# Sample Problem Solution

(a) Assume $c_1$ is rational. So we can write

$c_1 = \frac{p}{q}$ where $p$ and $q$ are integers, $0 \le p \le q$

We can use induction.

Suppose $c_n$ is rational, so $c_n = \frac{p}{q}$ for integers $p, q$ where $0 \le p \le q$

Then $c_{n+1} = \left|1 - \left|1 - \frac{2p}{q}\right|\right| = \left|1 - \left|\frac{q-2p}{q}\right|\right| = \frac{|q - |q-2p||}{q}$

If $p \le \frac{q}{2}$, then $c_{n+1} = \frac{|q-q+2p|}{q} = \frac{2p}{q} = 2c_n$

$p \to 2p$

If $p > \frac{q}{2}$, then $c_{n+1} = \frac{|q+q-2p|}{q} = \left|2 - 2\cdot\frac{p}{q}\right| = 2(1-\frac{p}{q}$

$p \to 2(q-p)$

In both cases, it is still true that $0 \le p \le q$

We rewrite the sequence as $\{c_n\} = \{\frac{p_n}{q}\}$

Integer $p_n$ only takes values $0, 1, 2, \ldots, q$

(not necessarily all values...)

Since the values of $p_n$ are limited, so are $c_n$, and at some point, the sequence will return to a $c_n$ already visited, at which point the sequence becomes periodic.

Therefore, if $c_1$ is rational, $\{c_n\}$ is periodic eventually.

Now assume $c_1$ is irrational. We need to show that $\{c_n\}$ is not periodic to finish this proof. (But I'm not sure how.)

Sample Problem Solution

(b) Examples:

$T=2: \frac{1}{5}, \frac{2}{5}, \frac{4}{5}, \frac{2}{5}, \frac{4}{5}, \ldots$

$T=3: \frac{1}{7}, \frac{2}{7}, \frac{4}{7}, \frac{6}{7}, \frac{2}{7}, \frac{4}{7}, \frac{6}{7}, \ldots$

$\frac{1}{9}, \frac{2}{9}, \frac{4}{9}, \frac{8}{9}, \frac{2}{9}, \ldots$

$T=4: \frac{1}{15}, \frac{2}{15}, \frac{4}{15}, \frac{8}{15}, \frac{14}{15}, \frac{2}{15}, \ldots$

$T=5: \frac{1}{11}, \frac{2}{11}, \frac{4}{11}, \frac{8}{11}, \frac{6}{11}, \frac{10}{11}, \frac{2}{11}, \ldots$

$T=6: \frac{1}{21}, \frac{2}{21}, \frac{4}{21}, \frac{8}{21}, \frac{16}{21}, \frac{10}{21}, \frac{20}{21}, \frac{2}{21}, \ldots$

$\vdots$

If $c_n \le \frac{1}{2}$, $c_{n+1} = 2c_n$. So there are actually <u>infinitely many $c_1$ for every $T$.</u>

For $T=2$, $c_1 = \frac{1}{5}$ or $\frac{1}{10}$ or $\frac{1}{20}$ or $\frac{1}{40}$ are all possible.

So given $c_1$, $(\frac{1}{2})^a \cdot c_1$ gives the same period sequence, (although after different thresholds N) for the infinitely many positive integers a.

Problem:

$$c_{n+1} = \left|1 - |1 - 2c_n|\right| \qquad 0 \le c_1 \le 1.$$

try ~~ex~~ examples: • $c_1 = \frac{1}{2}$ $\Rightarrow$ $|1 - 2c_1| = 0$

$\Rightarrow c_2 = 1 \Rightarrow |1 - 2c_2| = |1-2| = 1$

$\Rightarrow c_3 = 0 \Rightarrow |1 - 2c_3| = 1 \Rightarrow c_4 = 0$

$\Rightarrow$ we get

$c_n$-s are: $\frac{1}{2} \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad \ldots$

• other example: $c_1 = \frac{1}{4}$ $\Rightarrow$ $|1 - 2c_1| = \frac{1}{2}$.

$\Rightarrow c_2 = \frac{1}{2}$ $\Rightarrow$ get back to previous case!

$c_n$-s here are: $\frac{1}{4} \quad \frac{1}{2} \quad 1 \quad 0 \quad 0 \quad 0 \ldots$

• let's try $c_1 = \frac{1}{7}$ $\quad |1 - 2\frac{1}{7}| = \frac{5}{7} \Rightarrow c_2 = \frac{2}{7}$

$|1 - 2\frac{2}{7}| = \frac{3}{7} \Rightarrow c_3 = \frac{4}{7}$

$|1 - 2\frac{4}{7}| = \frac{1}{7} \Rightarrow c_4 = \frac{6}{7}$

$|1 - 2\frac{6}{7}| = \frac{5}{7} \Rightarrow c_5 = \frac{2}{7}$

$\Rightarrow$ back to earlier calculation

$c_6 = \left|1 - |1 - 2\frac{2}{7}|\right| = \frac{4}{7} \ldots$

$c_n$-s here are: $\frac{1}{7} \quad \frac{2}{7} \quad \frac{4}{7} \quad \frac{6}{7} \quad \frac{2}{7} \quad \frac{4}{7} \quad \frac{6}{7} \quad \frac{2}{7} \quad \ldots$

In all these cases: sequence becomes periodic.
This fits with what we should prove, since ~~the~~ the examples all started with rational $c_1$. So at least it is consistent.

In all cases all the $c_n$ are also in $[0,1]$, like the first $c_1$.
Is this always true?

In other words, if $x \in [0,1]$, must then necessarily follow that $|1 - |1-2x|| \in [0,1]$ ?

Yes! Proof: $x \in [0,1]$

$\Rightarrow 2x \in [0,2] \Rightarrow -2x \in [-2,0]$

$\Rightarrow 1-2x \in [-1,1]$

$\Rightarrow |1-2x| \in [0,1]$

$\Rightarrow 1 - |1-2x| \in [0,1]$

$\Rightarrow |1 - |1-2x|| = 1 - |1-2x| \in [0,1]$.

done.

Note: we didn't even need the "outer" absolute value. We can just say $c_{n+1} = 1 - |1-2c_n|$, and if $c_1 \in [0,1]$ then all $c_n \in [0,1]$ (and are positive automatically).

So: first fact: all the $c_n$ ~~are~~ always satisfy $0 \le c_n \le 1$.

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To compute $c_{n+1}$, we need to compute $|1-2c_n|$ first, which is also automatically in $[0,1]$.

$\Rightarrow$ give it a name and keep track of these?

$$c_n \longrightarrow d_n = |1-2c_n| \longrightarrow c_{n+1} = 1 - d_n.$$

in example with $c_1 = \frac{1}{7}$:

| | | | | | | |
|---|---|---|---|---|---|---|
| $c_n$ | $\frac{1}{7}$ | $\frac{2}{7}$ | $\frac{4}{7}$ | $\frac{6}{7}$ | $\frac{2}{7}$ | .. |
| $d_n$ | $\frac{5}{7}$ | $\frac{3}{7}$ | $\frac{1}{7}$ | $\frac{5}{7}$ | $\frac{3}{7}$ | .. |
| $n$ | 1 | 2 | 3 | 4 | 5 | ... |

does this help? not clear...

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let's try examples again.

we have tried $\frac{1}{2} \longrightarrow$ becomes periodic with period 1.

$\frac{1}{4} \longrightarrow$ " " " " 1.

$\frac{1}{7} \longrightarrow$ " " " " 3.

$$c_1 = \frac{1}{3} \Rightarrow |1 - 2c_1| = \frac{1}{3} \Rightarrow c_2 = \frac{2}{3}$$

$$\Rightarrow |1-2c_2| = \frac{1}{3} \Rightarrow c_3 = \frac{2}{3}.$$

$$\Rightarrow \frac{1}{3} \quad \frac{2}{3} \quad \frac{2}{3} \quad \frac{2}{3} \quad \dots$$

$\Rightarrow$ periodic with period 1.

$$c_1 = \frac{1}{5} \Rightarrow \left|1 - \frac{2}{5}\right| = \frac{3}{5} \Rightarrow c_2 = \frac{2}{5}.$$

$$|\cancel{4} - 2c_2| = \frac{1}{5} \Rightarrow c_3 = \frac{4}{5}$$

$$|1 - 2c_3| = \frac{3}{5} \Rightarrow c_4 = \frac{2}{5}$$

$$\Rightarrow \frac{1}{5} \quad \frac{2}{5} \quad \frac{4}{5} \quad \frac{2}{5} \quad \frac{4}{5} \quad \frac{2}{5} \quad \dots$$

$\Rightarrow$ periodic with period 2.

$$c_1 = \frac{1}{6} \Rightarrow. \quad |1 - 2c_1| = \frac{2}{3} \Rightarrow c_2 = \frac{1}{3}.$$

$\Rightarrow$ find cycle of $\frac{1}{3}$ again:

$$\frac{1}{6} \quad \frac{1}{3} \quad \frac{2}{3} \quad \frac{2}{3} \quad \frac{2}{3} \quad \dots$$

$\Rightarrow$ periodic with period 2.

We observe that several points can get to the same "tail". for instances $\frac{1}{2}$ and $\frac{1}{4}$, $\frac{1}{3}$ and $\frac{1}{6}$, ...

How does this work?

how can 2 different $c_1$ give the same $c_2$?

We saw that, for $c_1 \in [0,1]$, $\left|1 - |1-2c_1|\right| =$
$1 - |1-2c_1|$

so $\left|1 - |1-2c_1|\right| = \left|1 - |1-2\tilde{c}_1|\right|$

means that $1 - |1-2c_1| = 1 - |1-2\tilde{c}_1|$

or $|1-2c_1| = |1-2\tilde{c}_1|$.

If $c_1 \neq \tilde{c}_1$, then this is possible only if $1-2c_1$ and $1-2\tilde{c}_1$ have different signs.

$$\Rightarrow \quad 1-2c_1 = -(-2\tilde{c}_1+1) = 2\tilde{c}_1 - 1.$$

$$\Rightarrow \quad 2\tilde{c}_1 = 2-2c_1 \quad \text{or} \quad \tilde{c}_1 = 1-c_1.$$

So $c_1 = \frac{1}{3}$ and $c_1 = \frac{2}{3}$ must give the same "tail": we saw that.

$c_1 = \frac{1}{5}$ and $c_1 = \frac{4}{5}$ must also give the same tail?

<u>check:</u> $c_1 = \frac{1}{5} \Rightarrow c_2 = 1 - \left|1 - \frac{2}{5}\right| = 1 - \frac{3}{5} = \frac{2}{5}$

$c_3 = 1 - \left|1 - \frac{4}{5}\right| = 1 - \frac{1}{5} = \frac{4}{5}$

$c_4 = 1 - \left|1 - \frac{8}{5}\right| = 1 - \frac{3}{5} = \frac{2}{5}$

... (see also above)

$c_1 = \frac{4}{5} \Rightarrow c_2 = 1 - \left|1 - \frac{8}{5}\right| = 1 - \frac{3}{5} = \frac{2}{5}$.

$c_3 = \dots = \frac{4}{5}$

$\Rightarrow$ same tail as for $c_1 = \frac{1}{5}$.

So if $c_2 = \tilde{c}_2$ and $c_1 \neq \tilde{c}_1$, then $\tilde{c}_1 = 1 - c_1$.

What if $c_3 = \tilde{c}_3$ and $c_2 \neq \tilde{c}_2$? What are the possibilities for $\tilde{c}_1$, then?

$c_3 = \tilde{c}_3$ and $c_2 \neq \tilde{c}_2 \Rightarrow \tilde{c}_2 = 1 - c_2$. (by argument in top of page).

$$\Rightarrow 1 - |1-2\tilde{c}_1| = \tilde{c}_2 = 1 - c_2.$$

$$\Rightarrow |1-2\tilde{c}_1| = c_2 \Rightarrow$$ (continued on next page)

$\ldots \quad |1-2\tilde{c}_1| = c_2$ ~~other~~ $= 1 - |1-2c_1|$.

$\Rightarrow$ either $\quad 1-2\tilde{c}_1 = 1 - |1-2c_1| \Rightarrow 2\tilde{c}_1 = |1-2c_1|$

or $\quad 2\tilde{c}_1 - 1 = 1 - |1-2c_1| \Rightarrow 2\tilde{c}_1 = 2 - |1-2c_1|$.

for each there are <u>still</u> 2 possibilities:

$$\left\{\begin{array}{lll} 2\tilde{c}_1 = 1-2c_1 & \Rightarrow & \tilde{c}_1 = \frac{1}{2} - c_1 \\ 2\tilde{c}_1 = 2c_1 - 1 & \Rightarrow & \tilde{c}_1 = c_1 - \frac{1}{2}. \end{array}\right\}$$ depending on whether $c_1 > 1/2$ or $c_1 < 1/2$

$$\begin{array}{lll} 2\tilde{c}_1 = 2-(1-2c_1) = 1+2c_1 & \Rightarrow & \tilde{c}_1 = \frac{1}{2} + c_1 \\ 2\tilde{c}_1 = 2-(2c_1-1) = 3-2c_1 & \Rightarrow & \tilde{c}_1 = \frac{3}{2} - c_1. \end{array}$$ depending on whether $c_1 < 1/2$ or $c_1 > 1/2$.

so, if $c_1 < 1/2$, then $c_1$, $1-c_1$, $\frac{1}{2}-c_1$ and $\frac{1}{2}+c_1$ all lead to the same "tail"

if $c_1 > 1/2$, then $c_1$, $1-c_1$, $c_1 - \frac{1}{2}$ and $\frac{3}{2} - c_1$, all lead to the same "tail".

$c_1 < \frac{1}{2} \Rightarrow c_1 = \frac{1}{2} - \alpha$ for $0 < \alpha < \frac{1}{2}$.

$\Rightarrow c_1, 1-c_1, \frac{1}{2}-c_1, \frac{1}{2}+c_1$ become

$\frac{1}{2}-\alpha,\ \frac{1}{2}+\alpha,\ \alpha,\ 1-\alpha.$

$c_1 > \frac{1}{2} \Rightarrow c_1 = \frac{1}{2} + \alpha.$

$\Rightarrow c_1, 1-c_1, c_1 - \frac{1}{2}, \frac{3}{2} - c_1$ become

$\frac{1}{2}+\alpha,\ \frac{1}{2}-\alpha,\ \alpha,\ 1-\alpha.$

$\Rightarrow$ the two cases are really the same.

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<u>Example</u>: $\alpha = \frac{1}{8} \Rightarrow$ the choices $c_1 = \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}$ all lead to the same tail, namely $\ldots 000 \ldots$

these are all of the type $\frac{2l+1}{2^n}$ with $n=3$ and all possible values of $l$.

$\Rightarrow$ try other examples of this type?

ex. $c_1 = \frac{2l+1}{16}$ $\quad l = 0, \dots, 7$ $\quad$ (i.e. $\frac{1}{16}, \frac{3}{16}, \frac{5}{16}, \frac{7}{16}, \frac{9}{16}, \frac{11}{16}, \frac{13}{16}, \frac{15}{16}$)

$$\Rightarrow \quad c_2 = 1 - |1 - 2c_1| = 1 - \left|1 - \frac{2l+1}{8}\right|$$

whatever the result is, it will be
a fraction with denominator 8
$\Rightarrow$ will again lead, ultimately, to tail
$\dots 0000 \dots$

now that all fractions with denominator 16 also lead to this tail,
we can repeat the argument for $c_1 = \frac{2l+1}{32}$ : they lead to
$c_2 = 1 - |1 - 2c_1| = 1 - \left|1 - \frac{2l+1}{16}\right| = \frac{16 - |16 - (2l+1)|}{16}$ = fraction with denominator 16
$\rightarrow$ tail $\dots 0000 \dots$

we can repeat this, over and over, for all powers of 2 in denominator.

$\Rightarrow$ partial result: if $c_1 = \frac{k}{2^n}$, then the $c_n$ will eventually become all zero.

$$\Rightarrow \quad c_1 \dots c_{n-1} \, 0000 \dots$$

$\Rightarrow$ periodic tail with period 1.

Is the reverse true?

if $c_n = 0$, then $c_{n-1} = 1$ or $0$.

So if we take a $c_1$ that gives rise to a tail of zeros, and the tail begins at $c_n$, with $c_{n-1} \neq c_n$, then $c_{n-1} = 1$.
it then follows that $|1 - 2c_{n-2}| = 0$, so that $c_{n-2} = \frac{1}{2}$,
and this gives $1 - |1 - 2c_{n-3}| = \frac{1}{2}$ or $|1 - 2c_{n-3}| = \frac{1}{2}$,
so that $2c_{n-3} = \frac{1}{2}$ or $\frac{3}{2}$ $\Rightarrow$ $c_{n-3} = \frac{1}{4}$ or $\frac{3}{4}$.
We can keep going down in the index, and every time we have to divide by 2 to calculate the preceding $c_k$

$\Rightarrow$ we must get $c_1 = \frac{l}{2^{n-1}}$, $\Rightarrow$ reverse true as well.