Application Puzzlers

SWIM Princeton 2010

Instructions

Here we present three mathematical puzzles. As part of your application to SWIM Princeton 2009 you are asked to submit your work on **two** of these three problems: Taking a Prime, Finding the Maximum, Traveling in Triangle-Towns.

These problems are meant to be quite puzzling, and may take you days, possibly spread out over a couple weeks, to solve. When you write up your work for submission, we are interested not just in the final answer, but also in seeing how you think. For example, you could show us how you worked on several special cases, and indicate how this helped you develop the intuition to tackle the more general case as stated in the problem. Then you could show us how you approached the general case, and if you reach the final answer, show us how you did so and how you proved that it was correct. Even if you can't reach the final answer, we are still interested in your submission, and you should write up your work as far as you were able to proceed.

As a guide to show you the sort of submission we are looking for, we also give a fourth sample problem, for which we provide a variety of possible answers. We hope that this will help you in the process of documenting your work.

Good luck on these problems—we hope you enjoy working on them, and we look forward to reading your submission.

Problem 1: Taking a Prime

Alice and Bob play the following game. Between them is a pile of n stones, $n \geq 2$. They play alternately and Alice plays first. On each turn, the player can take a number of stones from the remaining pile; the number of stones he/she takes must be a prime number. The player who takes the last stone or leaves only one stone remaining wins the game.

We say that a player has a winning strategy if he can win no matter how his opponent plays.

Question (a) For $n = 10$ determine which player has a winning strategy.

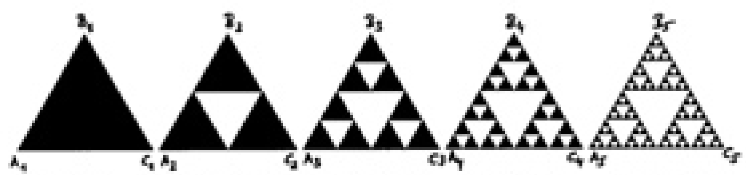
Question (b) For $n = 15$ determine which player has a winning strategy.

An important mathematical result says that for any two player game which certainly ends in finitely many steps and without the possibility of ending in a draw, one (and only one) of the two players has a winning strategy. As a consequence, we subdivide the set of all integers $n \geq 2$ into two families. We say that n belongs to Family A if Alice has a winning strategy; and we say that n belongs to Family B if Bob has a winning strategy if the initial pile consisted of n stones. (So any integer $n \geq 2$ belongs to either Family A or Family B.)

Question (c) Prove that both, Family A and Family B, consist of infinitely many integers.

Problem 2: Traveling in Triangle-Towns

The street maps of the Triangle Town are built inductively. For the triangle town T1, the street map is just one equilateral triangle with vertices A_1, B_1, C_1. In order to obtain the street map of triangle town T2, take three copies of the equilateral triangle and position the three triangles so that each triangle touches the two other triangles at a corner A_1, B_1or C_1. Call the new three outer corners A_2, B_2 and C_2. In general, in order to obtain the street map of triangle town Tn, take three copies of the street map of triangle town T(n-1) and position them so that each copy touches the other two copies at an outer corner A_{n-1}, B_{n-1} or C_{n-1}. Call the three outer corners A_n, B_n and C_n.



Map of town T1 Map of town T2 Map of town T3 Map of town T4 Map of town T5

Any edge of a triangle represents a street. All streets are one way streets and allow traffic to run only from west to east, from southwest to northeast or from southeast to northwest.

Question (a) How many ways are there to travel from A_3 to B_3 in triangle town T3?

Question (b) How many ways are there to travel from A_5 to B_5 in triangle town T5?

Question (c) How many ways are there to travel from A_n to B_n in triangle town Tn?

Problem 3: Finding the Maximum

Let a and b be two positive real numbers. We set $m(a,b) = \min\{a, \frac{1}{b}, \frac{1}{a} + b\}$ to be the minimum of the three positive real numbers a, $\frac{1}{b}$ and $\frac{1}{a} + b$. (So for example $m(3, 1.5) = \min\{3, \frac{2}{3}, \frac{11}{3}\} = \frac{2}{3}$.

Question (a) What is the maximal possible value $m(a,b)$ can assume?

Question (b) Determine all pairs of positive real numbers (a,b) for which $m(a,b)$ assumes this maximal value?

Sample Problem

To aid you in preparing your solutions to two of the above problems, we also provide several sample solutions to the following problem:

You are given a sequence of real numbers $\{c_n\}_{n=1}^{\infty}$ satisfying the relation:

$$c_{n+1} = |1 - |1 - 2c_n||$$

for all $n \geq 1$, and with the initial condition $0 \leq c_1 \leq 1$.

Question (a) Prove that if (and only if) c_1 is rational, the sequence $\{c_n\}_{n=1}^{\infty}$ becomes periodic after some entry c_N, for some N.

Question (b) Find out how many values c_1 can have such that after some term c_N (the threshold N may be different for different c_1), the sequence becomes periodic with period T. Answer this for each value of $T = 2, 3, 4, ...$

Terminology: A sequence is said to be periodic of period T after c_N, if for all $n \geq N$, $c_n = c_{n+T}$. A sequence is said to be periodic after some c_N if is is periodic of period T after c_N for some T.