Recent Progress on QUE

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In Figure 1 the domains $\Omega_E, \Omega_S, \Omega_B$ are an ellipse, a stadium and a "Barnett" billiard table respectively. Super-imposed on these are the densities of a sequence of high frequency eigenfunctions ("states", "modes") of the Laplacian. That is solutions to

$$\triangle \phi_j + \lambda_j \phi_j = 0, \; \int_\Omega \phi_j^2 \, dxdy = 1$$

$$\phi_j \big|_{\partial\Omega} = 0 \;\; \text{(Dirichlet boundary conditions)}$$

where $\triangle = \text{divgrad} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ and $\lambda_1 < \lambda_2 \leq \lambda_3 \cdots$. The sequences are of 12 consecutive modes around the 5600^{th} eigenvalue. They are ordered from left to right and then down and the grayscale represents the probability density $|\phi|^2$ with zero white and larger values darker.

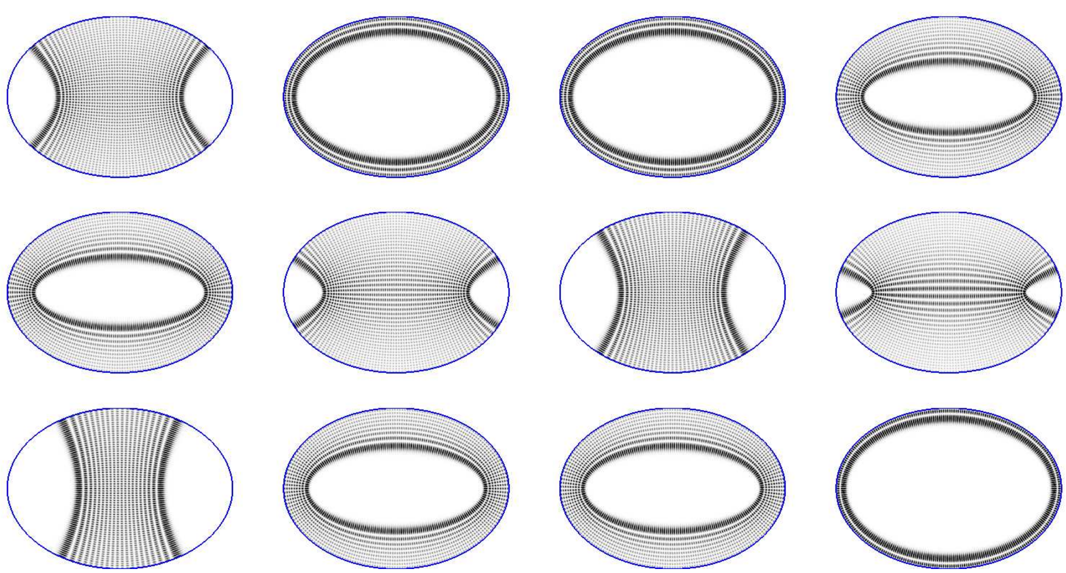

Figure 1E

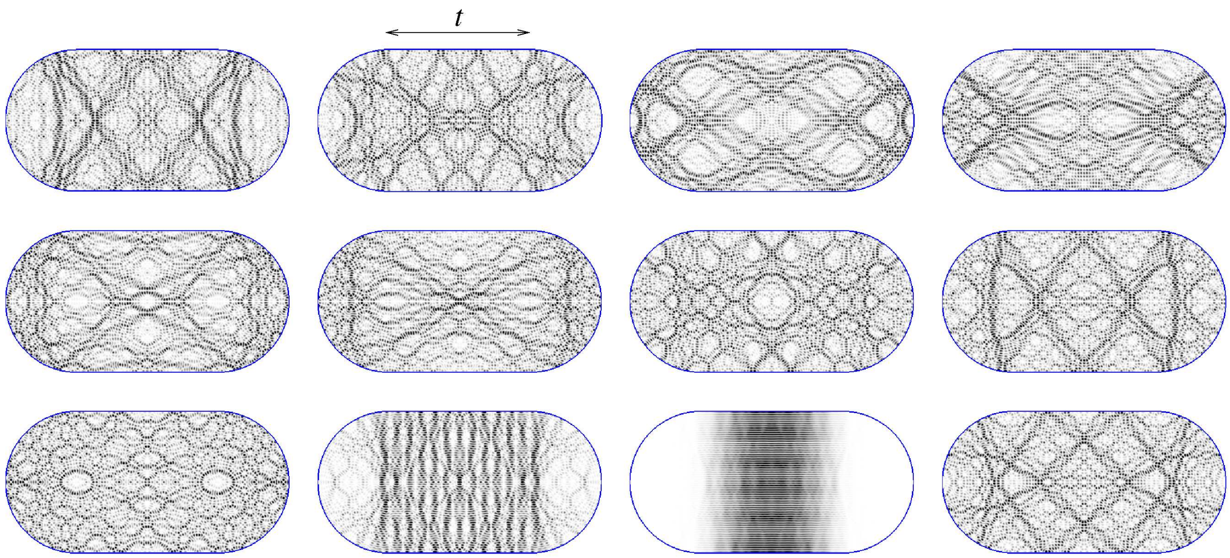

Figure 1S

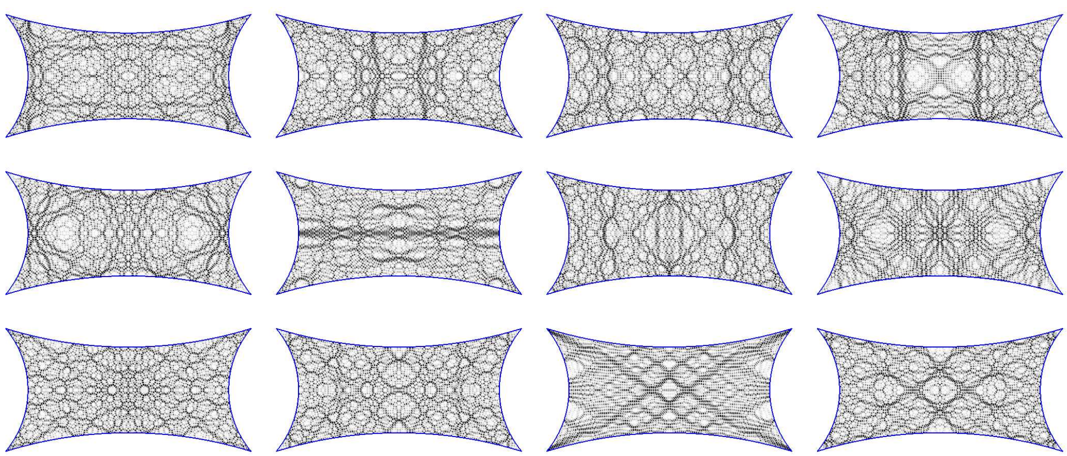

Figure 1B

The difference in the densities is striking and its source is well understood, either through the wave equation $\left(\frac{\partial^2 u}{\partial t^2} = \triangle u\right)$ on $\Omega \times \mathbb{R}$ and geometric optics or the Schrödinger equation $\left(i\,\frac{\partial u}{\partial t} = \triangle u\right)$ on $\Omega \times \mathbb{R}$ and its semiclassical analysis. Both point to the connection between high frequency states and the corresponding classical Hamiltonian dynamics which in this domain case is that of a billiard in Ω moving at unit speed and bouncing according to the law of angle of incidence equals angle of reflection. The differences between E and the other two is that for the ellipse this motion is integrable (once tangent to a confocal conic always so) while for S it is ergodic [Bu] and B being a dispersing Sinai billiard it is ergodic and strongly chaotic [Si][C-M]. In particular, for S and B almost all of the billiard trajectories are dense in the space of unit directions at the points of the domain. Moreover, these trajectories are equidistributed with respect to Liouville measure, $\mu = dxdy\,d\theta/(2\pi\text{Area }\Omega)$ where θ is the angle of the direction.

There are many questions that are asked about such high frequency eigenmodes, we focus on the most basic one concerning their distribution. The density $\nu_\phi := |\phi(x,y)|^2 dxdy$ is a probability measure on Ω which quantum mechanically is interpreted as the probability distribution "when is the state ϕ". Do these measures ν_ϕ equidistribute as $\lambda \to \infty$ or can they localize? In the case of E or more generally the quantization of any completely integrable Hamiltonian system, these measures (or rather their microlocal lifts, see below) localize on invariant tori in a well understood manner (see [La] and [CdV1] for example).[0] On the other hand for an ergodic and partially chaotic system like S or a hyperbolic and chaotic system like B, the familiar techniques from microlocal analysis (i.e., geometric optics and semiclassical analysis see for example [Mel]) say very little about individual high frequency states and a theoretical analysis is problematic.

There is a natural extension of the measures ν_ϕ to $T_1(\Omega)$, the space of unit directions over Ω, which measures their distribution in this larger "phase space". This extension is denoted by μ_ϕ and is called the microlocal lift of ν_ϕ and can be given explicitly as follows:

For a smooth function $f(x,z)$ on $T_1(\Omega) = \Omega \times S^1$ set

$$\mu_\phi(f) = \langle Op(f)\,\phi\,,\,\phi\rangle \quad \text{where}$$

$\langle\,,\,\rangle$ is the L^2 scalar product on Ω,

$$Op(f)\psi(x) = \int_{\mathbb{R}^2} e(\langle x,\,\xi\rangle)\,\hat{\psi}(\xi)\,f(x,\frac{\xi}{|\xi|})\,d\xi$$

$$\hat{\psi}(\xi) = \int_{\mathbb{R}^2} \psi(x)\,e(-\langle x,\xi\rangle)dx \quad \text{and}$$

[0]The remarkable correspondence between individual high frequency modes for an integrable system such as E and its invariant tori (which for E corresponds to confocal conics) is the reason that Bohr was able to develop quantum theory for the hydrogen atom using classical orbits. For helium he had little success as its classical mechanics is the nonintegrable and partially chaotic 3-body problem, as opposed to the integrable 2-body problem for hydrogen.

$$e(\langle x,\xi\rangle) = e^{2\pi i(x_1\xi_1+x_2\xi_2)}.$$

$Op(f)$ is a zeroth order pseudo differential operator with symbol f. Note that if $f(x,z)$ is a function of x alone then $Op(f)\,\psi(x) = f(x)\,\psi(x)$ and hence μ_ϕ is indeed an extension of ν_ϕ. It is known ([Sh1,2], [Zel1]) that μ_ϕ is a symptotically positive, that is if $f \geq 0$ then $\underline{\lim}\,\mu_\phi(f) \geq 0$ as $\lambda \to \infty$. Hence any weak limit of the μ_ϕ's is a probability measure. We call such a limit β, a quantum limit. As noted by Shnirelman [Sh1] it follows from Egorov's theorem in geometric optics that any quantum limit β is invariant under the Hamiltonian billiard flow on $T_1(\Omega)$[1]

The discussion above applies with almost no changes when Ω is replaced by a compact Riemannian manifold (M,g) which for simplicity we assume has no boundary. The Laplacian is replaced by the Laplace-Beltrami operator $\triangle_g$ for the metric and the classical mechanics is that of motion by geodesics on $X = T_1(M)$, the space of unit tangent vectors over M. For an eigenfunction ϕ of $\triangle_g$ on M we form as above the probability measure $\nu_\phi = |\phi(x)|^2 dV(x)$ on M (dV is the Riemannian volume element) and its microlocal lift[2] μ_ϕ to $T_1(M)$. A quantum limit β is a measure on $T_1(M)$ which is a weak limit of the μ_ϕ's as $\lambda \to \infty$ and as above such a measure is invariant under the geodesic flow.

We are interested in the case where the geodesic flow is ergodic meaning that the only flow invariant subsets of $T_1(M)$ are either of zero or full μ-measure where μ is the Liouville measure (i.e. Riemannian volume) on $T_1(M)$. In this case Birkhoff's ergodic theorem implies that μ-almost all geodesics are μ-equidistributed in $T_1(M)$. There is a corresponding high frequency analogue of ergodicity, called quantum ergodicity which is formulated and proven in [Sh1,2], [Zel1] and [CdV2]. It asserts that if the geodesic flow is ergodic then almost all (in the sense of density) of the eigenfunctions become equidistributed with respect to μ. That is if $\{\phi_j\}_{j=1}^\infty$ is an orthonormal basis of eigenfunctions with $\lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$, then there is a subsequence j_k of integers of full density, such that $\mu_{j_k} \to \mu$ as $k \to \infty$.[3]

The basic question is whether there can be other quantum limits, that is subsequences on which the μ_ϕ's behave differently. If M has (strictly) negative curvature then the geodesic flow is very well understood thanks to works of Hopf, Morse, Sinai, Bowen, ... It is ergodic and strongly chaotic in all senses. The periodic geodesics are isolated and are unstable and there is no restriction on how they may distribute themselves as their period increases. In this context Colin-de-Verdiere [CdV3] asks an insightful question as to whether the most singular flow invariant measure, that is the arc length measure on a closed geodesic can be a quantum limit.[4] In [R-S] this is answered for arithmetic surfaces (see below) and based on that and a careful examination of conflicting interpretations of numerical experiments ([Hel], [A-S], [Bo], [Be]) the Quantum Unique Ergodicity Conjecture ("QUE") was put forth: If M is a compact negativity curved manifold then $\mu_\phi \to \mu$ as $\lambda \to \infty$, put another way μ is the only quantum

[1] Actually [Sh1] is concerned with the case of no boundary, see [G-L] for this case.

[2] This microlocal lift is not unique but all choices lead to the same quantum limit [Sh1], [Zel1].

[3] One can see the domain version of this theorem [G-L] in action even in the small samples in Figures 1S and 1B.

[4] These have been called "strong scars" in the terminology of [Hel].

limit. If true this says that even in the semiclassical limit the quantum mechanics of such strongly chaotic systems does not reflect the finer features of the classical mechanics. More recently Barnett [Ba] developed a numerical method which allows him to compute modes for Ω_B (which is the domain analogue of negative curvature) as large as the 700000$^{\text{th}}$ one and his results confirm QUE for this system.

We report on progress on QUE focusing on recent advances (the discussion is not chronological). Unlike geodesic motion on a negatively curved M, billiards in the stadium have a family of periodic orbits of period twice the distance between the parallel sides and corresponding to billiards bouncing back and forth between these sides. The numerical computations (numbers 10 and 11 in Figure 1) as well as a direct construction of approximate eigenfunctions called quasimodes ([H-O], [Zel2], [Do]) indicates that there is a subsequence of modes whose μ_ϕ's converge to the singular measure corresponding to the totality of such bouncing balls. A rigorous proof that such "bouncing ball" modes exist in the limit, remained elusive until the recent work of Hassell [Ha]. Let S_t stadium with straight edge t as in Figure 1. By examining the variation of the eigenvalues as t varies he shows that the bouncing ball quasimodes impact the genuine modes. For almost all t the stadium S_t has a quantum limit β_t which gives positive mass to the bouncing ball trajectories and in particular S_t is not QUE.[5]

Turning to the negatively curved M's we point out the potential spoiling role that multiple eigenvalues may play. Let V_λ be the space of eigenfunctions on M with eigenvalue λ. If $m(\lambda) = \dim V_\lambda$ is very large (with λ large) one can choose ϕ in V_λ for which μ_ϕ is badly distributed. Thus implicitly the QUE conjecture asserts that these multiplicities cannot be very large and any proof of QUE would have to address this multiplicity issue, perhaps indirectly. The best known upper bound for $m(\lambda)$ is proven using the wave equation and geometric optics [Ber] and asserts that $m(\lambda) \ll \lambda^{(n-1)/2}/\log\lambda$, where $n = \dim M$. We expect say for $n = 2$ that $m(\lambda) = O_\epsilon(\lambda^\epsilon)$ for any $\epsilon > 0$.

An important step in understanding quantum limits in the negatively curved setting was taken by Anatharaman [An]. Given a quantum limit β on $T_1(M)$ one can ask about its entropy $h(\beta)$, that is the entropy of the dynamical system $(T_1(M), \mathcal{G}_t, \beta)$ where $\mathcal{G}_t$ is the geodesic flow.[6] $h(\beta)$ is a measure of the complexity of the β-flow. For example if β is the arc-length measure on a closed geodesic then $h(\beta) = 0$ while for the Liouville measure, $h(\mu) > 0$. In [An] Anantharaman gives an unexpected and striking proof that $h(\beta) > 0$ for any quantum limit β. Her direct estimation of the entropy involves a delicate combination of information about β that is gotten by pushing the known semiclassical asymptotics to their limit together with information obtained from the global hyperbolic dynamics of the geodesics. In particular her positive entropy theorem answers Colin-de-Verdiere's question emphatically: The arc-length measure on an unstable periodic orbit in such M is never a quantum limit. The proof that $h(\beta) > 0$ comes with an explicit lower bound which has been sharpened in [A-N] and [Ri]. This positive entropy theorem allows for the multiplicity $m(\lambda)$ to be as large as the upper bound mentioned earlier since it applies equally well with β replaced by weak limits of μ_ψ's

[5] In [B-Z] an elementary argument is given to show that ν_ϕ's cannot localize on a proper subset of the rectangular part of S.

[6] This question arose first in the context of arithmetic QUE discussed below.

where ψ is a function on M whose spectral projection lies in an interval $[\lambda, \lambda + H]$ with $H = \lambda^{1/2}/\log\lambda$.

The only cases for which QUE has been established are arithmetic manifolds and these are the subject of the rest of the report. We begin by restricting to dimension 2 and M of constant curvature, say $K \equiv -1$. The universal cover of M is the hyperbolic plane H with its line element $ds = |dz|/y$ and its orientation preserving isometry group $G = PSL_2(\mathbb{R})$ acting by fractional linear transformations. Thus M is realized as $\Gamma\backslash\mathbb{H}$ where Γ is a discrete subgroup of G. We also allow such quotients which are of finite area and non-compact. M is called arithmetic if the group Γ is constructed by arithmetic means (see [Ka]). The basic example is $\Gamma = PSL_2(\mathbb{Z})$, the 2×2 matrices with integer entries and determinant equal to 1. This quotient is called the modular surface Y and is non-compact (see [Se1] for example). Compact arithmetic surfaces are constructed using integral matrices associated with quaternion division algebras ([Ka]).[7] The eigenfunctions of the Laplacian for these arithmetic surfaces are automorphic forms called Maass forms and they are basic objects in modern number theory. As such one might expect and it is certainly the case, that this theory can be used to study the QUE question as well as many other interesting questions associated with high frequency states for arithmetic manifolds ([Sa1], [Sa2], [Mark], [Zel3]). The most important property that distinguishes arithmetic surfaces from the general constant curvature surface is that they carry a large family of algebraic correspondences which in turn give rise to the family of Hecke operators. These are linear operators on $L^2(M)$ which commute with each other and with $\triangle$. For example if Y is the modular surface then for $n \geq 1$ the Hecke operator T_n is defined by (see [Se1] for example.)

$$T_n\psi(z) \;=\; \sum_{\substack{ad=n \\ b \, mod \, d}} \psi\left(\frac{az+b}{d}\right). \tag{1}$$

One checks that if $\psi(\gamma z) = \psi(z)$ for $\gamma \in PSL_2(\mathbb{Z})$ then $T_n\psi$ is also $PSL_2(\mathbb{Z})$ invariant.

The T_n's are normal operators and hence this whole ring of Hecke operators together with $\triangle$ can be simultaneously diagonalized. If as is expected and is confirmed by numerical experiments, the Laplace spectrum of Y is simple, then any eigenfunction ϕ of $\triangle$ is automatically an eigenfunction of the full Hecke ring. In any case in this arithmetic setting we always assume that ϕ is an eigenfunction of the Hecke operators and one can always choose an orthonormal basis of such eigenfunctions. These are the eigenfunctions that are arithmetically interesting and this is the means by which we circumvent the issue of the unlikely, but potentially possible, high multiplicities $m(\lambda)$. It is known that the multiplicities of the spaces of such Hecke eigenfunctions is small and in particular for Y it is one.

The first results on QUE were obtained in [L-S1] and [Ja] where it is established for the continuous spectrum for a non-compact arithmetic surface. A key point in the analysis being an explicit relation between $\mu_\phi(f)$ (where ϕ and f are Hecke eigenforms) and special values of related automorphic L-functions on their critical lines. One of the primary reasons for studying automorphic forms is that they give rise to families of L-functions generalizing Riemann's

[7] Actually by arithmetic we will so mean that the group is a congruence subgroup [P-R].

Zeta function and having properties similar to it. Via this relation the QUE problem becomes one of estimating from above the corresponding special value. The "convexity bound" for such values is one that one gets from a simple complex analytic interpolation and it falls just short of what is needed. Any improvement of this bound by a factor of $\lambda^{-\epsilon}$ with $\epsilon > 0$ is called a subconvex bound and it suffices. For reasons such as the one at hand, supplying such subconvex bounds for various automorphic L-functions has become a central problem in the theory of L-functions ([Fr], [I-S]). There is no doubt about the truth of subconvexity since it and optimally sharp bounds, follow from the generalized Riemann Hypothesis for these L-functions. In this case of the continuous spectrum QUE, the required subconvex bounds were known and due to Weyl [We] for the Riemann Zeta function and to [Mu] and [Go] for the t-aspect of the degree 2 L-functions that present themselves here. In his thesis [Wa] Watson established a general explicit formula relating periods of 3 automorphic forms on an arithmetic surface to special values of degree 8 L-functions associated with these forms. His work builds an earlier work along these lines in [K-H]. Watson's explicit formula shows that the subconvexity feature is a general one and that the full QUE for all arithmetic surfaces would follow from subconvexity for these degree 8 L-functions. Moreover the Riemann Hypothesis for these yields the optimal rate of equatistribution of the measures μ_ϕ. With this there was no longer any doubt about the truth of QUE at least in this arithmetic surface setting. Subconvexity for certain degree 4 L-functions was established in [Sa3] and [L-L-Y] from which QUE followed for "dihedral forms". These forms are still special ones and are characterized by the degree 8 L-function factoring into ones of degree 4. While a lot of progress has been made on the subconvexity problem ([I-S], [M-V]) what is needed for the general arithmetic QUE remains out of reach and the solution of the problem took a quite different route.

The phase space $T_1(M) = T_1(\Gamma\backslash\mathbb{H})$ can be naturally identified with $\Gamma\backslash G$ and the geodesic flow $\mathcal{G}_t$ with the diagonal A-flow given by $\Gamma g \mapsto \Gamma g \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ $a \in \mathbb{R}^*$. As we noted before any quantum limit β on $\Gamma\backslash G$ is A-invariant but this alone is far from determining β. Motivated by ideas from measure rigidity for higher rank diagonal actions on homogeneous spaces which we discuss briefly below in the context of products of $PSL_2(\mathbb{R})$'s, Lindenstrauss [Li1] established the following striking classification (to be exact we are forced to get into more technical notions): We discuss it for the modular surface Y but it applies to any arithmetic surface. Fix a prime p and let $X = SL_2(\mathbb{Z}[\frac{1}{p}]) \setminus SL_2(\mathbb{R}) \times PGL_2(\mathbb{Q}_p)/(1 \times PGL_2(\mathbb{Z}_p))$. Here $\mathbb{Q}_p$ denotes the p-adic numbers, $\mathbb{Z}_p$ the p-adic integers and $SL_2\ (\mathbb{Z}\,[\frac{1}{p}])$ is embedded in the product $SL_2(\mathbb{R}) \times PGL_2(\mathbb{Q}_p)$ diagonally. X carries the natural A action by multiplication by $((a_{a^{-1}}), 1)$ on the right and it is naturally foliated by leaves isomorphic to the $p+1$ regular tree $\tau_p = PGL_2(\mathbb{Q}_p)/PGL_2(\mathbb{Z}_p)$ ([Se2]). Let β be a measure on X which is A-invariant and for which all ergodic components of β have positive entropy and which is τ_p recurrent (meaning that for $B \subset X$ with $\beta(B) > 0$ and β a.a. $x \in B$ there are infinitely many points x' in the τ-leaf of x which are also in B). The conclusion is that β in fact $SL_2(\mathbb{R})$ invariant. To apply this to the QUE, problem Lindenstrauss exploits the fact that our eigenstates on M are eigenfunctions of the Hecke operator T_p which allows one to consider the μ_ϕ's as measures on X and to consider their limits on this space. To verify the conditions in this measure classification one uses the full Hecke ring along the lines of [R-S] where these were used originally to show that the singular support of an arithmetic quantum limit cannot be a closed geodesic. In

[B-L] this argument is generalized vastly to show that the entropy of any ergodic component of β is positive. The recurrence property for the τ_p foliation is more elementary and putting all these threads together Lindenstrauss establishes QUE for compact arithmetic surfaces. In the non-compact case there is the possibility that some of the mass of μ_ϕ escapes in the limit into the cusp and the conclusion is that any quantum limit β is $c\,d\mu$ for some constant c in $[0, 1]$. Very recently Soundararajan [So1], using the multiplicativity of the Fourier coefficients in the expansion in the cusp of the forms ϕ, together with some clever elementary analytic arguments was able to show that there is no escape of mass in the non-compact cases. With this QUE for arithmetic surfaces is now fully proven.

The hyperbolic surfaces M are also complex analytic Riemann surfaces and the semiclassical analogue of high frequency modes are holomorphic sections of high tensor powers of the canonical line bundle over M or what is the same thing, holomorphic automorphic forms of large weight. There are however some fundamental differences as far as the analogue of QUE in this setting. A holomorphic form of weight k for $\Gamma \leq PSL_2(\mathbb{R})$ defines a probability density ν_f on $M = \Gamma\backslash\mathbb{H}$ using the Petersson inner product: $\nu_f = |f(z)|^2 y^k dA(z)$. If M is non-compact then we assume that f vanishes at the cusps (a "cusp form"). By the Riemann-Roch theorem the dimension of the space of such forms grows linearly with k. Hence it is not surprising that the analogue of QUE, that is the equidistribution of the ν_f's as $k \to \infty$, fails (take a fixed non-zero f_0 of weight k_0 and raise it to the k/k_0 power). By the same token there is no apparent A-invariant microlocal lift of ν_f to $\Gamma\backslash G$[8]. However as soon as the connection with special values of L-functions was made in the arithmetic Hecke eigenform setting, it was clear that QUE for holomorphic Hecke eigenforms should be equally valid. That is for such f's on an arithmetic M, $\nu_f \to dA/Area(M)$ as $k \to \infty$. Here too Watson's explicit formula together with subconvexity for corresponding degree 8 L-functions implies this holomorphic QUE. There is a very nice consequence in connection with the zeros of such f's. In [N-V] a general potential theoretic argument applied to $\partial\bar{\partial}\log|f(z)|$ is used to show that if $\nu_f \to dA/Area(M)$ and M is compact, then the zeros of f become equidistributed in M with respect to $dA/Area(M)$ as $k \to \infty$. This was extended in [Ru1] to the finite area non-compact cases. Thus the holomorphic QUE conjecture implies in particular that the zeros of such Hecke eigenforms are equidistributed in the large k limit. In the next paragraphs we report on recent works of Holowinsky [Ho], Soundararjan [So2] and [H-L] which establish this holomorphic QUE for M a non-compact arithmetic surface. Hence the zeros of a holomorphic cusp form of weight k for $SL_2(\mathbb{Z})$ are equidistributed with respect to hyperbolic area as $k \to \infty$. This basic and elegant result is a striking application of the theory developed for the QUE problem.

Soundararjan approaches the problem by seeking a more modest bound for the degree 8 L-functions in Watson's formula. Instead of improving the convexity bound by a factor of $k^{-\epsilon_0}$ he settles for $(\log k)^{-\epsilon_0}$ which he calls weak subconvexity. There are normalization factors in the explicit formula which involve special values of L-functions at $s = 1$ and which are potentially of size $\log k$ ([H-L-G-L]). Thus this weak subconvexity by itself cannot do the full job, however it is known that with very few exceptions these special values at $s = 1$ are well behaved. Thus the weak subconvexity allows him to prove QUE for all but $O_e(k^\epsilon)$ of

[8]For this reason no ergodic theoretic approach to this holomorphic QUE is known.

the Hecke eigen forms of weight k. Moreover he establishes this weak subconvexity in the full generality of L-functions associated automorphic forms on the general linear group,[9] the only assumption that he needs to make is that the corresponding forms satisfy the generalized Ramanujan Conjectures (see [Sa4]). Happily for the case of holomorphic forms on an arithmetic surface this is known by the celebrated theorem of Deligne [De]. To prove this general weak subconvexity Soundararajan develops a far reaching generalization of techniques in the theory of mean-values of multiplicative functions specifically showing that they don't vary much over certain ranges (see also [Hi] and [G-S]). This feature is the source of the sought after cancellation.

The starting point for Holowinsky is yet a third approach to QUE which depends critically on M being non-compact and was used in [L-S2], [LS-3] to investigate similar problems. Using the cusps of M one develops the holomorphic form in a Fourier expansion and the QUE problem can be reduced to estimating a shifted convolution (which are quadratic) sums involving these coefficients. One expects and the quantitative forms of QUE demand, that there is a lot of cancellation due to the signs of the coefficients of these varying forms. Holowinsky's novel idea is to forgo this cancellation and to exploit the fact that the mean values of the absolute values of these coefficients is of size $(\log k)^{-\delta}$. The source of this phenomenon is that if f is not dihedral (and we can assume this since QUE is known for these) then the distribution of coefficients at primes are expected to follow a Sato-Tate distribution and enough towards the latter is known by the work of Shahidi [K-S] to exploit this feature (see [E-M-S] for the case of f fixed). Using a sieving argument[10] Holowinsky is able to give a bound for the shifted convolution sums which improves the trivial bound by a factor of a small negative power of $\log k$. Here too Deligne's theorem is being used as a critical ingredient. To apply this bound to QUE, normalization factors which are values at $s = 1$ of associated L-functions intervene again. In this way Holowinsky is also able to establish QUE for all but $O_\epsilon(k^\epsilon)$ of the forms of weight k, for a non-compact arithmetic surface M.

The miracle and it is not uncommon for such "luck" to be at the heart of a breakthrough is that there are no common exceptions to Holowinsky and Soundararajan's treatments. Soundararajan's is unconditional as long as the value at $s = 1$ of a related L-function is not very small (that is essentially as small as $1/logk$) but if this is so then one can show that most of the Fourier coefficients of f are even smaller at primes and with this Holowinsky's treatment becomes unconditional.

The only case of QUE for surfaces that remains open at this time is that of holomorphic forms for the compact arithmetic surface. Soundararajan's arguments apply in this case but not Holowinsky's.

To end we discuss briefly some higher dimensional cases of arithmetic QUE. The Hilbert modular varieties are the closest to the arithmetic surface case. Let $n \geq 2$ and K a totally real (ie all the embedding of K, $\sigma_1, \ldots, \sigma_n$ are real) number field of degree n. Let $\mathcal{O}_k$ be its ring of integers and $\Gamma = SL_2(\mathcal{O}_k)$ the corresponding group of 2×2 matrices. Via the

[9] As such this result will no doubt find many further applications.

[10] See also [Na] for a general such inequality in the context of multiplicative functions.

embeddings Γ is a discrete subgroup of $G = SL_2(\mathbb{R}) \times SL_2(\mathbb{R}) \times \cdots SL_2(\mathbb{R})$. The quotient $M = \Gamma\backslash\mathbb{H} \times \mathbb{H} \times \cdots \times \mathbb{H}$ is a finite volume locally symmetric space and is also a complex manifold: the corresponding Hilbert modular variety. The QUE problem for high frequency eigenstates can be formulated in this context. In the philosophy of diagonalizing geometrically defined commuting operators we consider $\phi(z_1, \ldots, z_n)$ which is Γ-invariant and which is an eigenfunction of the full ring of differential operators on $\mathbb{H} \times \mathbb{H} \times \cdots \times \mathbb{H}$ which commute the action of G. That is ϕ is a simultaneous eigenfunction of $\triangle_{z_1}, \ldots, \triangle_{z_n}$ (and not just of $\triangle = \triangle_{z_1} + \cdots + \triangle_{z_n}$). The probability density $\nu_\phi = |\phi(z)|^2 dv(z)$ has a natural microlocal lift μ_ϕ to $\Gamma\backslash G$ ([Li2])[11] The new feature is that being an eigenfunction of each $\triangle_{z_i}$, the quantum limits, that is weak limits of the μ_ϕ's as $\min \lambda \to \infty$ where $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(n)})$, are invariant under the full multiparameter diagonal subgroup $A = \{(({}^{a_1}\, {}_{a_1^{-1}}), \ldots, ({}^{a_n}\, {}_{a_n^{-1}})) : a_j \neq 0\}$ of G. For $n \geq 2$ this is exactly the setting[12] of the measure rigidity conjecture of Margulis [Ma] (which in turn has its roots in the times 2 times 3 conjecture of Furstenberg) which asserts that any such measure on a homogeneous space $\Gamma\backslash G$ which is invariant under a rank 2 or higher diagonal group, should be rigid. While no cases of this conjecture are proven versions in which one assumes some form of positive entropy are known ([E-K-L], [Li1]) (and again these have their roots in similar such theorems for $\times 2 \times 3$ [Ru]). The positive entropy is established by using the full Hecke algebra as in [B-L] and QUE for these Hecke Maass forms on M is thus established.

The holomorphic QUE for these Hilbert modular varieties asserts that if $f(z) = f(z_1, \ldots, z_n)$ is a holomorphic Hecke cusp form of weight $k = (k_1, k_2, \ldots, k_n)$ than the probability densities $\nu_f = |f(z)|^2 y_1^{k_1} \cdots y_n^{k_n} dv(z)$ on M become equidistributed with respect to $dv(z)/vol(M)$ as $k_{\min} = \min(k_1, \ldots, k_n)$ goes to infinity. In his thesis [Mars] Marshall has shown that this is another setting where the Holowinsky-Soundararajan miracle occurs. Generalizing their arguments to this higher dimensional setting he establishes QUE for holomorphic Hilbert modular forms (the Ramanujan Conjectures are known in this case [Bl]). The potential theoretic argument showing that the equidistribution of the densities ν_f implies that of the zero divisor Z of f, is formulated and proven in the context of holomorphic sections of high tensor powers of a positive hermitian line bundle on a compact complex manifold in [S-Z]. Marshall shows that these arguments extend to the non-compact Hilbert modular setting and as a consequence of this and QUE he proves that the zero divisor $Z(f)$ of a Hecke cusp form f becomes equidistributed with respect to dv as $k_{\min} \to \infty$, either in the sense of $Z(f)$ being a real co-dimension 2 Riemannian submanifold of M or as a Lelong $(1,1)$ current; $\partial\bar{\partial} \log(|f(z)|^2 y_1^{k_1} \cdots y_n^{j_n})$. Again this is rather a basic fact about Hecke eigenforms in several variables which is a consequence of the QUE theory.

Unlike the periods to special values of L-functions relation which it appears are rather special, the ergodic approach of Lindenstrauss extends to quite general compact arithmetic manifolds as has been shown by Silberman and Venkatesh [S-V1], [S-V2]. The "micro-local" extension goes naturally from the locally symmetric space $M = \Gamma\backslash G/K$ to $\Gamma\backslash G$ rather than the unit tangent bundle of M and its construction requires some elaborate representation

[11] This is not the unit tangent bundle and corresponds to the geodesic flow not being ergodic.

[12] In fact this is the setting in which the connection between QUE and measure rigidity was first noted [Li2].

theory. Their proof of positive entropy for the corresponding quantum limits of Hecke eigenforms is general and robust and it clarifies the role played by homogeneous subvarieties in this connection.

To end we point out that in the symplectic setting the analogue of QUE for strongly chaotic transformations may fail. The mathematical model is that of quantizing a symplectic transformation of a compact symplectic manifold. The simplest (and very degenerate) case is that of a linear area preserving transformation of the torus $\mathbb{R}^2/\mathbb{Z}^2$ correspond to an A in $SL_2(\mathbb{Z})$. If $|\text{trace}A| > 2$ then the dynamics of iterating A is ergodic and strongly chaotic. In the literature this goes by the name "cat map"[13] The eigenstates of the corresponding quantization can be studied in depth [Ru2].[14]

In this setting when the eigenvalues of the quantization are maximally degenerate the analogue of QUE can fail ([F-N-D]). Some take this as a warning about the truth of the original QUE conjecture. Note that even though QUE fails here the positive entropy analogue of Anantharaman is still true ([Br]).

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References

[An] N. Ananthraman, *Annals of Math.*, **168** (2008), 435-475.

[A-S] R. Aurich and F. Steiner, *Physics D*, **48** (1991), 445-470.

[Ba] A.H. Barnett, *CPAM*, **59** (2006), 1457-1488.

[Be] M. Berry, *Proc. Roy. Soc.*, London, Ser A, **423** (1989), 219-231.

[Ber] P. Berard, *Mat Zeit*, **155** (1977), 2083-2091.

[Bl] D. Blasius, "Hilbert modular forms and the Ramanujan Conjecture", *ARXIV* (2005).

[Bog] E. Bogomolny, *Phys. D*, **31** (1988), 169-189.

[B-L] J. Bourgain and E. Lindenstrauss, *CMP*, **233** (2003), 153-171.

[Br] S. Brooks, *Entropy bounds for quantum limits*, Princeton University Thesis, (2009).

[Bu] L. Bunimovich, *Funct. Anal. and Appl.*, **8** (1974), 254.

[13] This silly but catchy name comes from a picture in the book "Ergodic problems of classical mechanics" by Arnold and Avez in which a figure of a cat and its deformation under a couple of iterates of A is depicted.

[14] In particular the analogues of Hecke operators and QUE for their eigenstates is known and proven [K-R].

[B-Z] N. Burq and M. Zworski, *SIAM Review*, **47** (2005), 43-49.

[C-M] N. Chernov and R. Markarian, *AMS*, Surveys 127 (2006).

[CdV1] Y. Colin de Verdiere, *Invent. Math.*, **3** (1977), 15-52.

[CdV2] Y. Colin de Verdiere, *Comm. Math. Phys.*, **102** (1985), 497-502.

[CdV3] Y. Colin de Verdiere, in Chaos and Quantum Physics, Le-Houches, (1989), 307-329.

[De] P. Deligne, *PUB IHES*, **43** (1974), 273-307.

[Do] H. Donnelly, *PAMS*, **131** (2003), 2945-2951.

[E-K-L] M. Einsiedler-A. Katok and E. Lindenstrauss, *Annals of Math.*, **164** (2006), 513-560.

[F-N-D] A. Faure, S. Nonnenmacher and S. DeBieure, *CMP*, **293** (2003), 449-492.

[Fr] J. Friedlander, *Proc. ICM* (1994), 363-373.

[G-H-L] D. Goldfeld, J. Hoffstein and D. Lieman, *Annals of Math.*, **140** (1994), 177-181.

[Go] A. Good, Mathematika, **29** (1982), 278-295.

[G-S] A. Granville and K. Soundararajan, *Can. J. Mah.*, **55** (2003), 992-1028.

[Hel] E. Heller, *Phys. Rev. Letters*, **53** (1984), 1515.

[H-O] E. Heller and P. O'Conner, *Phys. Lett. Rev.*, **61** (1988), 2288-2291.

[Hi] A. Hilderbrand, *C.R. Math. Rep. Acad. Sc.*, Canada, **8** (1986), 147-152.

[I-S] H. Iwaniec and P. Sarnak, *GAFA* (2000), II, 705-741.

[Ja] D. Jakobson, *Ann. Inst. Fourier*, **44** (1994), 1477-1504.

[Ka] S. Katok, "Fuchsian groups" University of Chicago Press.

[K-S] H. Kim and F. Shahidi, *D.M.J.*, **112** (2007), 177-197.

[K-H] S. Kudla and M. Harris, *Annals of Math.*, **133** (1991), 605-672.

[K-R] P. Kurlberg and Z. Rudnick, *D.M.J.*, **103** (2000, 47-77.

[L-L-Y] Y. K. Lau, J. Liu and Y. Ye, *Jnl.of No. Theory*, **121** (2006), 204-223.

[La] V. Lazutkin, *KAM theory and semiclassical approximations to eigenfunctions*, Springer Ergebnisse, Vol. 24, (1993).

[Li1] E. Lindenstrauss, *Annals of Math.*, **163** (2006), 165-219.

[Li2] E. Lindenstrauss, *IMRN* (2001), 913-933.

[L-S2] W. Luo and P. Sarnak, *CPAM* (2003), 977-891.

[L-S1] W. Luo and P. Sarnak, *Publ. IHES*, **81** (1995), 207-237.

[Ma] G. Margulis, "Problems and Conjectures in Rigidity Theory", *Frontiers (AMS)* (2000), 161-174.

[Mark] J. Marklof, *Encyc of Math. Phys.*, Vol 1, (2006), 212-220.

[Mars] S. Marshall, Thesis, Princeton University (2009).

[Mel] R. Melrose, *Adv. in Math.*, **32** (1979), 204-232.

[Mu] T. Muerman, *Colloq. Math. Soc. Janos Bolyai*, **51** (1990), 325-354.

[Na] M. Nair, *Acta Arith.*, **62** (1992), 257-269.

[N-V] S. Nonnenmacher and A. Voros, *J. Stat. Phys.*, **92** (1998), 431-518.

[P-R] V. Platonov and A. Rapinchuk, "Algebraic Groups and Number Theory", *A.P.* (1991), Chapter 9.

[Ri] G. Riviere, "Entropy of semiclassical measures in dimension 2", preprint (2009).

[Ru1] Z. Rudnick, *IMRN* (2005), 2059-2074.

[Ru2] Z. Rudnick, "The arithmetic theory of quantum maps" in *Nato Science Series*, Vol. 237 (2007).

[R-S] Z. Rudnick and P. Sarnak, *CMP*, **161** (1994), 195-213.

[Ru] D. Rudolph, *Erg. Th. and Dyn. Systems*, **10** (1990), 395-406.

[Sa1] P. Sarnak, "Arithmetic quantum chaos", Schur Lectures, Israel Math. Conf. Proc. (1995).

[Sa2] P. Sarnak, "Spectra of arithmetic manifolds", *BAMS* (2003), Vol. 40, 441-478.

[Sa3] P. Sarnak, *Jnl. Funct. Anal.*, **184** (2001), 419-453.

[Sa4] P. Sarnak, "Notes on the generalized Ramannjan Conjectures", *Clay Math. Proc.*, Vol. 4 (2005), 659-685.

[Se1] J.P. Serre, "A course in arithmetic".

[Se2] J.P. Serre, "Trees".

[S-Z] B. Shiffmann and S. Zelditch, *CMP*, **200** (1999), 661-683.

[Sh1] A. Shnirelman, *Uspenski Math. Nauk* **29/6** (1974), 181-182.

[Sh2] A. Shnirelman, Appendix in [La].

[S-V1] L. Silberman and A. Venkatesh, *GAFA*, **17** (2007), 960-998.

[S-V2] L. Silberman and A. Venkatesh, *GAFA*, (to appear).

[Si] Y. Sinai, *Russian Math. Surveys*, **25** (1970), 137-189.

[So1] K. Soundararajan, "Quantum Unique ergodicit;y for $SL_2(\mathbb{Z}\backslash\mathbb{H})$", *Annals of Math.*, (to appear).

[So2] K. Soundararajan, "Weak Subconvexity and central values of L-functions", *Annals of Math.*, (to appear).

[Wa] T. Watson, Princeton University Thesis (2001), see *Annals of Math.*, (to appear).

[We] H. Weyl, *Math Ann.*, **77** (1996), 313-352.

[Zel1] S. Zelditch, *Duke Math. Jnl.*, **55** (1987), 919-941.

[Zel2] S. Zelditch, *PAMS*, **132** (2004), 1869-1872.

[Zel3] S. Zelditch, *Encyc. of Math. Phys.*, Vol. 1, (2006), 183-196.