Introduction to number theory and Apollonian circle packings

In this class, we will explore a problem inspired by an old Greek construction which, in its simplicity, lends itself to surprisingly many still unanswered questions. Take three circles of any size – for example, a quarter, a nickel, and a dime – and arrange them so that each circle is tangent to the two others, as in Figure 1. Next draw a larger circle around these three, so that it's tangent to each of the three original circles. What we have now are four circles, all tangent to each other, and three curvy "triangles" as gaps.

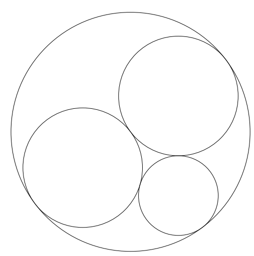

Figure 1: First generation

Our next step is to inscribe a circle in each of these triangular gaps – it turns out, due to a theorem of Apollonius, that this can be done in exactly one way, and we will see why (Figure 2). After this step, we again have many triangular gaps, which we again fill with more circles. This process can be continued indefinitely, and the resulting picture is called an Apollonian circle packing.

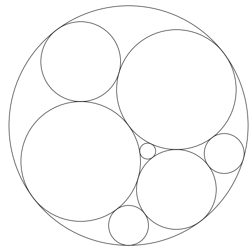

Figure 2: Second generation

While this picture is beautiful from a purely geometric standpoint, we will focus on the number theory involved in it. Namely, we will study the properties of the radii of the circles (there are infinitely many) in any given Apollonian circle packing. More concretely, we will consider the *curvatures* of the circles, which are simply the reciprocals of the radii (if a circle has radius r, its curvature is $\frac{1}{r}$). Figure 3 shows an Apollonian circle packing in which every circle is marked with its curvature.

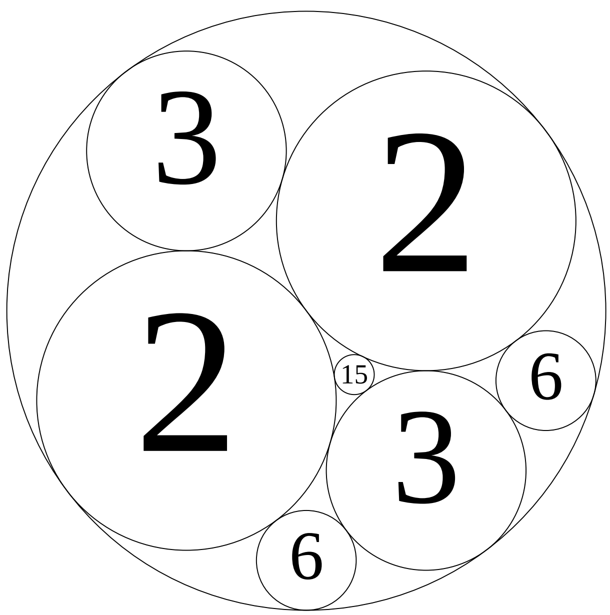

Figure 3: With curvatures

Our first question is whether we can write down a nice formulaic relation between the curvatures of any four tangent circles in the packing – it turns out we can! Descartes proved that the curvatures of any four tangent circles in the packing satisfy a simple quadratic equation, which we will derive in the course.

Armed with this formula, we will be able to attack many number-theoretic questions about Apollonian packings. Can we get a packing so that all of the circles in it have integer curvature? How many such packings are there? In these integer packings, can we find infinitely many prime curvatures? Even curvatures? While these questions are difficult to answer using geometry alone, we will approach them from many other angles, as well as experiment with the packings using a computer program called MATLAB.