Introduction to Abstract Algebra
with Applications to Social Systems

Women in Mathematics
SWIM
Summer Workshop in Mathematics
FOR HIGH SCHOOL STUDENTS
Princeton

Princeton SWIM 2010
Instructor: Taniecea A. Arceneaux
Teaching Assistants: Sarah Trebat-Leder and Amy Zhou
Communication Networks

Dominance Relations

Dominance Relation: For each pair $i, j$, with $i \neq j$, either $A_i \rightarrow A_j$ or $A_j \rightarrow A_i$, but not both; that is, in every pair of individuals, there is exactly one who is dominant.

Tournaments

$$D = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

NOT Symmetric
Communication Networks

One-stage vs. Two-stage Communication

One-Stage

\[
C = \begin{pmatrix}
0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

Two-Stage

\[
C^2 = CC = \begin{pmatrix}
0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix} \begin{pmatrix}
0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix} = \begin{pmatrix}
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

Graphical representations of the networks are shown below.
**Power**

**Dominance Matrices**

<table>
<thead>
<tr>
<th>One-Stage</th>
<th>Two-Stage</th>
</tr>
</thead>
</table>
| $D = \begin{pmatrix}
0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}$ | $D^2 = \begin{pmatrix}
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}$ |

**Power:** the total number of one-stage and two-stage dominances that an individual can exert. The power of individual $A_i$ is the sum of the entries in the $i$th row of the matrix

$$S = D + D^2$$
The results of a round-robin athletic contest are shown below. Using the power definition above, rank the four teams in terms of their athletic dominance.

Team A beats teams B and D.
Team B beats team C.
Team C beats team A.
Team D beats teams C and B.
Team A beats teams B and D.
Team B beats team C.
Team C beats team A.
Team D beats teams C and B.

\[
D = \begin{pmatrix}
  0 & 1 & 0 & 1 \\
  0 & 0 & 1 & 0 \\
  1 & 0 & 0 & 0 \\
  0 & 1 & 1 & 0
\end{pmatrix}
\]
Power

Example - Athletic Contest

One-Stage

\[ D = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \]

\[ S = D + D^2 = \begin{pmatrix} 0 & 2 & 2 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 2 & 0 \end{pmatrix} \]

Row Sum

\begin{array}{c|c}
\text{Row} & \text{Sum} \\
\hline
1 & 5 \\
2 & 2 \\
3 & 3 \\
4 & 4 \\
\end{array}

Two-Stage

\[ D^2 = \begin{pmatrix} 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \]

Ranking:
1. Team A
2. Team D
3. Team C
4. Team B
A **Markov process** is a stochastic (random) process with the following property:

The probability of any future behavior of the process depends only on the current state, not on its past behavior. (e.g., Markov property)
You are going to successively flip a quarter until the pattern HHT appears.
Markov Chains

Example - Flipping Coins

State Diagram

Transition Probability Matrix

\[
T = \begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & 0 & 0 \\
\frac{1}{2} & 0 & \frac{1}{2} & 0 \\
0 & 0 & \frac{1}{2} & \frac{1}{2} \\
0 & 0 & 0 & 1
\end{pmatrix}
\]
Markov Chains

Example - Flipping Coins

Question: On average, how many flips will be required until the pattern HHT appears?

Average number of flips required:

\[ v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} \]

We know that \( v_4 = 0 \). (Why?)

\[ v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ 0 \end{pmatrix} \]
Markov Chains

Example - Flipping Coins

**Question:** On average, how many flips will be required until the pattern HHT appears?

\[ v = 1 + \beta v \]

\[ v = 1 + \beta v = 1 + \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 1 + \begin{pmatrix} \frac{1}{2} v_1 + \frac{1}{2} v_2 \\ \frac{1}{2} v_1 + \frac{1}{2} v_3 \\ \frac{1}{2} v_3 \end{pmatrix} \]
Markov Chains

Example - Flipping Coins

Question: On average, how many flips will be required until the pattern HHT appears?

\[
\begin{pmatrix}
  v_1 \\
  v_2 \\
  v_3 \\
\end{pmatrix} = \begin{pmatrix}
  1 + \frac{1}{2} v_1 + \frac{1}{2} v_2 \\
  1 + \frac{1}{2} v_1 + \frac{1}{2} v_3 \\
  1 + \frac{1}{2} v_3 \\
\end{pmatrix}
\]

\[v_1 = 1 + \frac{1}{2} v_1 + \frac{1}{2} v_2\]
\[v_2 = 1 + \frac{1}{2} v_1 + \frac{1}{2} v_3\]
\[v_3 = 1 + \frac{1}{2} v_3\]

Solve this system of equations
Markov Chains

Example - Flipping Coins

**Question:** On average, how many flips will be required until the pattern HHT appears?

\[
\begin{pmatrix}
\nu_1 \\
\nu_2 \\
\nu_3
\end{pmatrix} =
\begin{pmatrix}
1 + \frac{1}{2} \nu_1 + \frac{1}{2} \nu_2 \\
1 + \frac{1}{2} \nu_1 + \frac{1}{2} \nu_3 \\
1 + \frac{1}{2} \nu_3
\end{pmatrix}
\]

\[
\nu_1 = 1 + \frac{1}{2} \nu_1 + \frac{1}{2} \nu_2 \\
\nu_2 = 1 + \frac{1}{2} \nu_1 + \frac{1}{2} \nu_3 \\
\nu_3 = 1 + \frac{1}{2} \nu_3
\]

\[
\nu_1 = 8 \\
\nu_2 = 6 \\
\nu_3 = 2 \\
\nu_4 = 0
\]
Markov Chains

Example - Flipping Coins

Question: In the long run, what fraction of time is spent in each state, no matter in which state the chain began at time 0?

THT HHT THT HTH TTH THT HTH THT THT THT HTH

\[ \pi^T = \pi \quad \text{Stationary Distribution} \]