Isomorphic Semigroups of Boolean Matrices

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Definitions

Semigroup: A set with the properties of a group but lacking identities and/or inverses

- Positive integers under addition
- Integers under multiplication
More definitions...

Isomorphism: a one-to-one and onto correspondence between two mathematical sets.

· ‘One, two three...’ and ‘Uno, dos, tres...’
3x3 boolean matrices

\[
A = \begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
1 & 1 & 1 \\
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
1 & 0 & 1 \\
1 & 1 & 0 \\
\end{bmatrix}
\]

\[
B^2 = \begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{bmatrix}
\]

\[
AB = \begin{bmatrix}
1 & 0 & 1 \\
1 & 1 & 1 \\
\end{bmatrix}
\]

\[
BA = \begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 0 \\
\end{bmatrix}
\]

6/28/10
Cayley table for $A$ and $B$

<table>
<thead>
<tr>
<th></th>
<th>$A$</th>
<th>$B$</th>
<th>$B^2$</th>
<th>$AB$</th>
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</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
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<td>$AB$</td>
<td>$B^2$</td>
<td>$AB$</td>
<td>$B^2$</td>
</tr>
</tbody>
</table>

6/28/10
More matrices

\[
\begin{align*}
C &= \begin{array}{ccc}
1 & 0 & 0 \\
1 & 1 & 0 \\
1 & 1 & 1 \\
0 & 1 & 1 \\
\end{array}
\begin{array}{cccccc}
& C & D & D^2 & CD & DC \\
C & C & CD & D^2 & DC & D^2 \\
D & DC & D^2 & D^2 & D^2 & D^2 \\
D^2 & D^2 & D^2 & D^2 & D^2 & D^2 \\
\end{array}

D &= \begin{array}{ccc}
1 & 1 & 1 \\
1 & 1 & 0 \\
\end{array}
\begin{array}{cccccc}
& CD & D^2 & D^2 & D^2 & D^2 \\
CD & D^2 & D^2 & D^2 & D^2 & D^2 \\
DC & DC & D^2 & D^2 & D^2 & D^2 \\
\end{array}
\end{align*}
\]
<table>
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<th>B^2</th>
<th>AB</th>
<th>BA</th>
<th>C</th>
<th>D</th>
<th>D^2</th>
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What is the same about these semigroups?
They’re isomorphic!

- Elements generated by A and B have the same role as those generated by C and D
- Each element in one semigroup has a corresponding element in the other

This is just one example of how isomorphism can relate to semigroups of matrices. So, we looked for other ways these concepts could be related...

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Even more definitions

(Note: not a generally accepted definition – we invented it)

· Order of a matrix: For matrix M of order \(n\),

\[ M_n = M_{n+1} \text{(where } n \text{ is minimum).} \]
Order 1

Order 2
Our theorem

- A matrix of order \( n \) generates a semigroup with exactly \( n \) elements.

- The semigroups generated by all matrices of order \( n \) are isomorphic to each other.

Proof: Take matrix \( M \) of order \( n \). Set \( m \) is defined as \( \{M, M2, \ldots, Mn\} \). This set will have \( n \) elements. Multiplication of matrices is associative, so the set is associative under multiplication. Any \( M_i \times M_j, i,j>0 \) (that is, any two members of the set multiplied) will give an already existing member of the set; if \( i+j<n \) then \( M_{i+j} \) is already in the set, and if \( i+j>n \) then \( M_{i+j} = M_n \). So set \( m \) is also closed, and therefore is a semigroup. Matrix \( N \) will generate the same set with \( N \) instead of \( M \), so the two sets are isomorphic.
Conjecture

- We conjectured that if two 3x3 matrices are of order 1, their product is also of order 1. More generally, if 3x3 matrices A, B of orders $c$, $d$ are multiplied, their product would be of order $\leq cd$. We haven’t found any counterexample so far, but we haven’t proved it either...