \frac{L}{3} \rceil - 1 \leq 11 + \lceil \frac{11}{3} \rceil - 1 = 14$$

which is a contradiction to Lemma 2. Similarly, if $S(2,1) = L \leq 8$ then by Lemma 1

$$S(3,1) \leq L + \lceil \frac{L}{2} \rceil - 1 \leq 8 + \lceil \frac{8}{2} \rceil - 1 = 11$$

violates the first half of our lemma. □

Lemma 4. *If $l > 4$ then for region T, $S(l,1) > (2l+6)$.*

The proof for this lemma is in Appendix B. It should be noted that proving this lower bound for T also works for R, in particularly this works for infinite and finite region (as in Figure 1).

4.1 $k_2 = 1$

For any two hexagons u and v, if $D(u,v) = 1$ then their channels differ by at least k ($k_1 = k$) for any positive integer k, and $k_2 = 1$. With this generalization, we would like to see how the span relates to k_1.

Lemma 5. *For $k_1 \geq 4$, a width of the interval of the frequency spectrum in region R is less than or equal to $2k_1 + 7$.*

Proof. We will prove by induction. We will also use the set defined in Definition 9. First we will show that $k_1 = 4$ satisfies Lemma 5. If $k_1 = 4$, then for all u, v such that $D(u,v) \leq 1$ we know that $|C(u) - C(v)| \geq 4$ by definition. By Lemma 2 we know that $S(4,1) > 14$. To see that for $k_1 = 4$ a frequency width is $2(4) + 7 = 15$, use the channel assignment set $\mathcal{N}_4 = \{1,2,3,7,8,9,13,14,15\}$. As shown in Figure 3, the channel assignment set satisfies the constraints. Also, by further examination of Figure 3, we can see that the channel assignments tessellate, and the resulting tessellated pattern always meets the constraints. To see this, translate the channel assignments in figure 3 from A to B. After translation we have a repeated pattern with no gaps, and the constraints still hold. Now instead translate from A to C, and again we have a repeated pattern with no gaps while keeping to all constraints. Since these are the only two possible kinds of translation, we have shown that the pattern is a tessellation. We are able to tessellate this pattern to cover all of region R. Since the maximum channel assigned is 15, the width of the frequency spectrum is $15 = 2(4) + 7$.

Next, let k be any integer such that $k \geq 4$, assume that Lemma 5 holds true for $k_1 = k$ with channel assignment set $\mathcal{N}_k$. This generates a tessellation as illustrated in Figure 4. It is easy to see that this pattern tessellates and meets the constraints.

Now we need to prove that Lemma 5 holds for $k_1 = k + 1$. To do this we need to generate a tessellation pattern from $\mathcal{N}_{k+1}$ in region R that satisfies the constraints. From our hypothesis that Lemma 5 holds for k, we can replace all k with $k + 1$ in Figure 4. The results is the tessellation in Figure 5. It is clear that in Figure 5 the tessellation meets all the constraints.

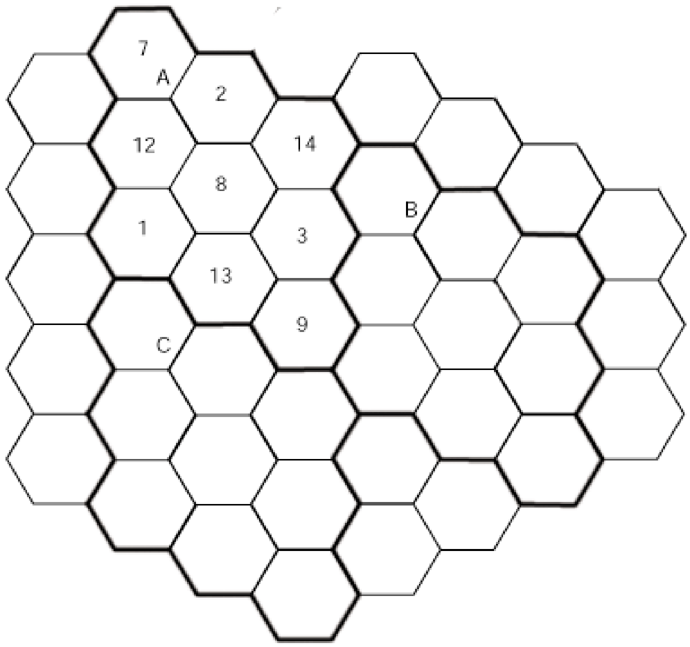

Figure 3: Channel Assignment for $k_1 = 4$.

Also, we can see that the maximum frequency used is $2k + 9 = 2(k + 1) + 7$. That is, for $k + 1$ the width of the interval of the frequency spectrum is $2(k + 1) + 7$. Since this matches our inductive hypothesis for $k + 1$ we have proven the lemma. $\square$

Lemma 5 is a very nice result. We now have a way of constructing a tessellation under the constraints that $k_1 \geq 4$ and $k_2 = 1$, and we can make this pattern using the channel assignment set $\mathcal{N}_{k_1}$. Most importantly, we can assign the channels with a frequency width of $2k_1 + 7$. Next we will prove that this width is a lower bound for any $k_1 \geq 4$.

Theorem 1. *Let $k_1 \geq 4$, then $S(k_1, 1)$ of a region R is $2k_1 + 7$.*

We have decided to insert the proof in the Appendix C.

With Theorem 1 and Lemma 5 we know how to form a repeating pattern for the given constraints, and we also know the span over the region R. A very nice outcome from these results is that for **any** $k_1 \geq 4$, we can choose nine connected hexagons and produce a channel assignment with $S(k_1, 1) = 2k_1 + 7$. By looking at Figure 4 we can see that for large k values we are going to have a larger spread in frequencies. That is, for larger k_1 we will have a more efficient system of transmitters in terms of interference because the frequency width is large. Its now time to attend to the two cases we have not talked about, we will look at $k_1 = 2$ and $k_1 = 3$.

Theorem 2. *For $k_1 = 2$ the channel assignment set is $\mathcal{N}_2 = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ with $S(2, 1) = 9$. For $k_1 = 3$ the channel assignment set is $\mathcal{N}_3 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ with $S(3, 1) = 12$.*

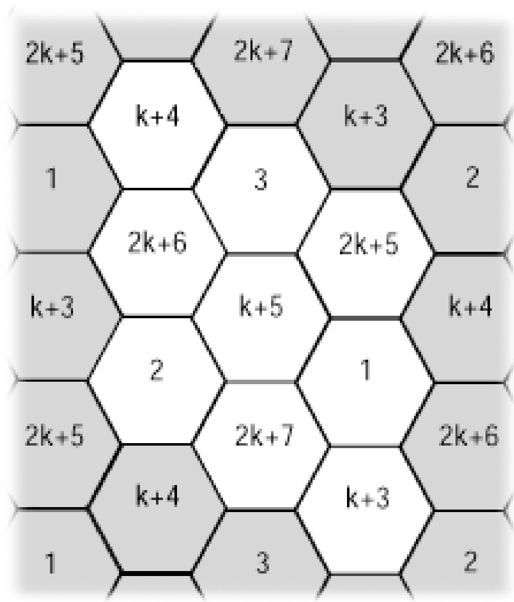

Figure 4: Channel Assignment for $k_1 = k$.

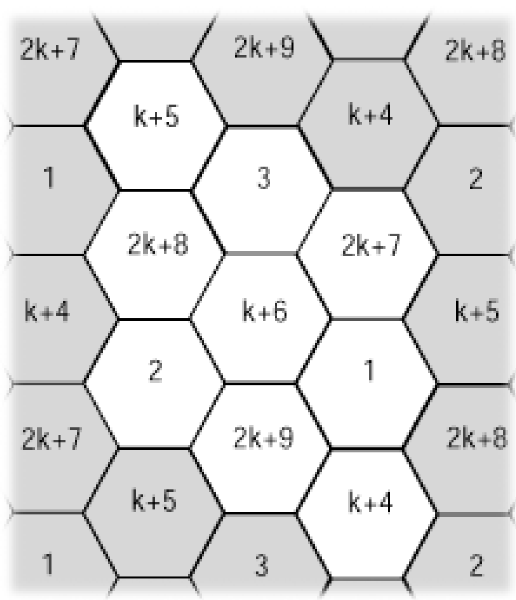

Figure 5: Channel Assignment for $k_1 = k + 1$.

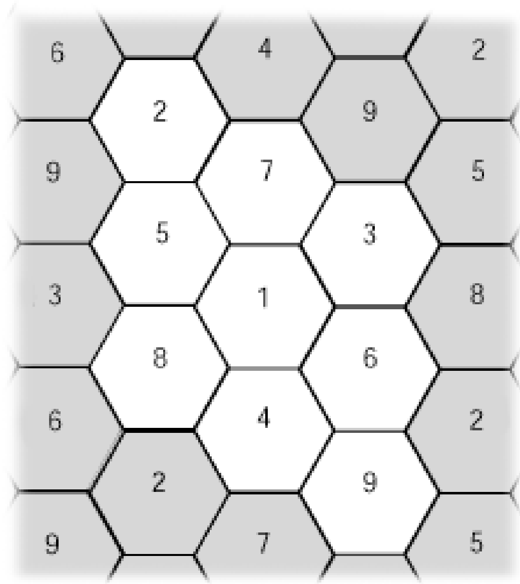

Figure 6: Channel Assignment for $k_1 = 2$.

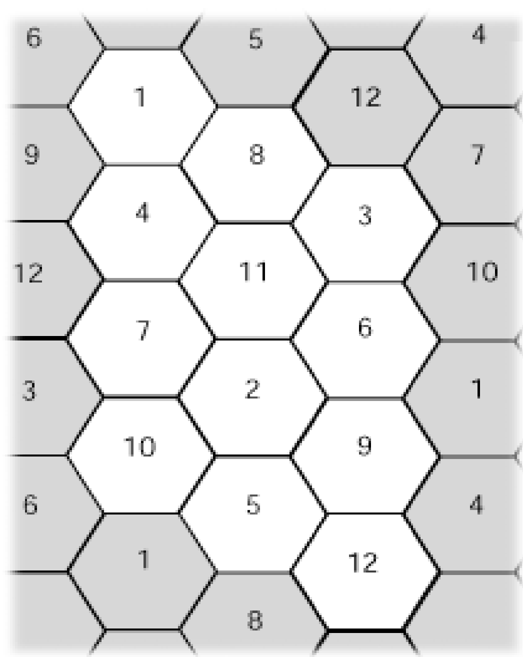

Figure 7: Channel Assignment for $k_1 = 3$.

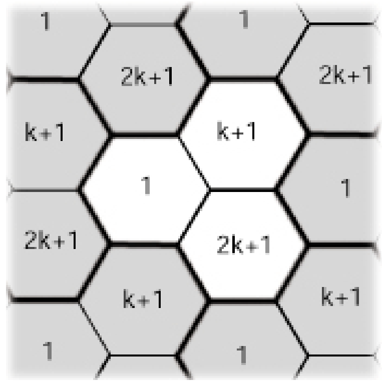

Figure 8: Channel Assignment for $k_1 = k$ and $k_2 = 0$.

Proof. By Lemma 3 we know that $S(2,1) > 8$ and $S(3,1) > 11$. In Figure 6 we have constructed a tessellation pattern for $k_1 = 2$ with channel assignment set as C_2. By inspection, $S(2,1) = 9$, the lowest possible value. For $k_1 = 3$ we have a similar argument, only we use the channel assignment set C_3. By inspection of Figure 7, $S(3,1) = 12$, the lowest possible value. □

4.2 $k_1 = k$ and $k_2 = 0$

For this case, we just made a channel assignment by inspection. This can be seen in Figure 8. The values in the illustration meet the constraints clearly. Therefore the span over a region R for this case is $2k+1$. To see this, try for $k-1$, then the channel assignment set is $\{1, k, 2k-1\}$, but k and 1 must be at least k apart. Hence $2k+1$ is the span.

4.3 $k_1 = k_2 = k$

Again, we have developed a general channel assignment by inspection. This is illustrated in Figure 9 where all values meet the constraint. Hence, the span over a region R is $6k+1$. To see this, as above try for $k-1$, then we have a contradiction in Figure 8 with the hexagon containing 1 and $(k-1)+1 = k$. Therefore $2k+1$ is the span.

4.4 General Case

For this section we have that k_1 and k_2 can be any positive integer.

Theorem 3. *Given a region, R, that contains region T. As long as $k_1 \geq 4k_2$ then*

A) *if k_2 divides k_1 then $S(k_1, k_2) = 2k_1 + 6k_2 + 1$,*

B) *if $k_1 > 6k_2 + 1$ then $S(k_1, k_2) = 2k_1 + 6k_2 + 1$.*

The proof of this theorem is inserted in Appendix D.

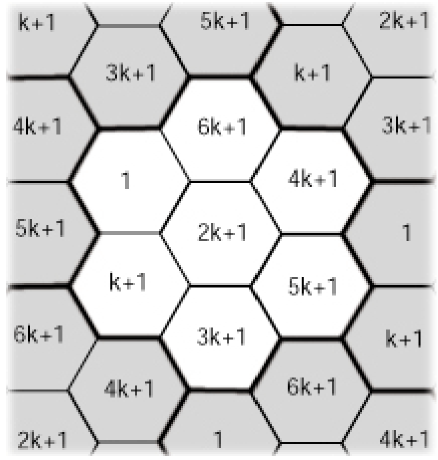

Figure 9: Channel Assignment for $k_1 = k_2 = k$

Theorem 4. *Let* $3k_2 \le k_1 \le 4k_2$*. Then*

$$S(k_1, k_2) \le 3k_1 + 2k_2 + 1.$$

Proof. We will prove by construction. Consider the tiling given in Figure 10. As long as $3k_2 \le k_1 \le 4k_2$, the channel assignment holds. Then by construction,

$$S(k_1, k_2) \le 3k_1 + 2k_2 + 1.$$

As shown by the hi-lighted tiles in Figure 10, this tiling only works if $2k_2 + 1$ and $k_1 + 1$ differ by at least k_2(by definition of k_2). It follows that

$$\begin{aligned}
(k_1 + 1) - (2k_2 + 1) &\ge k_2, \\
k_1 - 2k_2 &\ge k_2, \\
k_1 &\ge 3k_2.
\end{aligned}$$

Yet we know from Theorem 3 that for $k_1 \ge 4k_2$ we have a strict lower bound, therefore we must have a strict upper bound, that is

$$k_1 \le 4k_2.$$

Hence we have that if $3k_2 \le k_1 \le 4k_2$ then $S(k_1, k_2) \le 3k_1 + 2k_2 + 1$. □

5 Conclusion

The results are summarized in Tables 1 and 2. In Table 1 we have tabulated some of the results that we have proven, as well as other cases that we have proven but whose proofs have not been included in this report. For the cases $k_2 = 2$, and k_1 equal to 9, 11 or 13, we were unable to determine $S(k_1, k_2)$. However, we were able to find bounds for those values by Lemma 2. We

Table 1: Compilation of Spans for Different k_1 and k_2 Values

k_1	k_2	$S(k_1, k_2)$
1	1	7
2	1	9
3	1	12
4	1	15
5	1	17
$l > 5$	1	2l + 7
2	2	13
3	2	17
4	2	17
5	2	21
6	2	23
7	2	26
8	2	29
9	2	30, 31 or 32
10	2	33
11	2	34, 35 or 36
12	2	37
13	2	39 or 40
$l > 13$	2	2l + 13

Table 2: General Results for k_1 and k_2 Values

Constraints	Span
Any k_1 , $k_2 = 0$	$2k_1 + 1$
$k_1 = k_2$	$6k_1 + 1$
$k_1 = 2$, $k_2 = 1$	9
$k_1 = 3$, $k_2 = 1$	12
$k_1 \geq 4$, $k_2 = 1$	$2k_1 + 7$
$k_1 \geq 4k_2$	$\leq 2k_1 + 6k_2 + 1$
$3k_2 \leq k_1 \leq 4k_2$	$\leq 3k_1 + 2k_2 + 1$
$k_1 > 4$, $k_2 = 1$	$> 2k_1 + 6$
$3k_2 \geq 2k_1$	$4k_1 + 3k_2$

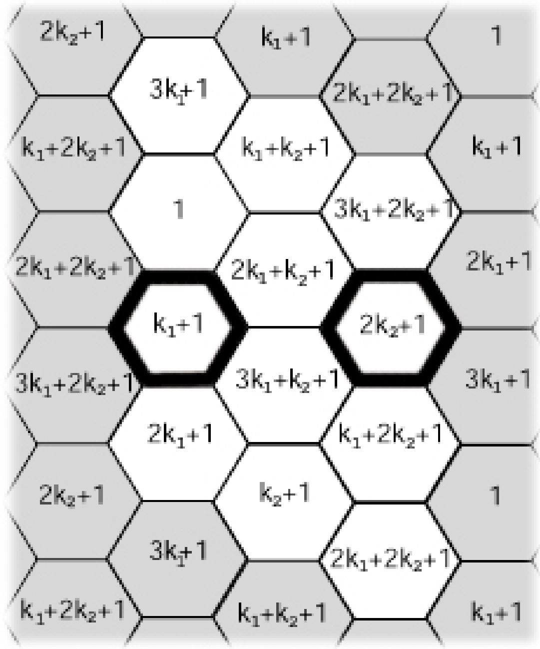

Figure 10: Channel Assignment for Theorem 4.

have proven that $S(9,2) = 30, 31$ or 32, $S(11,2) = 34, 35$ or 36 and lastly that $S(13,2) = 39$ or 40.

In Table 2 we have our general results. It should be noted that the last row in Table 2 was not proven by us; Mark Shepherd [1] proved this in his thesis. For selected values of k_1 and k_2 we have proven that the span over an arbitrarily sized planar hexagonal region which includes T. In general, for all combinations we can find a pattern that repeats, that is we can find a tessellation of frequencies. This is a major result because we know how to construct a frequency assignment based on the values of k_1 and k_2 through a simple formula, as shown in Figure 4 for $k_1 \geq 4$ and $k_2 = 1$.

A Proof of Lemma 2

Lemma 6. *On region ,R, which contains T (see Figure 2),* $S(4,1) > 14$.

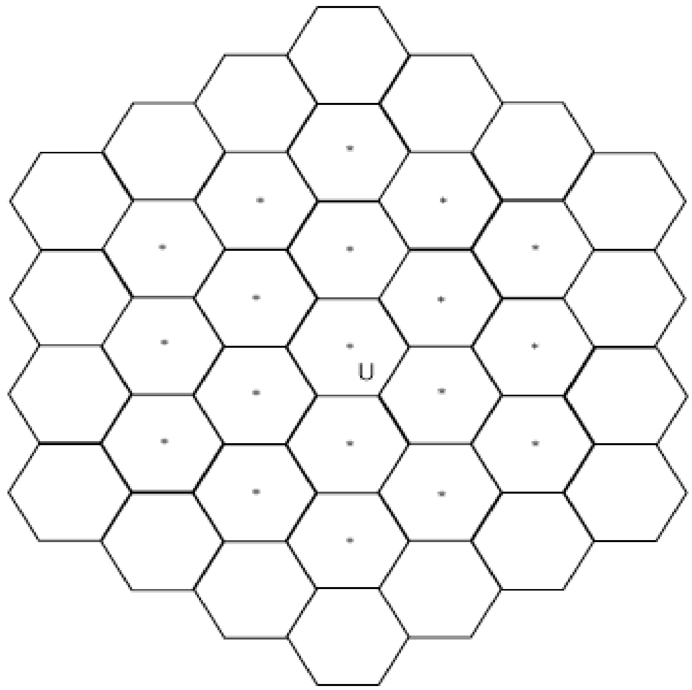

Figure 11: Region R for Lemma 2

Proof. For this proof, we will be need to be able to succinctly refer to several hexagons. Thus, we will star about half the hexagons as shown in figure 11. We will prove this by showing that there can not be a channel assignment which has a largest frequency used ≤ 14. We will do so by method of contradiction. We will assume that there is a channel assignment obeying constraints $k_1 = 4$ and $k_2 = 1$ with largest channel ≤ 14. Next, we will show that none of the starred hexagons can have a frequency of four. Then, by the same method, we will show there are other frequencies that cannot be used, and finally we will be able to conclude that there is no such channel assignment.

First, assume there one of the starred hexagons has frequency $= 4$. Then, the 6 hexagons adjacent to it have to have frequencies between 8 and 14 inclusive, and at most one of those frequencies may not be used. However, none of the hexagons can have frequency 12 or 13. This is because with either frequency, there is only one legal frequency for a hexagon to have which is adjacent to both that hexagon and the hexagon of frequency 4 (frequency 8 and 9 respectively). Therefore, one cannot assign frequencies to the 6 surrounding hexagons if one of the starred hexagons has frequency 4. This also implies that none of the starred hexagons can have frequency 11. If there were, then you could change the channel assignments of the hexagons by the map $C(u) \to (15 - C(u))$ and obtain a legal channel assignment with one of the starred hexagons with channel assignment 4.

If one of the starred hexagons had frequency 5, then the 6 surrounding hexagons would have to have frequencies from the set $\{1, 9, 10, 12, 13, 14\}$ (there cannot be one of channel 11 by our above argument). However, if one of those has channel 12, then there is only one legal channel assignment for the two hexagons which are adjacent to that one and the one of frequency 5. Thus, the channels cannot be assigned legally, and so none of the starred hexagons can have a frequency 5, or consequently $15 - 5 = 10$. The exact same logic shows that none of the starred hexagons can have channel assignment of 6 or 9.

This only allows a channel assignment set of $\{1, 2, 3, 7, 8, 12, 13, 14\}$ to the starred hexagons. However, no hexagon which is u, or is adjacent to u, can have channel assignment set of $\{1, 2, 3, 12, 13, 14\}$ because if it did, then there would not be six more legal channel assignments for the other starred hexagons which are adjacent to that one. This leaves us with the only legal channel assignment for the hexagons adjacent to to u are 7 and 8. However, such an assignment is not possible since each hexagon has to have a different channel assignment. Thus, this shows that there cannot be a legal channel assignment to the region T, and consequently to any region containing T, with $k_1 = 4$ and $k_2 = 1$.

$\square$

B Proof for Lemma 4

Lemma 7. *If $l > 4$ then for region T, $S(l, 1) > (2l + 6)$.*

Proof. We will prove this by showing that there can not be a channel assignment which has a largest frequency used of $\leq 2l + 6$. We will do so by method of contradiction. We will assume that there is such a channel assignment. Then, we will show that none of the starred hexagons (see Figure 11) can have a frequency of four. We will extend this idea and then show that none of the starred hexagons can have channels assignments from $[4, \dots, l + 2]$ or from $[l + 5, \dots, 2l + 3]$.

First, assume that one of the starred hexagons, w, has frequency $= 4$. There cannot be any hexagons adjacent to that one with channel assignment between $l+6$ and $2l+4$. If there were a hexagon, v, with such a channel assignment, then it would not be possible to assign channels to the two hexagons which are adjacent to both v and w without going over our allowed largest frequency of $2l + 6$. However, this only leaves three allowed frequencies, so it cannot be done.

This exact same argument shows that none of the starred hexagons can have a frequency of anything from $[4, \dots, l]$. Now, assume one of the starred hexagons, w, has frequency $l + 1$. Of the hexagons bordering w, can have frequency 1, and at most three of the other ones can have frequencies between $2l+1$ and $3l$ (since two of those channels cannot be adjacent to each other). That means that there have to be two hexagons adjacent to w which have channels larger than $3l$. The smallest such channel that can be used is $3l + 1$, and so there has to be a hexagon with channel at least $3l + 2$ used. However, since $l \geq 5$ this channel is larger than our allowed maximum of $2l + 7$.

Similarly, if there is a hexagon, w which has a channel of $l + 2$, then of the six adjacent hexagons, two can have channels of either 1 or 2, and three more can have channels between $2l + 2$ and $3l + 1$. This still leaves one hexagon which must have a channel which is at least $3l + 2$ which is not allowed. Thus, we have shown that none of the starred hexagons can have a channel which is from $[4, \dots, l + 2]$. This also shows that none of the starred hexagons can have a channel assignment from $[l+5, \dots, 2l+3]$. This is because if there were such a hexagon, then we could remap the channels of all the hexagons by mapping channel $C \to (2l + 7 - C)$ and then we would have a hexagon with an unallowed channel assignment.

Thus, now we have shown that the only allowed frequencies for the starred hexagons are 1, 2, 3, $l+3$, $l+4$, $2l+4$, $2l+5$, and $2l+6$. Now, assume for the sake of contradiction that either hexagon u or a hexagon that is adjacent to has channel 2. If that is the case, then none of the hexagons which are adjacent to it can have channel 1, 2, or 3 since adjacent channels have to differ by at least l. However, this leaves only five different channels for the 6 adjacent hexagons which is not sufficient since $k_2 = 1$ forces those hexagons to have different channels. The exact same argument shows that neither u or any of the hexagons adjacent to u can have channel 1, 2, 3, $2l + 4$, $2l + 5$ or $2l + 6$. However, this leaves only two allowed channel assignments ($l + 3$ and $l + 4$) for those 7 hexagons, which is not possible since they have to have distinct channels. Thus, we have shown that there cannot be a legal channel assignment to region T which satisfies $k_1 = l$ and $k_2 = 1$ and has width $\leq 2l + 6$, and hence for any region containing T. Ergo, we have shown that $S(l, 1) \geq 2l + 7$ as desired.

□

C Proof of Theorem 1

Theorem 5. *Let $k_1 \geq 4$, then $S(k_1, 1)$ of a region R is $2k_1 + 7$.*

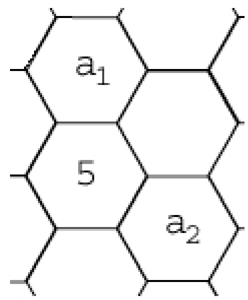

Figure 12: For a_1 not adjacent to a_2 but both are adjacent to 5.

Proof. We will prove this by contradiction. First, we have shown from Lemma 2 and from the proof of Lemma 5 that $2k_1 + 7$ is the span for $k_1 = 4$ and $k_2 = 1$. Then, for this we proof will prove for $k_1 \geq 5$.

Let l be the smallest number such that there is a channel assignment with $k_1 = l$ and $k_2 = 1$ such that $S(k_1, 1) \leq 2l + 6$. Assume that five is a channel assignment. Then let a_1 and a_2 be the smallest and next smallest channel adjacent to five respectively. We know then that $a_1 \geq l + 5$ and $a_2 \geq l + 6$.

Case 1

If a_1 is not adjacent to a_2, as in Figure 12, then there has to be two channels larger than a_1 next to a_2, that is $2l + 6$ and $2l + 7$. But this contradicts our assumption that $S(l, 1) = 2l + 6$.

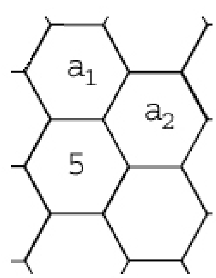

Figure 13: For a_1 adjacent to a_2 with both adjacent to 5.

Case 2

Let a_1 be adjacent to a_2, as in Figure 13. Since a_2 is adjacent to a_1, we know that the channel in a_2 is at least $2l+5$. However, there are still four other hexagons which are adjacent to the one with channel five which have to have larger channels that a_2's. Thus, one of those has to have a channel larger than $2l+6$. That would contradict our assumption that $S(l,1)=2l+6$. Thus, if there is such a channel assignment, then there cannot be any hexagon with channel 5.

Let a_3, a_4 and a_5 be as illustrated in Figure 13. Then we have that $a_3 \geq 2l+5$ which implies that $a_4 \geq 3l+5$ and $a_5 \geq 2l+6$. This contradicts our assumption that $S(l,1)=2l+6$.

We have shown that there cannot be a channel assignment of five, which was also illustrated in Lemma 2. This also implies that there cannot be a channel assignment of $2l+2$, since if there is a channel assignment of $2l+2$ then we can come up with a new channel assignment by changing the channel used by each hexagon according to the map $C \to 2l+7-C$. Thus, if there is a channel assignment which has a channel of $2l+2$, then you can construct a channel assignment which uses channel five which we showed is impossible.

Define a channel assignment with $k_1 = l-1$ by changing the channels by the following mapping

$$x \to \begin{cases} x & : \quad x \in [1,\ldots,4] \\ x-1 & : \quad x \in [5,\ldots,2l+2] \\ x-2 & : \quad x \in [2l+2,\ldots,2l+6]. \end{cases}$$

To finish our induction, we need to show that this assignment of the channels satisfies the constraints $k_1 = l-1$ and $k_2 = 1$. Let $S_1 = [1,\ldots,4]$, $S_2 = [6,\ldots,2l+1]$ and $S_3 = [2l+3,\ldots,2l+6]$. It is easy to see that $k_2 = 1$ is still satisfied because if two hexagons had different channels assignments originally, they still have different channels. This is because the only way for u and v to go from different assignments to the same assignment would be if their channels differed by one, and one was subtracted from the channel of one of the hexagons, but not the other. This would imply they were in different S sets, but then their original channels can not have differed by exactly one.

Now we will show that the new bound for $k_1 = l-1$ is satisfied. If you examine two adjacent hexagons, u and v, then we know that their original assignments differed by at least l. If they differed by at least $l+1$ then we know that our new bound of $k_1 = l-1$ is still satisfied since the difference in the amount their channels were changed by is at most 2. If the channels of u and v differed by exactly l, then it is not possible for one of their channels to have originally been in S_1 and the other originally been in S_3 (since the smallest difference that can be obtained from one element of S_3 and one element of S_1 is $(2l+3)-4 = 2l-1 > l$.) Therefore, the difference in the amounts that the channel assignments changed by is at most 1, and therefore, our new value of $k_1 = l-1$ is valid. Therefore $S(l-1,1) = 2l+6-2 = 2(l-1)+6$ which is a contradiction by our choice of l. □

D Proof of Theorem 3

Theorem 6. *Given a region, R, that contains region T. As long as $k_1 \geq 4k_2$ then*

A) *if k_2 divides k_1 then $S(k_1, k_2) = 2k_1 + 6k_2 + 1$,*

B) *if $k_1 > 6k_2 + 1$ then $S(k_1, k_2) = 2k_1 + 6k_2 + 1$.*

Proof. To prove part A, we will first construct a channel assignment on R with that span, then show that no better channel assignment can be made. Let $\frac{k_1}{k_2} = l$. We know from Theorem 1 that $S(l, 1) = 2l + 7$. Let C_1 be a channel assignment which demonstrates that span is achievable. In order to get show our new span is attainable, on the region R, define a new channel assignment, C_2 by $C_2(u) = (k_2)(C_1(u) - 1) + 1$. The largest channel that this new assignment uses is $(k_2)(2l+7-1)+1 = 2k_2l+6k_2+1 = 2k_1+6k_2+1$ as desired. If two hexagons, u and v, are adjacent, then their channel assignments from C_1 differed by at least l, and so their new channel assignments differ by at least $lk_2 = k_1$. Similarly, if $D(u, v) = 2$, then we know that they had different channel assignments in C_1, and so in C_2 their channel assignments differ by at least k_2. Thus, this channel assignment shows that $S(k_1, k_2) \leq 2k_1 + 6k_2 + 1$.

Now, assume for the sake of contraction, there is a channel assignment, C_2, on R that obeys those constraints, but has no channel $\geq 2k_1 + 6k_2 + 1$ used. Now, define a new channel arrangement, C_1 by $C_1(u) = \lceil \frac{C_2(u)}{k_2} \rceil$. Let u and v be two hexagons. Without loss of generality, assume that $C_1(u) \geq C_2(v)$. We know that if $D(u, v) = 1$ then $C_2(u) - C_2(v) \geq k_1$. This implies that

$$C_1(u) - C_1(v) = \frac{C_2(u)}{k_2} - \frac{C_2(v)}{k_2} \geq \frac{k_1}{k_2} = l.$$

Similarly, if $D(u, v) = 2$, then $C_2(u) - C_2(v) \geq k_2$. Which means that

$$C_1(u) - C_2(v) = \frac{C_2(u)}{k_2} - \frac{C_2(v)}{k_2} \geq \frac{k_2}{k_2} = 1.$$

Thus, we have shown that C_1 is a legal channel assignment with no adjacent hexagons having channels differing by less than l and no hexagons with $D(u, v) = 2$ having the same channel. However, the largest channel used here is $\lceil \frac{2k_1+6k_2}{k_2} \rceil = 2l+6$, and this is a contradiction according to Theorem 1 which showed $S(l, 1) = 2l + 7$. Thus, we have shown that if k_2 divides k_1 then $S(k_1, k_2) = 2k_1 + 6k_2 + 1$ as wanted.

Part B follows easily from Part A. First, let $l = \lfloor \frac{k_1}{k_2} \rfloor$, and $m = k_1 - lk_2$. Then, apply Lemma 1 with $M = m$ to our solution in Part A, and, using the fact that $k_1 > 6k_2 + 1$, we can obtain

$$\begin{aligned} S(k_1, k_2) &\leq S(k_1 - m, k_2) + m(\lceil \frac{S(k_1 - m, k_2)}{k_i - m} \rceil - 1) \\ &= 2(k_1 - m) + 6k_2 + 1 + 2m + m\lceil \frac{6k_2 + 1}{k_1 - m} \rceil - 1 \\ &= 2k_1 + 6k_2 + 1. \end{aligned}$$

Now, let $m' = (l+1)k_2 - k_1$. Assume for the sake of contradiction that $S(k_1, k_2) < 2k_1 + 6k_2 + 1$. If that is true, then we can apply Theorem 1 with $M = m'$ and using the fact that $k_1 > 6k_2 + 1$ and find that

$$\begin{aligned} S(k_1 + m', k_2) &\leq S(k_1, k_2) + m'(\lceil \frac{S(k_1, k_2)}{k_1} \rceil - 1) \\ &< 2(k_1) + 6k_2 + 1 + 2m' + m' \lceil \frac{6k_2 + 1}{k_1} \rceil - 1 \\ &= 2k_1 + 6k_2 + m' + 1 \\ &= 2(k_1 + m') + 6k_2 + 1. \end{aligned}$$

which contradicts our results in part A. Thus, $S(k_1, k_2)$ is not less than $2k_1 + 6k_2 + 1$. Since we have already shown that $S(k_1, k_2) \leq 2k_1 + 6k_2 + 1$ we know have that $S(k_1, k_2) = 2k_1 + 6k_2 + 1$. Therefore, we have proven our theorem $\square$

References

[1] Shepherd, Mark *Radio Channel Assignment* (Oxford University, Ph.D. Thesis), *http://www.maths.ox.ac.uk/combinatorics/thesis.html*