Number variance for $SL(2,\mathbb{Z})\backslash\mathbb{H}$

Xiaoqing Li*
Department of Mathematics,
Columbia University, New York, NY, 10027, U.S.A.

Peter Sarnak†
Department of Mathematics,
Princeton University & Courant Institute of Math. Sciences,
Princeton, NJ, 08544, U.S.A

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Abstract: We examine the variance of the number of eigenvalues of the Laplacian for the modular surface in a short interval. The analysis allows for the interval to be small enough so that the size of the variance is Poissonian. The starting point for the investigation is the Kuznetsov formula and the body of the work consists of studying the complicated off-diagonal contributions which are responsible for the shape of the final asymptotics. A consequence of the main result is a slight improvement of the known lower bound for the remainder term in the Weyl counting function.

Contents:

*E-mail: xqli@math.columbia.edu
†E-mail: sarnak@math.princeton.edu

§1. Introduction

Let $X = SL(2,\mathbb{Z})\backslash\mathbb{H}^2$ be the modular surface and $\lambda_j = \frac{1}{4} + t_j^2$, $0 < \lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \ldots$ be the eigenvalues of the Laplacian $\triangle$ on the cuspidal subspace of $L^2(X)$ [Sa3]. Selberg [Se] showed that these obey a Weyl law:

$$N(T) = \sum_{0 \leqslant t_j \leqslant T} 1 \sim \frac{T^2}{12} \tag{1.1}$$

as $T \to \infty$. Define the remainder term $S(T)$ by

$$N(T) = \frac{T^2}{12} - \frac{2T\log T}{\pi} + \left(\frac{2+\log\pi - \log 2}{\pi}\right) T + \frac{13}{144} + S(T) \tag{1.2}$$

(See [He2, pp. 466] and [St] for these lower order terms which come from the trace formula).

The graph of $S^{\pm}$, where $\pm$ denotes the even and odd parts of the spectrum corresponding to $z \to -\bar{z}$, was calculated by Steil [St] for $T \leqslant 3000$ and is reproduced in Figure 1.

We write (1.2) as

$$N(T) = N_{\mathsf{smooth}}(T) + S(T) \tag{1.3}$$

where $N_{\mathsf{smooth}}(T)$ is the "smooth" contribution to the Weyl count and it includes the smaller and well understood contribution $\omega(T)$ from the continuous spectrum. $S(T)$ is the oscillatory part, about which much less is known. A simple application of the Selberg trace formula shows that

$$S(T) = O\left(\frac{T}{\log T}\right). \tag{1.4}$$

On the other hand, Selberg established the lower bound (see [He1, pp. 303]) for the mean square*

$$\frac{1}{T}\int_T^{2T} (S(t))^2\,dt \gg \frac{T}{(\log T)^2}. \tag{1.5}$$

It follows in particular that

$$S(T) = \Omega\left(\frac{T^{1/2}}{\log T}\right). \tag{1.6}$$

There have been many conjectural and related numerical developments concerning this modular spectrum (see [Sa3], [BLS] [St]). For example, it is believed that the spectrum is simple and that the local scaled spacing distributions are "Poissonian" rather than the Gaussian Orthogonal distribution which is what is expected for the generic hyperbolic surface. However, progress on $S(T)$ remains elusive and (1.4) and (1.5) are all that were known concerning $S(T)$. One of the consequences of the analysis of the number variance below is the following modest improvement of (1.5).

*The proof given there is for quaternion groups and applies equally well to X.

Theorem 1.1.
$$\frac{1}{T}\int_T^{2T} (S(t))^2\,dt \gg \frac{T}{(\log T)^2}\exp\left((\log\log T)^{5/17}\right)$$
and
$$S(T) = \Omega\left(\frac{T^{1/2}}{\log T}\exp\left(\frac{1}{2}(\log\log T)^{5/17}\right)\right).$$

Our main results concern the smoothed number variance. Fix $h \in \mathcal{S}(\mathbb{R})$ a Schwartz function on $\mathbb{R}$ with $\int_{-\infty}^{\infty} h(x)\,dx = 1$ and $h(x) = h(-x)$. We assume further that the support of $\hat{h}$ is contained in $(-1,1)$ where $\hat{h}(\xi) = \int_{-\infty}^{\infty} h(x)e^{-2\pi i x\xi}\,dx$ is the Fourier transform of h. We use this h to define a smoothed count of the eigenvalues in a short interval. For $1 \ll L \ll t^{1-\epsilon}$ set

$$N_h(t,L) = \sum_{j\geqslant 1} h(L(t-t_j)). \tag{1.7}$$

Thus, N_h counts the number of eigenvalues within $1/L$ of t and the larger we can take L the more information about the local distribution of the spectrum can be determined. A simple application of the trace formula shows that for $1 \leqslant L \leqslant \log t/\pi$, N_h is asymptotic to the smooth part in Weyl's law,

$$N_h(t,L) \sim \frac{t}{6L}, \text{ for } 1 \leqslant L \leqslant \frac{\log t}{\pi}. \tag{1.8}$$

This range for L falls just short of being critical for the number variance. The following extends the range suitably

Theorem 1.2.
$$\text{For } 1 \leqslant L \leqslant \frac{2\log t}{\pi}, \text{ we have that}$$
$$N_h(t,L) \sim \frac{t}{6L} \text{ as } t \to \infty.$$

■

Corollary 1.3. *Let $m(t)$ be the multiplicity of the eigenvalue $\frac{1}{4}+t^2$, then*

$$\overline{\lim_{t\to\infty}}\,\frac{m(t)\,\log t}{t} \leqslant \frac{\pi}{12}.$$

This is embarrassingly far from the believed bound $m(t) \leqslant 1$ but it is the best bound that we know.

We turn to the variance of $N_h(t,L)$ (the "smoothed number variance"). Again for $1 \leqslant L \leqslant \frac{\log T}{\pi}$ one can use the trace formula (the point being that in this range the off-diagonal terms don't contribute significantly) to show that as $T \to \infty$

$$\begin{aligned}\sum(T,L) &:= \frac{1}{T}\int_T^{2T} \left(N_h(t,L) - \frac{t}{6L}\right)^2 dt \\ &\sim \frac{1.328\ldots}{2\pi L}\int_0^\infty |\hat{h}(\xi)|^2\, e^{\pi L\xi}\, d\xi \\ &= O(T^\alpha)\,,\ \text{for some } \alpha < 1\,. \end{aligned} \tag{1.9}$$

This was shown by Rudnick [Ru] (see also [LS] where the stronger lower bound as in (1.5) is established for the number variance) who shows further more, by computing the higher moments, that $N_h(t,L)$, for $T \leqslant t \leqslant 2T$, has a Gaussian distribution if $L = o(\log T)$ and $L \to \infty$. The constant 1.328... is the one obtained by Peter [Pe1] for the mean square of the multiplicity of the lengths of closed geodesics on X. Thus in the range $L \leqslant \frac{\log T}{\pi}$ the variance is much smaller than the Poisson variance whose order of magnitude is T/L. Our main result is the determination of the number variance $\sum(T,L)$ in a window $L \in \left[\frac{(1+\delta)}{\pi}\log T\,, \frac{(1+\frac{1}{121})}{\pi}\log T\right]$ for any given $\delta > 0$. The result indicates a Poissonian number variance which emerges from a detailed analysis of the off-diagonal terms whose contribution turns out to be significant.

Throughout the paper we denote by $S(u;v)$ the Kloosterman sum

$$S(u;v) = \sideset{}{^*}\sum_{a(\mathrm{mod}\,v)} e\left(\frac{au + \bar{a}u}{v}\right), \ \text{with} \ \ a\,\bar{a} \equiv 1(v)\,,\ \text{where } * \text{ means } (a,v) = 1\,. \tag{1.10}$$

Theorem 1.4. *Let $\psi \geq 0$ be a fixed smooth function with support in (1,2) with $\int_0^\infty \psi(x)\,dx = 1$ and fix $\delta > 0$. Then for $\frac{(1+\delta)}{\pi}\log T \leqslant L \leqslant \frac{\left(1+\frac{1}{121}\right)}{\pi}\log T$*

$$\begin{aligned}\sum\nolimits_h(T,L) &:= \frac{1}{T}\int_0^\infty \psi\left(\frac{t}{T}\right)\left(N_h(t,L) - \frac{t}{6L}\right)^2 dt \\ = \frac{\int_0^\infty \xi^5\psi(\xi)\,d\xi}{\pi^6}\,&\frac{T}{L^2}\sum_{v\geq 1}\sum_{(u,v)=1}\prod_{p|v}\left(1-p^{-2}\right)^{-2}\frac{S^2(u;v)}{u^2v^2}\left|\hat{h}\left(\frac{\log\frac{Tv}{u}}{\pi L}\right)\right|^2 \\ &+ O\left(\frac{T}{L^2}\right).\end{aligned}$$

Note that the series on the right hand side above consists of positive terms. Thus its asymptotic behavior depends on the average sizes of Kloosterman sums. This is a quite subtle issue and it has been addressed recently by Fouvry and Michel [FM]. They show that

$$\exp\left[(\log\log x)^{5/17}\right] \ll \sum_{v\leqslant x}\frac{|S(1;v)|^2}{v^2} \ll (\log x)(\log\log x)^3 . \tag{1.11}$$

From this and similar bounds (6.6) and (6.7), one deduces that if the support of $\hat{h}$ is close enough to ± 1 then the first term on the right in Theorem 1.4 satisfies

$$\frac{T}{L^2}\exp\left((\log\log T)^{5/17}\right) \ll R \ll \frac{T}{L}(\log L)^3 . \tag{1.12}$$

In particular, it is the main term!

As a consequence we have for such h

Corollary 1.5. *Fix $\delta > 0$, then for*

$$\frac{(1+\delta)}{\pi}\log T \leqslant L \leqslant \frac{\left(1+\frac{1}{121}\right)}{\pi}\log T ,$$

$$\frac{T}{L^2}\exp\left((\log\log T)^{5/17}\right) \ll \sum\nolimits_h (T,L) \ll \frac{T}{L}(\log L)^3$$

Theorem 1.1 then follows from the lower bound in this Corollary.

It seems reasonable to conjecture that as $x\to\infty$,

$$\sum_{v\leqslant x}\frac{|S(u;v)|^2}{v^2} \sim A\log x , \ \text{ for a non-zero } A . \tag{1.13}$$

This combined with Theorem 1.4 would lead to $\sum_h(T,L)\sim cT/L$, for a non-zero constant c (and L restricted as in Theorem 1.4). That is to say that at least for L in this window the number variance is Poissonian. In the same way (1.13) would lead to the lower bound of $T/\log T$ in (1.5) which could well be the true order of magnitude for the mean square of $S(t)$. The extension of the range for L to the window specified in Theorem 1.4 is the analogue of extending the range in Montgomery's pair correlation conjecture for the zeros of the zeta function [Mo] to the region $\alpha > 1$ (see [Pe2] for the analogue of Montgomery's analysis in the context of the eigenvalues of a hyperbolic surface). In the case of the zeros of zeta such an extension would follow from a quantitative version of the Hardy-Littlewood prime 2-tuple conjectures. In our case of the eigenvalues of X we have to handle similar off-diagonal shifted sums as described briefly in the next paragraph.

We end the introduction by outlining the proofs of the results. Instead of using the Selberg trace formula, we use the Kuznetzov formula (see Section 2). The latter involves sums over the spectrum weighted by Fourier coefficients of eigenfunctions. These weights need to be removed (which turns out to be non-trivial) since the count in $N(T)$ involves no weights. The gain in using the Kuznetzov formula over the trace formula is that the sums on the geometric side involve Kloosterman sums and integrals which apparently package certain cancellations in a more transparent way than do the sums involving class numbers which appear naturally from the Selberg trace formula. In fact the doubling of the range of L that is the content of Theorem 1.2, is achieved in this fashion without too much trouble. The idea of introducing these weights and then removing them is not new. It was used by Iwaniec [Iw1] in connection with improving the error term in counting closed geodesics on X and we also use some other technical devices introduced in that paper. As we noted earlier, Theorem 1.4 involves extending L to be large enough to see the Poissonian number variance. Not surprisingly this analysis requires understanding the contributions from off-diagonal terms. These do in fact contribute to the main term and certain further cancellations among these are crucial. We handle these off-diagonal terms emerging from the Kuznetzov formula using the circle method and in particular the smooth "δ-method" developed in [DFI]. It is possible that one could also obtain Theorem 1.4 by making the analysis in Peter [Pe1] (specifically the shifted sums) effective by obtaining a uniform power saving in the error terms. The quality of the result (i.e. doubling the window length) in Theorem 1.2 would appear to be more difficult to achieve without using the Kuznetzov formula.

§2. Some Technical Tools

We review the Kuznetzov formula as well as some facts about Rankin-Selberg L-functions which will be used later on. Our notations and set up agree with that in [Iw1] and [Iw2]. The Eisenstein series for X is given by

$$E(z,s) \;=\; \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} (y(\gamma z))^s \tag{2.1}$$

for $\Re(s)>1$, $\Gamma = PSL(2,\mathbb{Z})$ and $\Gamma_\infty = \left\{\left(\begin{smallmatrix}1 & m\\ 0 & 1\end{smallmatrix}\right) : m\in\mathbb{Z}\right\}$. $E(z,s)$ has a meromorphic continuation to the entire complex plane and has Fourier expansion

$$E(z,s) \;=\; y^s \,+\, \phi(s)y^{1-s} \,+\, \sum_{n\neq 0}\phi(n,s)\,W_s(|n|z) \tag{2.2}$$

where $\phi(s)$ and $\phi(n,s)$ are given by

$$\phi(s) \;=\; \frac{\xi(2s-1)}{\xi(2s)}\,,\; \xi(s) \;=\; \pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right)\,\zeta(s)$$

$$\phi(n,s) \;=\; \pi^s(\Gamma(s)\,\xi(2s))^{-1}\,|n|^{1/2}\sum_{ab\,=\,|n|}\left(\frac{a}{b}\right)^{s-\frac12}\,, \tag{2.3}$$

and $W_s(z)$ is the Whittaker function

$$W_s(z) \;=\; 2y^{1/2}\,K_{s-\frac12}\,(2\pi y)\,e(x)\,. \tag{2.4}$$

$E(z,s)$ is an eigenfunction of $\triangle$ with eigenvalue $s(1-s)$ and it furnishes the continuous spectrum for $\triangle$ on $L^2(X)$ when $s=\frac12+it$, $t\in\mathbb{R}$. Other than a simple pole at $s=1$, $E(z,s)$ has no poles in $\Re(s) \geq \frac{1}{2}$. The cuspidal subspace $L^2_{\mathsf{cusp}}(X)$ consists of all functions in $L^2(X)$ which have

$$\int_0^1 f(z)\, dx \;=\; 0 \text{ for almost all } y\,. \tag{2.5}$$

This is a closed $\triangle$ invariant subspace which is the orthogonal complement in $L^2(X)$ of the continuous spectrum and the constant function $u_0(z) = (\text{Area } X)^{-1/2} = \left(\frac{\pi}{3}\right)^{-1/2}$. The spectrum of $\triangle$ on $L^2_{\mathsf{cusp}}(X)$ is discrete and let $\{u_j\}_{j=1}^{\infty}$ be a corresponding orthonormal basis. These functions have Fourier expansions

$$u_j(z) \;=\; \sum_{n\neq 0} \rho_j(n)\, W_{s_j}(|n|z) \tag{2.6}$$

where $s_j(1-s_j) \;=\; \lambda_j$ (i.e. $s_j \;=\; \frac{1}{2} + it_j$) and $\rho_j(n)$ are the corresponding Fourier coefficients.

Let $p(z)$ be an even test function which is holomorphic in $|\Im(z)| \;\leqslant\; \frac{1}{2} + \epsilon$ and which is $O((1+|z|)^{-2-\epsilon})$ in this region.

Set

$$p_0 \;=\; \frac{1}{\pi} \int_{-\infty}^{\infty} y\, \tanh{(\pi y)}\, p(y)\, dy \tag{2.7}$$

and

$$p^+(x) \;=\; \frac{2i}{\pi} \int_{-\infty}^{\infty} J_{2iy}\,(x)\, \frac{p(y)y}{\cosh \pi y}\, dy\,. \tag{2.8}$$

Define the normalized coefficients

$$\nu_j(n) \;=\; \left(\frac{4\pi|n|}{\cosh\, \pi\, t_j}\right)^{1/2} \rho_j(n) \tag{2.9}$$

and

$$\eta(n,t) \;=\; \left(\frac{4\pi|n|}{\cosh\, \pi\, t}\right)^{1/2} \phi\left(n, \frac{1}{2} + it\right)\,. \tag{2.10}$$

With these notations, we have the following form of the Kuznetzov formula that we will need (see [Iw2]).

Proposition 2.1. *For any* $n \geq 1$

$$\sum_{j\geq 1} p(t_j)\, |\, \nu_j(n)|^2 \,+\, \frac{1}{4\pi} \int_{-\infty}^{\infty} p(t) |\eta(n,t)|^2\, dt$$

$$=\; p_0 \,+\, \sum_{c=1}^{\infty} \frac{S(n;c)}{c}\, p^+\left(\frac{4\pi n}{c}\right)\,.$$

Here $S(n;c)$ is the Kloosterman sum defined in (1.10).

Next, we review the Rankin-Selberg L-functions. We assume as we may, that the u_j's are eigenforms of the Hecke operators T_n. Thus, for $n \geqslant 1$ (see [Iw2])

$$\rho_j(n) \;=\; \rho_j(1)\,\lambda_j(n) \tag{2.11}$$

where $\lambda_j(n)/\sqrt{n}$ is u_j's eigenvalue for T_n. In particular, these satisfy

$$\lambda_j(n)\lambda_j(m) \;=\; \sum_{d|(n,m)} \lambda_j\left(\frac{nm}{d^2}\right). \tag{2.12}$$

For $\Re(s) > 1$ we define the Rankin-Selberg L-functions $R_j(s)$ by

$$R_j(s) \;=\; \sum_{n\geqslant 1} |\nu_j(n)|^2\, n^{-s}\,. \tag{2.13}$$

It is known (Rankin and Selberg) that $L_j(s) = \zeta(2s)\,R_j(s)$ has an analytic continuation to the complex plane with a simple pole at $s = 1$ and residue 2. Furthermore, $L_j(s)$ satisfies the functional equation

$$\Lambda_j(s) \;:=\; L_j(\infty,s)\,L_j(s) \;=\; \Lambda_j(1-s) \tag{2.14}$$

where

$$L_j(\infty,s) \;=\; \pi^{-2s}\,\Gamma^2\!\left(\frac{s}{2}\right)\Gamma\left(\frac{s}{2}+it_j\right)\Gamma\left(\frac{s}{2}-it_j\right).$$

In the critical strip $0 < \Re(s) < 1$ we have the convexity bounds:

Proposition 2.2. *For $0 < \beta < 1$ and $\Re(s) = \beta$ we have*

$$|R_j(s)| \ll_{\epsilon} |t_j|^{1-\beta+\epsilon}\,|s|^{2(1-\beta)+\epsilon}\,.$$

Proof: It is known ([Iw2 pp. 130]) that

$$\sum_{1\leqslant m\leqslant M} |\nu_j(m)|^2 \ll_{\epsilon} M\,|t_j|^{\epsilon}\,. \tag{2.15}$$

Hence for $\Re(s) = 1+\epsilon$ we have

$$|L_j(s)| \;\ll\; |R_j(s)| \ll_{\epsilon} |t_j|^{\epsilon}\,. \tag{2.16}$$

Now applying the functional equation (2.14) and Stirling's formula gives for $\Re(s) = \epsilon$

$$|L_j(s)| \ll_{\epsilon} |t_j|^{1+3\epsilon}\,|s|^{2+4\epsilon}\,. \tag{2.17}$$

Applying the Phragmen-Lindelöf principle and Stirling's formula we get

$$|R_j(s)| \underset{\epsilon}{\ll} |t_j|^{1-\beta+6\epsilon}\,|s|^{2(1-\beta)+6\epsilon}\,,$$

on the line $\Re(s)=\beta$.

■

We will also make use of Luo's zero density theorem for the family $L_j(s)$ (see [Lu]). Specifically, we appeal to the following consequences of his results.

Proposition 2.3. *Let $\eta>0$ be a sufficiently small constant, then at most $t^{1/5}$ of the R_j's with $|t_j|\leqslant t$ have a zero in the rectangle*

$$1-\eta\leqslant\Re(s)\leqslant 1\,,\quad |\Im(s)|\leqslant\log^3 t\,.$$

Furthermore, all but at most $t^{1/5}$ of these R_j's with $|t_j|\leqslant t$ satisfy

$$|(s-1)\,R_j(s)|\underset{\epsilon}{\ll}(|s|\,|t|)^{\epsilon}$$

for

$$1-\eta/2\leqslant\Re(s)<1\,,\quad |\Im s|\leqslant(\log t)^2\,.$$

§3. Multiplicity Bounds

We turn to the proofs of Theorem 1.2 and Corollary 1.3. Let $h(x)$ satisfy the conditions stated before (1.7) and let

$$h_{t,L}(x):=h(L(t-x))+h(L(t+x))\text{ where } t \text{ is large and } L\ll t^{\epsilon}\,. \tag{3.1}$$

$h_{t,L}$ is even and satisfies the conditions imposed on p in Proposition 2.1. Note that

$$N_h(t,L)=\sum_{j\geqslant1}h_{t,L}(t_j)+O(1)\,. \tag{3.2}$$

In order to apply Proposition 2.1, we introduce the weights $|\nu_j(n)|^2$ and then remove them by averaging over $n\leqslant N$ (with N to be chosen). According to Proposition 2.3 with $2t$ instead of t, we split the set of $|t_j|\leqslant 2t$ into two sets G_1 and G_2. G_1 contains those t_j for which $R_j(s)$ has no zeros in the rectangle described in the Proposition and G_2 contains the rest. By the Proposition $|G_2|\leqslant(2t)^{1/5}$.

Now consider

$$\Omega_j(N):=\sum_{n=1}^{\infty}|\nu_j(n)|^2\,e^{-n/N}=\frac{1}{2\pi i}\int_{\sigma=2}\Gamma(s)R_j(s)N^s\,ds\,. \tag{3.3}$$

Shifting the contour to $\Re(s) = \beta = 1-\delta$ with $\delta < \eta/2$ and η as in Proposition 2.3 and using the properties of $R_j(s)$ discussed in Section 2, we obtain

$$\Omega_j(N) \,=\, \frac{12N}{\pi} \,+\, I_j(N) \tag{3.4}$$

with

$$I_j(N) \,=\, \frac{1}{2\pi i}\int_{\Re(s)=\beta} \Gamma(s)R_j(s)N^s\,ds\,. \tag{3.5}$$

For $t_j \in G_1$ we apply Proposition 2.3 which gives

$$I_j(N) \underset{\epsilon}{\ll} N^\beta\, t^\epsilon\,. \tag{3.6}$$

For $t_j \in G_2$ we simply apply the convexity bound in Proposition 2.2 and find that

$$I_j(N) \underset{\epsilon}{\ll} N^\beta |t_j|^{1-\beta+\epsilon}\,. \tag{3.7}$$

Hence, combining (3.4), (3.6) and (3.7) gives

$$\begin{aligned} M_h(t,L) \;&:=\; \frac{1}{N}\sum_j h_{t,L}(t_j)\,\Omega_j(N) \\ &=\; \frac{12}{\pi}\,N_h(t,L) \,+\, \frac{1}{N}\sum_j h_{t,L}(t_j)\,I_j(N) \,+\, O(1) \\ &=\; \frac{12}{\pi}\,N_h(t,L) \,+\, \frac{1}{N}\sum_{j\in G_1} \,+\, \frac{1}{N}\sum_{j\in G_2} \,+\, O(1) \end{aligned} \tag{3.8}$$

$$=\; \frac{12}{\pi}\,N_h(t,L) \,+\, O_\epsilon\left(N^{-\delta}t^{1+\epsilon} \,+\, 1\right)\,. \tag{3.9}$$

In order to use Proposition 2.1, we note that the analogous contribution of the continuous spectrum to the sum via in (3.9) is

$$\frac{1}{4\pi N}\sum_{n\ge 1} e^{-n/N}\int_{-\infty}^{\infty} h_{t,L}(x)\,|\eta(n,x)|^2\,dx$$

$$\ll\; N^\epsilon \int_{-\infty}^{\infty} |h_{t,L}(x)|\,\log^2(1+2|x|)\,dx$$

$$\ll\; N^\epsilon\, L^{-1}\,\log(1+2|t|)\,, \tag{3.10}$$

by the well-known bounds $\zeta(1+2ix) \gg \log(1+2|x|)^{-1}$ and $\tau(n) \ll n^\epsilon$, where $\tau(n)$ is the divisor function.

We apply Proposition 2.1 to the sums

$$\sum_{j\geqslant 1} h_{t,L}(t_j)|\nu_j(n)|^2 \;=\; \sum_{j\geqslant 1} h(L(t_j-t))|\nu_j(n)|^2 \;+\; O(1)\,.$$

The main term comes from $h_0 = p_0$ which gives

$$\frac{1}{\pi}\int_{-\infty}^{\infty} x\,\tanh(\pi x)\,[h(L(t-x))\;+\;h(L(t+x))]\,dx\;\sim\;\frac{2t}{\pi L} \tag{3.11}$$

(recall we normalized h so that $\displaystyle\int_{-\infty}^{\infty} h(x)\,dx \;=\; 1$).

Hence this contribution from h_0 to the N-sum is

$$\sim\;\frac{1}{N}\sum_{n\geqslant 1} e^{-n/N}\,\frac{2t}{\pi L}\;\sim\;\frac{2t}{\pi L}\,. \tag{3.12}$$

The other terms that arise applying Kuznetzov formula to (3.11) involve the sum over c and in particular $h^{+}_{t,L}$ and $S(n;c)$. We will estimate these.

First, we need the behavior of $J_{2iy}(2x)$ for $y\;\geqslant\;cx$ and c any positive number. Let $z\;=\;\sqrt{x^2+y^2}$, then we have the asymptotic expansion (see [Er, pp.87])

$$\begin{aligned} J_{2iy}(2x)\;=\;&(2\pi^{1/2})^{-1}\,z^{-1/2}\,e^{-\frac{\pi}{4}i}\,\exp(\pi y)\,\cdot\;\; e\left(\tfrac{z}{\pi}\;-\;\tfrac{y}{\pi}\,\log\left(\tfrac{z-y}{x}\right)\right)\,\cdot\\ &\cdot\left\{1\;+\;\tfrac{1}{2iy}\left(\tfrac{1}{8}\,\tfrac{y}{z}\;-\;\tfrac{5}{24}\,\left(\tfrac{y}{z}\right)^3\right)+\tfrac{1}{(2iy)^2}\,\left(\tfrac{9}{128}\,\left(\tfrac{y}{z}\right)^2\;-\;\tfrac{231}{576}\,\left(\tfrac{y}{z}\right)^4\;+\;\tfrac{1155}{3456}\,\left(\tfrac{y}{z}\right)^6\;+\;\cdots\right)\right\}\,. \end{aligned} \tag{3.13}$$

Also since $J_{-2iy}(x)=\overline{J_{2iy}(x)}$ it follows that (recall (2.8)) $h^{+}_{t,L}(x)=2\Re[h^{+}_{*}(x)]$ where $h_*(y) = h(L(t-y))$.

Proposition 3.1. *For $x \ll t$*

$$h^{+}_{*}(x)\;\sim\;i\pi^{-3/2}\,e^{\frac{-i\pi}{4}}\,\frac{t^{1/2}}{L}\,\left(\frac{ex}{4t}\right)^{2it}\,\hat{h}\,\left(\frac{\log\frac{4t}{ex}}{\pi L}\right)\;+$$

$$+L^{-1}t^{1/2}\,\left(\frac{ex}{4t}\right)^{2it}\,\sum_{k\geq 1} t^{-k}\sum_{m\geq 0}\alpha_m(L^{-1},t^{-1})\,\cdot\,\hat{h}^{(m)}\,\left(\frac{\log\frac{4t}{ex}}{\pi L}\right)$$

where $\alpha_m(L^{-1},t^{-1})$ are polynomials in L^{-1} and t^{-1} and the asymptotic expansion when terminated at say $k\leqslant B$, leaves a remainder of $O(t^{-B})$.

For the rest of the paper, when $t \gg x$ we will only examine the leading term in the above series, the higher order terms can be handled similarly. We always terminate at some fixed order B which is large enough so that the remainder is negligible for our purpose.

Our original test function h satisfies support $\hat{h} \subset [-b, b] \subset (-1, 1)$. Let δ_1 be small with $0 < \delta_1 < 1 - b$ and let $N = t^{\delta_1}$ and $1 \leqslant L \leqslant \frac{2\log t}{\pi}$.

Then

$$\begin{aligned} &\frac{1}{N} \sum_{n \geqslant 1} e^{-n/N} \sum_{c \geqslant 1} \frac{S(n;c)}{c}\, h^{+}_{t,L}\left(\frac{4\pi n}{c}\right) \\ &\ll \frac{1}{N} \sum_{n \leqslant N^{1+\epsilon}} \sum_{c \geqslant 1} \frac{t^{1/2}}{L} \frac{(n,c)^{1/2}}{c^{1/2}}\, \tau(c) \left| \hat{h}_{t,L}\left(\frac{\log \frac{ct}{e\pi n}}{\pi L}\right)\right| \qquad (3.14) \\ &\ll_{\epsilon} L^{-1} N^{\epsilon}\, t^{b+\frac{\delta_1}{2}} \;=\; o\left(\frac{t}{L}\right) \end{aligned}$$

where we have invoked Weil's bound

$$S(n;c) \;\ll\; c^{1/2}(n,c)^{1/2}\, \tau(c)\,. \qquad (3.15)$$

Combining (3.14), Proposition (2.1), (3.11), (3.10) yields

$$M_h(t,L) \;=\; \frac{2t}{\pi L} \;+\; o\left(\frac{t}{L}\right)\,. \qquad (3.16)$$

This, together with (3.9) yields (with our choice of N) that for $1 \leqslant L \leqslant (2\log t)/\pi$,

$$N_h(t,L) \;=\; \frac{t}{6L} \;+\; o\left(\frac{t}{L}\right)\,. \qquad (3.17)$$

This completes the proof of Theorem 1.2.

To deduce Corollary 1.3 from this, let h be as above with $h(x) \geqslant 0$. Then (recall $\hat{h}(0) = 1$)

$$h(0)m(t) \;\leqslant\; \sum_{j \geqslant 1} h((t_j - t)L) \;\sim\; \frac{t}{6L}\, \hat{h}(0)\,.$$

Taking $L = \frac{2\log t}{\pi}$ (that is as large as is allowed)

$$\varlimsup_{t \to \infty} \frac{m(t)\log t}{t} \;\leqslant\; \frac{\pi}{12} \frac{\hat{h}(0)}{h(0)}\,. \qquad (3.18)$$

As shown in [ILS pp. 115]

$$\min_{\substack{h \geqslant 0 \\ \mathrm{supp}\, \hat{h} \subset [-1,1]}} \frac{\hat{h}(0)}{h(0)} \;=\; 1\,.$$

Hence

$$\overline{\lim_{t\to\infty}}\frac{m(t)\log t}{t} \leqslant \frac{\pi}{12}.$$

This proves Corollary 1.3.

Note that the right sides of (35) and (36) of [Sa3] and the corresponding bound in [Sa2] should be multiplied by 2.

§4. Smoothed Number Variance

The final three sections are concerned with proving Theorem 1.4. In this section we use Proposition 2.1 to bring the number variance into a form that will allow us to determine its asymptotic behavior. Fix $\phi(x)$ and $\psi(x)$ smooth test functions which are supported in $(1,2)$ and which satisfy $\int_{-\infty}^{\infty}\psi(x)dx = \int_{-\infty}^{\infty}\phi(x)dx = 1$. Define the weighted number variance $\sum_h^{\mathrm{w}}(T,L)$ by

$$\textstyle\sum_h^{\mathrm{w}}(T,L) := \frac{1}{T}\int_0^{\infty}\left(M_h^{\phi}(t,L) - \frac{t}{6L}\right)^2 \psi\left(\frac{t}{T}\right)dt \tag{4.1}$$

where

$$M_h^{\phi}(t,L) = \sum_{j\geqslant 1} h_{t,L}(t_j)\left(\frac{t\pi}{12NT}\sum_{n\geqslant 1}|\nu_j(n)|^2\,\phi\left(\frac{nt}{NT}\right)\right), \tag{4.2}$$

and N is to be determined as a function of T. As in the last section, we have

$$\sum_n|\nu_j(n)|^2\,\phi\left(\frac{nt}{NT}\right) = \frac{12}{\pi}\,\frac{NT}{t} + I_j^{\phi}(N), \tag{4.3}$$

where

$$I_j^{\phi}(N) = \frac{1}{2\pi i}\int_{\Re(s)=\frac{1}{2}}\left(\frac{NT}{t}\right)^s R_j(s)\,\Omega(s)\,ds$$

and

$$\Omega(s) = \int_0^{\infty}\phi(\xi)\,\xi^s\,\frac{d\xi}{\xi}.$$

Hence, using the convexity bound for $R_j(s)$ in Proposition 2.2, we have

$$I_j^{\phi}(N) = O_{\epsilon}\left(N^{1/2}\,|t_j|^{1/2+\epsilon}\right). \tag{4.4}$$

Thus,

$$\textstyle\sum_h^{\mathrm{w}}(T,L) = \sum_h(T,L) + O\left(N^{-1}\,T^{3+\epsilon} + N^{-1/2}\,T^{5/2+\epsilon}\right). \tag{4.5}$$

For the rest of the paper we choose $N = T^{100}$. With this it clearly suffices to study $\sum_h^{\mathrm{w}}(T,L)$ rather than $\sum_h(T,L)$.

Apply the Kuznetzov formula to the j sum in (4.2). One checks that the continuous spectrum contribution is $O_\epsilon(N^\epsilon)$. The contribution from the h_0 term is

$$\begin{aligned}&\frac{\pi t}{12NT}\sum_n \phi\left(\frac{nt}{NT}\right)\frac{2}{\pi}\int_{-\infty}^{\infty} x\tanh(\pi x)h_{t,L}(x)\,dx\\ &\qquad\sim \frac{t}{6L}\,,\quad \text{with a negligible error term.}\end{aligned}\tag{4.6}$$

Hence, with our choice of N and the above comment about the continuous spectrum, we have

$$\textstyle\sum_h^{\mathrm{w}}(T,L) \;=\; \sigma(T,L)\;+\;O\;\left(\sigma(T,L)^{1/2}\,N^\epsilon\right)\tag{4.7}$$

where

$$\sigma(T,L)\;=\;\frac{1}{T}\int_0^\infty\left|\frac{\pi t}{12NT}\sum_c\sum_n\frac{S(n;c)}{c}\,h^+_{t,L}\left(\frac{4\pi n}{c}\right)\phi\left(\frac{nt}{NT}\right)\right|^2\cdot\,\psi\left(\frac{t}{T}\right)dt\tag{4.8}$$

and $h^+_{t,L}(x)=2\Re(h^+_*(x))$ with

$$h^+_*(x)\;=\;\frac{2i}{\pi}\int_{-\infty}^{\infty}J_{2iy}(x)\,\frac{yh(L(y-t))}{\cosh\pi y}\,dy\,.\tag{4.9}$$

So for our purposes it is sufficient to investigate $\sigma(T,L)$.

Note that the integrand in (4.9) is negligible unless y is near t. Consider first the range of summation for c where $\frac{N}{c}\geqslant T$. In this case the argument $\frac{4\pi n}{c}$ in h^+ is $\geqslant 4T$ and we use the following estimates for J.

For $x\geq 2y$

$$J_{2iy}(x)\;=\;\frac{1}{\sqrt{2\pi x}}\;\left(W_1(2iy,x)e^{ix}\;+\;W_2(2iy,x)\,e^{-ix}\right)\tag{4.10}$$

where

$$\frac{\partial^{(j)}}{\partial x^j}W_i\ll_j(1+|x|)^{-j}\,\cosh\pi y\qquad \text{for } i\;=\;1,2,\;j\geq 0\;\;\text{(see[Wa], pp. 205).}$$

Hence, applying Poisson summation

$$\begin{aligned}\sum_n S(n;c)\,J_{2iy}\left(\frac{4\pi n}{c}\right)\phi\left(\frac{nt}{NT}\right)\;&=\;\sum_{d(\mathrm{modc})}S(d;c)\,e\left(\frac{2d}{c}\right)\sum_m e\left(\frac{-md}{c}\right)\\ &\cdot\int_{-\infty}^{\infty}x^{-1/2}\,W\left(2iy,\,\frac{4\pi x}{c}\right)e\left(\frac{mx}{c}\right)\phi\left(\frac{xt}{NT}\right)dx\\ \sim\;\sqrt{c}\sum_{d(\mathrm{modc})}&S(d;c)\,e\left(\frac{2d}{c}\right)\int_{-\infty}^{\infty}x^{-1/2}\,W\left(2iy,\,\frac{4\pi x}{c}\right)\cdot\,\phi\left(\frac{xt}{NT}\right)dx\\ &\ll\;c^{1/2}\,N^{1/2}\,\cosh\pi y\,.\end{aligned}\tag{4.11}$$

Hence for $c \leqslant NT^{-1}$

$$\sum_n S(n;c)\, h^+_{t,L}\left(\frac{4\pi n}{c}\right)\,\phi\left(\frac{nt}{NT}\right) \ll \frac{t}{L}\, c^{1/2}\, N^{1/2}$$

and

$$\frac{\pi t}{12NT}\sum_{c\leqslant NT^{-1}}\sum_n \frac{S(n,c)}{c}\, h^+_{t,L}\left(\frac{4\pi n}{c}\right)\,\phi\left(\frac{nt}{NT}\right)$$

$$\ll \frac{t}{NL}\, N^{1/2}\sum_{c\leqslant NT^{-1}} c^{-1/2} = \frac{t}{LN^{1/2}}\cdot\frac{N^{1/2}}{t^{1/2}} = t^{1/2}/L\,. \tag{4.12}$$

Hence, we have

$$\sum\nolimits_h^{\mathrm{w}}(T,L) = \frac{1}{T}\int_0^\infty \left|\frac{\pi t}{12NT}\sum_{c\geqslant NT^{-1}}\sum_n \frac{S(n;c)}{c}\, h^+_{t,L}\left(\frac{4\pi n}{c}\right)\,\phi\left(\frac{nt}{NT}\right)\right|^2\cdot\,\psi\left(\frac{t}{T}\right)dt$$

$$+\,O\left(\frac{T}{L^2}\right) \tag{4.13}$$

Applying Proposition 3.1 to $h^+_*\left(\frac{4\pi n}{c}\right)$ with $c \geqslant NT^{-1}$ yields

$$\sum\nolimits_h^{\mathrm{w}}(T,L) =$$

$$\frac{4}{\pi^3L^2}\int_0^\infty \left|\frac{t^{1/2}\pi t}{12NT}\,\Re\left[-i\, e^{\frac{\pi i}{4}}\, e^{\frac{\pi}{42}}\sum_{c\geqslant NT^{-1}}\sum_n \frac{S(n;c)}{c}\left(\frac{e\pi n}{tc}\right)^{-2it}\cdot\right.\right.$$

$$\left.\left.\cdot\,\phi\left(\frac{nt}{NT}\right)\,\hat{h}\left(\frac{\log\frac{ct}{en}}{\pi L}\right)\right]\right|^2\,\psi\left(\frac{t}{T}\right)\,\frac{dt}{T} + O\left(\frac{T}{L^2}\right) \tag{4.14}$$

In order to execute the n sum in (4.14) we write

$$S(n;c) = \sum_{-\frac{c}{2}<a\leqslant\frac{c}{2}} \rho(c,a)\, e\left(\frac{na}{c}\right) \tag{4.15}$$

where $\rho(a,c)$ denotes the number of solutions $d(\bmod\ c)$ of

$$d^2 - ad + 1 \equiv 0(c)\,. \tag{4.16}$$

Now applying the Euler-Maclaurin formula gives

$$\sum_n e\left(\frac{an}{c}\right) n^{-2it}\,\hat{h}\left(\frac{\log\frac{ct}{n}}{\pi L}\right)\phi\left(\frac{nt}{NT}\right)$$
$$= \int_{\mathbb{R}} e(f(x))\,\hat{h}\left(\frac{\log\frac{ct}{x}}{\pi L}\right)\phi\left(\frac{xt}{NT}\right)dx + O\left(\frac{1}{N}\right) \tag{4.17}$$

where

$$f(x) = \frac{ax}{c} - t\,\frac{\log x}{\pi}\,.$$

From now on, set

$$U = e^{\pi L} \tag{4.18}$$

(so that $U \leqslant T^{1+\delta}$ for $\delta > 0$ but small as in Theorem 1.4). If $a = 0$ or if $|a| \geq 100U$ then there is no stationary phase point in the integral in (4.17) and one sees that the integral is $O_B(NT^{-B})$ for any positive B. In particular, this restricts the range of a's that we need to consider. For $0 < |a| \leqslant 100\,U$ we use the stationary phase method ([Hu]) and find that

$$\sum_n e\left(\frac{an}{c}\right) n^{-2it}\,\hat{h}\left(\frac{\log\frac{ct}{n}}{\pi L}\right)\phi\left(\frac{nt}{NT}\right)$$
$$= \frac{e(f(x_0) + \frac{1}{8})}{\sqrt{f''(x_0)}}\,\hat{h}\left(\frac{\log\frac{ct}{x_0}}{\pi L}\right)\phi\left(\frac{x_0 t}{NT}\right) + O\left(\frac{N}{T^{3/2}}\right) \tag{4.19}$$

with

$$x_0 = \frac{tc}{a\pi}\,.$$

By the mean value theorem ([Iw]) for $\rho(c,a)$:

$$\sum_{1\leqslant c\leqslant C}\ \sum_{A\leqslant a\leqslant 2A} \rho(c,a) = \frac{6}{\pi^2}\,AC + O_\epsilon\left(\left(A^{\frac{11}{6}}C + AC^{1/2}\right)C^{\epsilon}\right) \tag{4.20}$$

and hence the contribution to $\sum_h^{\mathrm{w}}(T,L)$ from the error term in (4.19) is at most $(UT^{-1}L^{-1})^2$. We arrive at

$$\textstyle\sum_h^{\mathrm{w}}(T,L) =$$
$$\frac{4}{144\pi^2L^2}\int_0^\infty \left|\frac{t^2}{NT}\,\Re\left[\sum_{a>0} a^{2it-1}\,W_a\,\hat{h}\left(\frac{\log a}{\pi L}\right)\right]\right|^2 \psi\left(\frac{t}{T}\right)\frac{dt}{T}$$
$$+\,O\left(TL^{-2} + U^2T^{-2}L^{-2}\right)\,, \tag{4.21}$$

where

$$W_a = \sum_c \rho(c,a)\,\phi\left(\frac{cT}{aN}\right)\,. \tag{4.22}$$

Squaring out in (4.21) leads to

$$\sum_h^{\mathrm{w}}(T,L) \;=\; \frac{T^2}{576\pi^6.L^2N^2}\sum_{|k|\leqslant K} D(k,T) \;+\; O\left(TL^{-2} \;+\; U^2T^{-2}L^{-2}\right) \tag{4.23}$$

where $K \;=\; UT^{-1+\epsilon}$ and (assuming $k \geqslant 0$ without loss of generality)

$$\begin{aligned} D(k,T) \;=\; \sum_a W_a\, W_{a+k}\, \hat{h}\left(\frac{\log a}{\pi L}\right) \hat{h}\left(\frac{\log(a+k)}{\pi L}\right) \\ \cdot\, \hat{\psi}^{(4)}\left(\frac{T}{\pi}\,\log\,\frac{a+k}{a}\right)\, a^{-1}(a+k)^{-1}\,. \end{aligned} \tag{4.24}$$

Note that $\rho(c,a)$ is multiplicative in c. Following [Iw1, pp. 154] we factor c as $k\ell$ with $(k,4\ell)=1$ and k being the square-free part of c, we have $\rho(c,a)=\rho(k,a)\rho(\ell,a)$ and

$$\rho(k,a) \;=\; \sum_{r|k}\left(\frac{a^2-4}{r}\right)\,. \tag{4.25}$$

Let $\mathcal{L}$ be the set of integers ℓ s.t. $p|\ell \Rightarrow p^2|\ell$, then using (4.25) we may write for R a large parameter which will be chosen shortly:

$$W_a \;=\; W_{a,1} \;+\; W_{a,2}\,. \tag{4.26}$$

where

$$W_{a,1} \;=\; \sum_{\substack{\ell\\ \ell\in\mathcal{L}}}\;\sum_{\substack{r\\ \ell r\leqslant R}}\;\sum_{\substack{s\\ (rs,4\ell)=1}} \mu^2(rs)\,\rho(\ell,a)\,\left(\frac{a^2-4}{r}\right)\,\phi\left(\frac{\ell rsT}{aN}\right) \tag{4.27}$$

and

$$W_{a,2} \;=\; \sum_{\substack{\ell\\ \ell\in\mathcal{L}}}\;\sum_{\substack{r\\ \ell r\geqslant R}}\;\sum_{\substack{s\\ (rs,4\ell)=1}} \mu^2(rs)\rho(\ell,a)\,\left(\frac{a^2-4}{r}\right)\,\phi\left(\frac{\ell rsT}{aN}\right)\,. \tag{4.28}$$

In order to estimate the r-sum in (4.28), we use the zero density theorem for Dirichlet L-functions as a substitute for the Lindelöf hypothesis. Let

$$L(a,s) \;=\; \sum_{r\geq 1}\left(\frac{a^2-4}{r}\right)\,r^{-s}\,.$$

Choosing an exponential smoothing (we could of course use any smoothing essentially),

we have

$$I(a) := \sum_{r\geq 1}\left(\frac{a^2-r}{r}\right)\,e^{-r/R_1} \;=\; \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\Gamma(s)\,L(a,s)\,R_1^s ds\,. \tag{4.29}$$

Move the line of integration to $\beta = 1-\delta$ with $0<\delta\leqslant \frac{1}{2}$, this yields

$$I(a) \;=\; \frac{1}{2\pi i}\int_{\beta}\Gamma(s)\,L(a,s)\,R_1^s ds\,. \tag{4.30}$$

Let D_{R_1} be the rectangle

$$1-\frac{1}{30}\;<\;\Re(s)<1\,,\;-2\log R_1\;<\;\Im(s)\;<\;2\log R_1\,. \tag{4.31}$$

Then according to Barban ([Ba], Lemma 5.3) and Stirling's formula, if $L(a,s)$ has no zeroes in D_{R_1}, in which case we say $a\in G(R_1)$, we have

$$I(a) \ll_{\epsilon} a^{\epsilon} R_1^{\frac{29}{30}}\,. \tag{4.32}$$

Let $B(R_1)$ be the complement of $G(R_1)$ with $|a|\leqslant 100\,U$. Using standard density theorems (see for example [Sa1, pp. 342]) we have

$$\#B(R_1)\;\ll\;U^{1/2}\,. \tag{4.33}$$

For $a\in B(R_1)$ we employ the convexity bound for $L(a,s)$

$$L(a,s) \ll_{\epsilon} |a|^{1/2}\,|s|^M$$

with $M>0$ and $\Re(s)\;=\;\frac{1}{2}$. This yields

$$I(a) \ll_{\epsilon} |a|^{1/2+\epsilon}\,R_1^{1/2} \qquad \text{for } a\in B(R_1). \tag{4.34}$$

One can remove the weights e^{-r/R_1} by Fourier analysis (see for example [Iw1] pp. 146) to get

Lemma 4.1. *For $a\in G(R_1)$,*

$$\sum_{1\leqslant r\leqslant R_1}\left(\frac{a^2-4}{r}\right) \ll_{\epsilon} a^{\epsilon}\,R_1^{\frac{29}{30}}\,,$$

while for any $a\in B(R_1)$

$$\sum_{1\leqslant r\leqslant R_1}\left(\frac{a^2-4}{r}\right) \ll_{\epsilon} a^{1/2+\epsilon}\,R_1^{1/2}\,.$$

Adding the condition that r be square free and relatively prime to a given q by the device in [IS, pp. 329], we obtain

Corollary 4.2. *Let $q \geq 1$,*

for $a \in G(R_1)$

$$\sum_{\substack{r \leqslant R_1 \\ (r,q)=1}} \mu^2(r) \left(\frac{a^2-4}{r}\right) \underset{\epsilon}{\ll} R_1^{\frac{29}{30}+\epsilon} (aq)^\epsilon$$

while for $a \in B(R_1)$

$$\sum_{\substack{r \leqslant R_1 \\ (r,q)=1}} \mu^2(r) \left(\frac{a^2-4}{r}\right) \underset{\epsilon}{\ll} |a|^{1/2+\epsilon} R_1^{\frac{1}{2}+\epsilon} q^\epsilon \,.$$

It is elementary that

$$\sum_{\substack{s \leqslant S \\ (s,q)=1}} \mu^2(s) \sim \frac{6}{\pi^2} \Pi_{p|q} \left(1+p^{-1}\right)^{-1} S \,. \tag{4.35}$$

We break the r-sum in (4.28) into dyadic boxes $[R_1, 2R_1]$ and apply Corollary 4.2 and (4.35). For $a \in G(R)$ the contribution to (4.28) is

$$\underset{\epsilon}{\ll} \sum_{\substack{\ell \geq 1 \\ \ell \in \mathcal{L}}} \ell^{-29/30} |a|^{1+\epsilon} N^{1+\epsilon} T^{-1} R^{-\frac{1}{30}} \underset{\epsilon}{\ll} |a|^{1+\epsilon} N^{1+\epsilon} T^{-1} R^{-\frac{1}{30}} \,. \tag{4.36}$$

While for $a \in B(R)$ it is

$$\underset{\epsilon}{\ll} |a|^{3/2+\epsilon} N^{1+\epsilon} T^{-1} R^{-1/2} \,. \tag{4.37}$$

Consider now

$$\sum_a W_{a,2} W_{a+k,1} \, \hat{h}\left(\frac{\log a}{\pi L}\right) \hat{h}\left(\frac{\log(a+k)}{\pi L}\right) \hat{\psi}^{(4)}\left(T \log \frac{a+k}{a}\right) (a(a+k))^{-1} \,. \tag{4.38}$$

We split it into $\sum_{a \in B(R)}$ and $\sum_{a \in G(R)}$ and use the trivial bound

$$W_{a+k,1} \ll |a+k| N T^{-1}$$

and (4.36), (4.37) and (4.33), we get that

$$\begin{aligned} \sum_{a \in G(R)} &\underset{\epsilon}{\ll} (U+k) N T^{-1} U^{-1} (U+k)^{-1} U^{1+\epsilon} N^{1+\epsilon} T^{-1} R^{-\frac{1}{30}} U \\ &\underset{\epsilon}{\ll} U^{1+\epsilon} T^{-2} N^{2+\epsilon} R^{-\frac{1}{30}+\epsilon} \end{aligned} \tag{4.39}$$

and

$$\sum_{a\in B(R)} \ll_{\epsilon} U^{1+\epsilon}\, T^{-2}\, N^{2+\epsilon} R^{-1/2}\,. \tag{4.40}$$

On summing $|k| \leqslant K$ (K as in (4.23)), it follows that the contribution to (4.23) of the terms in (4.38) is

$$\ll\; N^{\epsilon} U^{2+\epsilon}\, R^{-\frac{1}{30}}\, T^{-1}\,. \tag{4.41}$$

The same applies to the contributions from $W_{a,1}W_{a+k,2}$ and $W_{a,2}\; W_{a+k,2}$. Hence if we choose

$$R \;=\; U^{(60+\frac{1}{10^5})} T^{-60} \tag{4.42}$$

then we have that

$$\textstyle\sum_h^{\mathrm{w}}\,(T,L) \;=\; \dfrac{T^2}{576\pi^6 L^2 N^2} \displaystyle\sum_{|k|\leqslant K} D_1(k,T) \;+\; O(T/L^2) \tag{4.43}$$

where

$$D_1(k,T) \;=\; \sum_{a}\sum_{\substack{b\\ b-a=k}} W_{a.1}\, W_{b,1}\hat{h}\left(\frac{\log a}{\pi L}\right)\, \hat{h}\left(\frac{\log b}{\pi L}\right)\,\cdot \hat{\psi}^{(4)}\left(\frac{T}{\pi}\,\log\,\frac{a}{b}\right)\,(ab)^{-1} \tag{4.44}$$

and where

$$T \;<\; U \;<\; T^{\frac{122}{121}}\,. \tag{4.45}$$

The condition (4.45) is ensured by the choice (4.18) and the support condition of h in Theorem 1.4.

In the next section, we use the "δ-method" to study the asymptotic behavior of the shifted sum (4.44). Note for later that with U satisfying (4.45)

$$R \leqslant T^{1/2-\delta_1} \text{ with } \delta_1 \text{ fixed and positive.} \tag{4.46}$$

§5. The δ-Method

We recall the flexible variant of Kloosterman's circle method, due to Duke-Friedlander and Iwaniec [DFI], known as the δ-method. It uses Fourier analysis to isolate the terms $b-a=k$ in the shifted sum (4.44). Let $w(u)$ be a smooth even test function supported in $V<|u|<2V$ where V is a large parameter. Assume further that w satisfies

$$w^j(u) \ll_j V^{-j-1} \text{ for each } j\geq 0\,. \tag{5.1}$$

Normalize $w(u)$ by requiring that

$$\sum_{v\geqslant 1} w(v) \;=\; 1\,. \tag{5.2}$$

Then for any $n \in \mathbb{Z}$

$$\delta(n) = \sum_{u \geqslant 1} \sideset{}{^*}\sum_{u(\mathrm{mod} v)} e\left(\frac{un}{v}\right) \triangle_v(n) \tag{5.3}$$

with

$$\triangle_v(u) = \sum_{r \geqslant 1} (vr)^{-1} \left(w(vr) - w\left(\frac{u}{vr}\right)\right) . \tag{5.4}$$

Now $\triangle_v(u)$ is an approximate δ-function as the following shows (see [DFI]):

Lemma 5.1. *For $f \in C_0^\infty(\mathbb{R})$ and any $j \geq 1$ we have*

(i) $\displaystyle\int_{-\infty}^{\infty} f(u) \triangle_v(u) du = f(0) + O\left(V^{-1} v^j \int_{-\infty}^{\infty} \left(V^{-j}|f(u)| + V^j |f^{(j)}(u)|\right) du\right)$

(ii) $\triangle_v(u) \ll (vV + V^2)^{-1} + (vV + |u|)^{-1}$

(iii) $\frac{\partial^a}{\partial u^a} \triangle_v(u) \ll_a (vV)^{-a-1}$, *for $a \geq 0$.*

Note that (i) is only useful if $v \ll V^{1-\epsilon}$, in the case $v \gg V^{1-\epsilon}$ we use (ii).

We study the shifted sums in (4.44) in more general form: Let $f(x,y)$ be a smooth function of x and y satisfying:

$$x^{i+1} y^{j+1} f^{(i,j)}(x,y) \ll_{i,j} \left(1 + \frac{|x|}{U}\right)^{-B} \left(1 + \frac{|y|}{U}\right)^{-B} \tag{5.5}$$

for each $i, j \geq 0$, where $f^{(i,j)}$ is the mixed (i,j)-th partial derivative, B is a large constant and U a large parameter.

In our application we take

$$f(x,y) = \frac{1}{xy} \hat{h}\left(\frac{\log x}{\pi L}\right) \hat{h}\left(\frac{\log y}{\pi L}\right) \hat{\psi}^{(4)}\left(\frac{T}{\pi} \log \frac{x}{y}\right) . \tag{5.6}$$

We can assume that $\hat{h}$ is supported in a small neighborhood of $\{-1, 1\}$ and so $U = e^{\pi L}$ as in (4.18) and L is as in Theorem 1.4. By a partition of unity argument we can reduce to the case that f is supported in $[U, 2U] \times [U, 2U]$. The shifted sums that we study are

$$D_f(k,T) := \sum_a \sum_{a-b=k} W_{a,1} W_{b,1} f(a,b) \tag{5.7}$$

where

$$W_{z,1} = \sum_{\substack{\ell \\ \ell \in \mathcal{L}, \ell r \leqslant R \\ (rs, 4\ell)=1}} \sum_r \sum_s \mu^2(rs) \rho(\ell, z) \left(\frac{z^2-4}{r}\right) \phi\left(\frac{\ell r s T}{zN}\right) . \tag{5.8}$$

In applying the δ-method to (5.7) we take the parameter V in the definition of $w(u)$ to be

$$V \;=\; U^{1/2}\,, \tag{5.9}$$

so that

$$\triangle_v(u) \;=\; 0 \;\text{ if }\; |u| \leqslant U \;\text{ and }\; v \geq 2V\,. \tag{5.10}$$

Thus

$$D_f(k,T) \;=\; \sum_{1\leqslant v\leqslant 2V}\;\sideset{}{^*}\sum_{u(v)} e\left(\frac{-ku}{v}\right)\;\sum_a\sum_b W_{a,1}\,W_{b,1}\,e\left(\frac{ua-ub}{v}\right)\;E(a,b) \tag{5.11}$$

with

$$E(x,y) \;=\; f(x,y)\,\triangle_v(x-y-k)\,. \tag{5.12}$$

Note that from Lemma 5.1 and (5.5)

$$E(x,y) \;\ll\; U^{-2}(vV)^{-1} \tag{5.13}$$

and

$$E^{(i,j)}(x,y) \;\ll\; (vV)^{-1-i-j}\,U^{-2}\,,\; i,j\;\geq 0\,. \tag{5.13$'$}$$

Next we carry out a and b sums in (5.11). We begin with the a sum. Let

$$F(u,v,b) \;=\; \sum_a W_{a,1}\,e\left(\frac{ua}{v}\right)\;E(a,b) \tag{5.14}$$

$$= \sum_{\substack{\ell_1 \\ \ell_1\in\mathcal{L}\,,\,\ell_1 r_1\leqslant R_1 \\ (r_1s_1,4\ell_1)=1}}\;\sum_{r_1}\;\sum_{s_1}\;\mu^2(r_1s_1)\sum_a \rho\,(\ell_1,a)\;\left(\frac{a^2-4}{r_1}\right)\;e\left(\frac{ua}{v}\right)\;I_1\,(a,b,\ell_1r_1s_1) \tag{5.15}$$

where

$$I_1(a,b,c_1) \;=\; E(a,b)\,\phi\left(\frac{c_1T}{aN}\right)\,. \tag{5.16}$$

Splitting the a sum in (5.15) into residue classes $\alpha(\bmod\ \ell_1)$ and applying Poisson summation yields

$$\begin{aligned}\sum_{a\equiv\alpha(\ell_1)}\;\left(\frac{a^2-4}{r_1}\right)\;e\left(\frac{ua}{v}\right)\;I_1(a,b,\ell_1r_1s_1)&\\ =\;\frac{1}{\ell_1r_1}\sum_{s(r_1)}\;\left(\frac{s^2-4}{r_1}\right)\;\sum_{h_1} e&\left(\frac{-h_1(\alpha r_1\bar r_1+s\ell_1\bar\ell_1)}{\ell_1r_1}\right)\\ &\cdot\,\widehat{I}_1\;\left(\frac{h_1}{\ell_1r_1}+\frac{u}{v},b,\ell_1r_1s_1\right)\end{aligned} \tag{5.17}$$

where $\bar r_1$ and $\bar\ell_1$ are defined by

$$r_1\bar r_1 \;\equiv\; 1(\mod \ell_1) \text{ and } \ell_1\bar\ell_1 \;\equiv\; 1(\mod r_1)$$

and $\widehat{I}_1$ denotes Fourier transform in the first variable. Hence, the a sum in (5.15) is

$$\begin{aligned}\sum_a \;&=\; \sum_a \rho(\ell_1,a)\left(\frac{a^2-4}{r_1}\right) e\left(\frac{ua}{v}\right) I_1\left(a,b,\ell_1r_1s_1\right)\\ &= \frac{1}{\ell_1r_1}\sum_{h_1} S(-h_1\bar{r}_1;\ell_1)\, H(h_1,\ell_1,r_1)\,\widehat{I}_1\left(h_1',b,\ell_1r_1s_1\right)\end{aligned} \tag{5.18}$$

where

$$H(h_1,\ell_1,r_1) \;=\; \sum_{s(r_1)}\left(\frac{s^2-4}{r_1}\right) e\left(\frac{-h_1\bar{\ell}_1 s}{r_1}\right), \tag{5.19}$$

$$h_1' \;=\; \frac{h_1}{\ell_1r_1} \;+\; \frac{u}{v} \tag{5.19$'$}$$

and where we have used (4.15). We define $H(h_2,\ell_2,r_2)$ similarly.

Next, we apply Poisson summation in b,

$$\begin{aligned}\sum_b \;&=\; \sum_b \rho(\ell_2,b)\left(\frac{b^2-4}{r_2}\right) e\left(\frac{ub}{v}\right)\widehat{I}_1\left(h_1',b,\ell_1r_1s_1\right)\phi\left(\frac{c_2T}{bN}\right)\\ &= \frac{1}{\ell_2r_2}\sum_{h_2} S(-h_2\bar{r}_2;\ell_2)\, H(h_2,\ell_2,r_2)\, I_2(h_1',h_2',c_1,c_2)\end{aligned} \tag{5.20}$$

where $h_2' \;=\; \frac{h_2}{\ell_2r_2} \;+\; \frac{u}{v}$ and

$$\begin{aligned}I_2(h_1',h_2',c_1,c_2) \;&=\; \int\!\!\int I_1(x,y,c_1)\,\phi\left(\frac{c_2T}{yN}\right) e(h_1'x+h_2'y)\,dxdy\\ &= \int\!\!\int E(x,y)\,\phi\left(\frac{c_1T}{xN}\right)\;\phi\left(\frac{c_2T}{yN}\right)\; e\left(h_1'x+h_2'y\right)\;dxdy\end{aligned} \tag{5.21}$$

with $c_i \;=\; \ell_ir_is_i\,,\; i \;=\; 1,2.$

Now if $h_j' \neq 0$ then according to (5.19$'$) and (4.46)

$$|h_j'vV| \;\gg\; \frac{vV}{\ell_jr_jv} \;\gg\; \frac{U^{1/2}}{R} \;\gg\; T^{\delta_1} \text{ with } \;\delta_1>0 \text{ and fixed}\,. \tag{5.22}$$

Hence integrating by parts M times in (5.21) and using (5.13$'$) we conclude that if one of h_1' or h_2' is not zero then the integral I_2 satisfies

$$I_2 \underset{M}{\ll} T^{-\delta_1M} \;\text{ for any }\; M>0\,.$$

Combining (5.18) and (5.20) we arrive at

$$\ell_1\ell_2 r_1 r_2 \sum_b \sum_a =$$

$$S(-h_1^0\,\bar{r}_1;\ell_1)\,S(-h_2^0\,\bar{r}_2;\ell_2)\,H(h_1^0,\ \ell_1,r_1)\,H(h_2^0,\ \ell_2,r_2)\,I_2(0,0,c_1,c_2)\ +\ \text{negligible} \tag{5.23}$$

where

$$\frac{h_i^0}{\ell_i r_i} + \frac{u}{v} = 0 \ \text{ for } \ i = 1,2\,. \tag{5.24}$$

(Here, negligible means it is $O(T^{-M})$ for any M.)

To evaluate the main term in (5.23) we need to evaluate various complete character sums.

Lemma 5.2.

$$\sum_{s(q_1q_2)} \left(\frac{s^4-4}{q_1}\right) e\left(\frac{ts}{q_1q_2}\right) = 0$$

if

$$(tq_1,q_2) = 1 \ \textit{and} \quad q_2 > 1\,.$$

Proof: Write

$$s = s_1q_1\bar{q}_1 + s_2q_2\bar{q}_2\,.$$

Then

$$\sum_{s(q_1q_2)} \left(\frac{s^2-4}{q_1}\right) e\left(\frac{ts}{q_1q_2}\right)$$

$$= \sum_{s_2(q_1)} \left(\frac{s_2^2-4}{q_2}\right) e\left(\frac{ts_2\bar{q}_2}{q_1}\right) \sum_{s_1(q_2)} e\left(\frac{ts_1\bar{q}_1}{q_2}\right) = 0 \quad \text{since} \quad (t,q_2) = 1\,.$$

Lemma 5.3. *Let u,v and h_i^0 be as in (5.24), then*

$$S(-h_i^0 r_i;\ell_i)\,H(h_i^0,\ell_i,r_i) = \mu(r_i')\,\phi(\ell_i')\,m_in_i\,\frac{S(u;v)}{v}$$

where for $i=1$ or 2

$$r_i = m_ir_i',\ \ell_i = n_i\ell_i' \ \textit{and}$$

$$m_in_i = vv_i' \ \textit{where} \ v_i'|m_i$$

and if p is a prime and $p|v_i'$ then $p|v$ (i.e. m_i is the part of r_i in v and n_i the part of ℓ_i in v).

(Here μ is the Möbius function and ϕ the Euler function).

Proof: The Kloosterman sum satisfies the multiplicativity

$$S(-h_i^0\bar{r}_i;\ell_i) \;=\; S(-h_i^0\bar{r}_i\overline{\ell_i'}\,;n_i)\,S(-h_i^0\bar{r}_i\overline{n_i}\,;\ell_i')$$

where

$$S(-h_i^0\bar{r}_i\bar{n}_i;\ell_i') \;=\; \phi(\ell_i')$$

since $\ell_i'|h_i^0$, while

$$\begin{aligned} S(-h_i^0\bar{r}_i\bar{\ell}_i';n_i) \;&=\; \sum_{d(n_i)}{}^{*}\, e\left(\frac{-h_i^0\bar{r}_i\bar{\ell}_i'(d+\bar{d})}{n_i}\right) \\ &=\; \sum_{d(n_i)}{}^{*}\, e\left(\frac{u\ell_i'r_i'v_i'\overline{\ell_i'r_i}(d+\bar{d})}{n_i/v_i'\cdot v_i'}\right) \\ &=\; \sum_{d(n_i)}{}^{*}\, e\left(\frac{u\bar{m}_i(d+\bar{d})}{n_i/v_i'}\right) \\ &=\; v_i'\,S(u\bar{m}_i;n_i/v_i')\,. \end{aligned} \tag{5.25}$$

On the other hand,

$$H(h_i^0,\ell_i,r_i) \;=\; H(h_i^0,\ell_ir_i',m_i)\,H(h_i^0,\ell_im_i,r_i')$$

and

$$H(h_i^0,\ell_im_i,r_i') \;=\; \sum_{s(r_i')}\left(\frac{s^2-4}{r_i'}\right)\,e\left(\frac{-h_i^0\overline{\ell_im_i}s}{r_i'}\right) \;=\; \mu(r_i')$$

since $r_i'|h_i^0$.

Finally,

$$\begin{aligned} H(h_i^0,\ell_ir_i',m_i) \;&=\; \sum_{s(m_i)}\left(\frac{s^2-4}{m_i}\right)\,e\left(\frac{-h_i\overline{\ell_ir_i'}s}{m_i}\right) \\ &=\; \sum_{s(m_i)}\left(\frac{(s\ell_ir_i')^2-4}{m_i}\right)\,e\left(\frac{-h_i^0s}{m_i}\right) \\ &=\; \sum_{s(m_i)}\left(\frac{(s\ell_ir_i')^2-4}{m_i}\right)\,e\left(\frac{u\ell_ir_i'v_i's}{m_i}\right) \end{aligned}$$

$$= \sum_{s(m_i)} \left(\frac{s^2-4}{m_i}\right) e\left(\frac{ur_i'v_i'\ell_i's}{m_i}\right)$$

$$= \sum_{s(m_i)} \left(\frac{s^2-4}{m_i}\right) e\left(\frac{uv_i'\bar{n}_i s}{m_i}\right)$$

$$= S(u\bar{n}_i/v_i'; m_i) \tag{5.26}$$

where we have used Lemma 5.2.

Lemma 5.3 now follows from (5.25) and (5.26) and the multiplicativity property of S.

Using (5.11), (5.23) and these lemmas, we conclude that

$$D_f(k,T) = \sum_{1\leqslant v\leqslant 2V} \sideset{}{^*}\sum_{u(v)} e\left(\frac{-uk}{v}\right) \cdot$$

$$\cdot \sum_{\substack{\ell_1 \\ \ell_2 \\ \ell_i r_i \leqslant R}} \sum_{\substack{r_1 \\ r_2}} \sum_{\substack{s_1 \\ s_2}} (\ell_1\ell_2 r_1 r_2)^{-1} \; \mu(r_1')\,\mu(r_2')\,\phi(\ell_1')\,\phi(\ell_2')$$

$$\cdot \; \mu^2(r_1s_1)\mu^2(r_2s_2)m_1n_1\,m_2n_2\,I_2(0,0,c_1,c_2)\,\frac{S^2(u,v)}{v^2} \quad + \text{ negligible}\,, \tag{5.27}$$

recall that R is given by (4.42).

To evaluate the main term in (5.27), set

$$\Omega_{c_2}(s_1) = \int_0^\infty I_2(0,0,\ell_1 r_1 x, c_2) x^{s_1}\,\frac{dx}{x}\,.$$

Then we have by Mellin inversion

$$\sum_{\substack{s_1 \\ (s_1, 4\ell_1 r_1)=1}} \mu^2(s_1)\,I_2(0,0,\ell_1 r_1 s_1, c_2)$$

$$= \frac{1}{2\pi i}\int_{(2)} \Omega_{c_2}(s_1)\,\frac{\zeta(s_1)}{\zeta(2s_1)} \prod_{p|4\ell_1 r_1} (1+p^{-1})^{-s_1}\,ds_1$$

$$= \frac{6}{\pi^2}\prod_{p|4\ell_1 r_1}(1+p^{-1})^{-1}\,\Omega_{c_2}(1) + O\left(\left(\frac{UN}{T\ell_1 r_1}\right)^{1/2}\right)$$

on moving the line of integration from $\Re(s_1)=2$ to $\Re(s_1)=\frac{1}{2}$.

Summing on s_2 in the same way, we have

$$\begin{aligned}
&\sum_{\substack{s_2\\(s_2,4\ell_2r_2)=1}} \mu^2(s_2) \sum_{\substack{s_1\\(s_1,4\ell_1r_1)=1}} \mu^2(s_1)\, I_2(0,0,\ell_1r_1s_1,\ell_2r_2s_2)\\
&= \left(\frac{6}{\pi^2}\right)^2 \prod_{p|4\ell_1r_1}\left(1+\frac{1}{p}\right)^{-1} \prod_{p|4\ell_2r_2}\left(1+\frac{1}{p}\right)^{-1} I(\ell_1r_1,\ell_2r_2)\\
&\quad + O\left(N^{3/2}U^{1/2}T^{-3/2}(\ell_2r_2)^{-1}(\ell_1r_1)^{-1/2}\right.\\
&\quad \left. + N^{3/2}U^{1/2}T^{-3/2}(\ell_1r_1)^{-1}\left(\ell_2r_2\right)^{-1/2}\right)
\end{aligned}$$

where

$$I(\ell_1r_1,\ell_1r_2) \;=\; \int\!\!\int I_2(0,0,\ell_1r_1s_1,\ell_2r_2s_2)\,ds_1ds_2\,. \tag{5.28}$$

Hence,

$$\begin{aligned}
D_f(k,T) \;&=\; \left(\frac{6}{\pi^2}\right)^2 \sum_{1\leqslant v\leqslant 2V}\ \sideset{}{^*}\sum_{u(v)} e\left(\frac{-uk}{v}\right)\\
&\cdot \sum\sum_{\ell_1r_1\leqslant R}\ \sum\sum_{\ell_2r_2\leqslant R}\left(\prod_{p|4\ell_1r_1}\left(1+p^{-1}\right)^{-1}\right)\left(\prod_{p|4\ell_2r_2}\left(1+p^{-1}\right)^{-1}\right)\\
&\cdot \frac{1}{\ell_1r_1\ell_2r_2}\,\mu(r_1')\,\mu(r_2')\,\phi(\ell_1')\,\phi(\ell_2')\,\mu^2(r_1)\,\mu^2(r_2)m_1n_1m_2n_2\,I(\ell_1r_1,\ell_2r_2)\,\left(\frac{S(u;v)}{v}\right)^2\\
&+ O\left(N^{3/2}T^{-3/2}R^{1/2}\right)
\end{aligned} \tag{5.29}$$

$$\begin{aligned}
=\;& \left(\frac{6}{\pi^2}\,\frac{N}{T}\right)^2 \sum_{1\leqslant v<2V}\ \sideset{}{^*}\sum_{u(v)} e\left(\frac{-uk}{v}\right)\prod_{p|v}\left(1-\frac{1}{p^2}\right)^{-2}\\
&\left[\sum_{\ell_1'}\sum_{r_1'}\sum_{\ell_1'}\sum_{r_2'}\!_{\ell_i'r_i'\leqslant R/v}\quad\quad \prod_{p|4\ell_1'r_1'}\left(1\,+\,\frac{1}{p^2}\right)^{-1}\prod_{p|4\ell_2'r_2'}\left(1+\frac{1}{p^2}\right)^{-1}\ \frac{\mu(r_1')\mu(r_2')\phi(\ell_1')\phi(\ell_2')}{(\ell_1'\ell_2'r_1'r_2')^2}\right]\\
\cdot\;\;& \frac{S(u;v)^2}{v^4}\,I\,+\,O\left(N^{3/2}T^{-3/2}R^{1/2}\right)
\end{aligned} \tag{5.30}$$

where

$$I \;=\; \iint xy\,E(x,y)\,dxdy \tag{5.31}$$

and we have used

$$\begin{aligned} I(\ell_1 r_1, \ell_2 r_2) &= \iiiint E(x,y)\, \phi\left(\frac{\ell_1 r_1 s_1 T}{xN}\right) \phi\left(\frac{\ell_2 r_2 s_2 T}{yN}\right) ds_1 ds_2 dx dy \\ &= \left(\frac{N}{T}\right)^2 \frac{1}{\ell_1 r_1 \ell_2 s_2}\, I\,. \end{aligned}$$

In (5.30), we extend the summation ranges for $\ell_1', r_1', \ell_2', r_2'$ to infinity. This introduces an error of $O(N^2T^{-2}R^{-1/2}U^{1/4})$ which is admissible for our existing error term. Let α be the sum

$$\alpha = \sum_{\substack{\ell_i' \geqslant 1 \\ \ell_i' \in \mathcal{L}}} \sum_{\substack{r_i' \geqslant 1 \\ (r_i', 4\ell_i') = 1}} \prod_{p \mid 4\ell_i' r_i'} \left(1 + \frac{1}{p}\right)^{-1} \frac{\mu(r_i')\phi(\ell_i')}{(\ell_i' r_i')^2}\,. \tag{5.32}$$

One can show (see [Iw1] pp. 156) that

$$\alpha = 1\,. \tag{5.33}$$

We have shown that

$$\begin{aligned} D_f(k,T) &= \\ &\left(\frac{6N}{\pi^2 T}\right)^2 \sum_{1 \leqslant v < 2V} \sum_{u(v)}^{*} e\left(\frac{uk}{v}\right) \prod_{p \mid v} \left(1 - \frac{1}{p^2}\right)^{-2} \frac{S^2(u;v)}{v^4}\, I \\ &\quad + O\left(N^2 T^{-2} R^{-1/2} U^{1/4} + N^{3/2} T^{-3/2} R^{1/2}\right). \end{aligned} \tag{5.34}$$

If $v < V^{1-\epsilon}$ then by Lemma 5.1 (i), we have

$$\begin{aligned} I &= \iint xy\, f(x,y)\, \triangle_v(x-y-k)\, dx dy \\ &= \iint (x+y+k)\, y\, f(x+y+k, y)\, \triangle_v(x)\, dx dy \\ &= \int (y+k)\, y\, f(y+k,y)\, dy + \text{ negligible} \end{aligned} \tag{5.35}$$

If $v > V^{1-\epsilon}$ then by part (ii) of Lemma 5.1, $I \ll v^{-1}U^{3/2}$. We extend the summation over v in (5.34) to infinity making an error of at most $(NT^{-1})^2\, U^{1/2+\epsilon}$. We have established the following theorem:

Theorem 5.4. *For $|k| \leqslant UT^{-1+\epsilon}$ we have that*

$$D_f(k,T) = \left(\frac{6N}{\pi^2 T}\right)^2 \sum_{v \geqslant 1} \sum_{u(v)}^{*} e\left(\frac{uk}{v}\right) \prod_{p \mid v} \left(1 - \frac{1}{p^2}\right)^{-2} \frac{S^2(u;v)}{v^4}$$

$$\cdot \int (y+k)\, y\, f(y+k,y)\, dy + O_\epsilon\left(N^2 T^{-2}\, U^{\frac{1}{2}+\epsilon} + N^{3/2} T^{-3/2} R^{1/2}\right).$$

Applying the above theorem with f as in (5.6) and using

$$\hat{h}\left(\frac{\log(y+k)}{\pi L}\right) = \hat{h}\left(\frac{\log y}{\pi L}\right) + O\left(\frac{U^\epsilon}{T}\right)$$

and

$$\hat{\psi}^{(4)}\left(T\log\left(1+\frac{k}{y}\right)\right) = \hat{\psi}^{(4)}\left(\frac{Tk}{y}\right) + O\left(\frac{U^\epsilon}{T}\right)$$

gives together with (4.44)

$$D_1(k,T) = D^*(k,T) + O\left(N^2T^{-2}U^{1/2+\epsilon} + N^{3/2}T^{-3/2}R^{1/2}\right) \tag{5.36}$$

where

$$D_1^*(k,T) = \left(\frac{6N}{\pi^2 T}\right)^2 \sum_{v\geq 1}\sum_{u(v)}{}^{*} e\left(\frac{ku}{v}\right)\prod_{p|v}\left(1-p^{-2}\right)^{-2}\frac{S^2(u;v)}{v^4} \cdot \int \hat{h}^2\left(\frac{\log y}{\pi L}\right)\hat{\psi}^{(4)}\left(\frac{Tk}{y}\right)dy\,. \tag{5.37}$$

§6. Completion of Proofs

We can now combine the results of Sections 4 and 5 and deduce the main Theorem 1.4. According to (4.43), (4.44), (5.36) and (5.37) we have that with our choice of parameters,

$$\textstyle\sum_h^{\mathrm{w}}(T,L) = \frac{T^2}{576\pi^6L^2N^2}\sum_{|k|\leqslant K} D_1^*(k,T) + O\left(\frac{T}{L^2}\right) \tag{6.1}$$

$$= \frac{1}{16\pi^{10}L^2}\sum_{v\geqslant 1}\sum_{u(v)}{}^{*}\prod_{p|v}\left(1-p^{-2}\right)^{-2}\frac{S^2(u;v)}{v^4} \cdot \int\left|\hat{h}\left(\frac{\log y}{\pi L}\right)\right|^2\sum_{|k|\leqslant K} e\left(\frac{ku}{v}\right)\hat{\psi}^{(4)}\left(\frac{Tk}{y}\right)dy + O(T/L^2)\,. \tag{6.2}$$

Now for $v\geq 1$ fixed, we have

$$\sum_{u(v)}{}^{*}\sum_{|k|\leqslant K} S^2(u;v)\,e\left(\frac{-ku}{v}\right)\hat{\psi}^{(4)}\left(\frac{Tk}{y}\right) = \sum_{u(v)}{}^{*}\sum_{k} S^2(u;v)\,e\left(\frac{ku}{v}\right)\hat{\psi}^{(4)}\left(\frac{Tk}{y}\right) + \text{ negligible}\,.$$

Applying Poisson summation in k this becomes

$$\begin{aligned}
&= \sum_{u(v)}{}^{*} S^2(u;v) \sum_{\ell\in\mathbb{Z}} \int_{-\infty}^{\infty} e\left(\frac{xu}{v} + x\ell\right) \hat{\psi}^{(4)}\left(\frac{Tx}{y}\right) dx \\
&= \sum_{(u,v)=1} S^2(u;v) \int_{-\infty}^{\infty} e\left(\frac{xu}{v}\right) \hat{\psi}^{(4)}\left(\frac{Tx}{y}\right) dx \\
&= (2\pi)^4 \frac{y}{T} \sum_{(u,v)=1} S^2(u;v) \int_{-\infty}^{\infty} e\left(\frac{yu\xi}{Tv}\right) \hat{\psi}^{(4)}(\xi)\, d\xi \\
&= (2\pi)^4 \frac{y}{T} \sum_{(u,v)=1} S^2(u;v) \left(\frac{yu}{Tv}\right)^4 \psi\left(\frac{yu}{Tv}\right) \qquad (6.3)
\end{aligned}$$

Hence,

$$\begin{aligned}
\textstyle\sum_h^{\mathrm{w}}(T,L) &= \frac{1}{\pi^6 L^2} \sum_{v\geqslant 1} \sum_{(u,v)=1} \prod_{p|v} \left(1-p^{-2}\right)^{-2} \\
&\quad \cdot \frac{S^2(u;v)}{v^4} \cdot \int \left|\hat{h}\left(\frac{\log y}{\pi L}\right)\right|^2 \frac{y}{T} \left(\frac{yu}{Tv}\right)^4 \psi\left(\frac{yu}{Tv}\right) dy + O\left(\frac{T}{L^2}\right) \qquad (6.4)
\end{aligned}$$

$$\begin{aligned}
&= \frac{T}{\pi^6 L^2} \sum_{v\geqslant 1} \sum_{(u,v)=1} \prod_{p|v} \left(1-p^{-2}\right)^{-2} \\
&\quad \frac{S^2(u;v)}{v^4} \int \frac{v\xi}{u} \xi^4 \psi(\xi) \frac{v}{u} \left|\hat{h}\left(\frac{\log Tv\xi/u}{\pi L}\right)\right|^2 d\xi + O(T/L^2) \\
&= \frac{T}{\pi^6 L^2} \sum_{v\geqslant 1} \sum_{(u,v)=1} \prod_{p|v} \left(1-p^{-2}\right)^{-2} \\
&\quad \frac{S^2(u;v)}{u^2 v^2} \left(\int_0^{\infty} \xi^5 \psi(\xi)\, d\xi\right) \left|\hat{h}\left(\frac{\log(Tv/u)}{\pi L}\right)\right|^2 + O\left(\frac{T}{L^2}\right). \qquad (6.5)
\end{aligned}$$

Combining this with (4.5) completes the proof of Theorem 1.4.

To prove Corollary 1.5 we need the following bounds due to Fouvry and Michel [FM]:

$$\psi_1(x) = \sum_{v\leqslant x} \frac{|S(1;v)|^2}{v^2} \gg \exp\left((\log\log x)^{5/17}\right) \qquad (6.6)$$

and

$$\psi_u(x) = \sum_{u\leqslant x} \frac{|S(u;v)|^2}{v^2} \ll \eta(u)\, (\log\log x)^3 \log x \qquad (6.7)$$

where $\eta(u) = O_\epsilon(u^\epsilon)$, the implied constants being absolute.

We begin with the lower bound for the (to be shown) main term R in (6.5).

Clearly,

$$R \gg \frac{T}{L^2}\sum_v \frac{S^2(1;v)}{v^2}\left|\hat{h}\left(\frac{\log Tv}{\pi L}\right)\right|^2$$
$$\gg \frac{T}{L^2}\sum_{v\leqslant U/T}\frac{S^2(1;v)}{v^2}\,.$$

Since we are assuming that the support of $\hat{h}$ comes very close to 1, $U/T \geq T^\delta$ with $\delta > 0$ and hence by (6.6)

$$R \gg \frac{T}{L^2}\ \exp\left((\log\log T)^{5/17}\right)\,. \tag{6.8}$$

This proves the claimed lower bound in Corollary 1.5 and it also shows that R is the main term in Theorem 1.4.

For the upper bound we have

$$R \ll \frac{T}{L^2}\sum_u\sum_v\frac{S^2(u;v)}{u^2v^2}\left|\hat{h}\left(\frac{\log Tv/u}{\pi L}\right)\right|^2$$
$$\leqslant \frac{T}{L^2}\sum_u\sum_{v\leqslant T^\delta U}\frac{S^2(u;v)}{v^2}$$
$$\ll \frac{T}{L^2}\sum_u\frac{\eta(u)}{u^2}(\log\log T^\delta u)^3(\log T^\delta u)\,,\ \text{by (6.7)}$$
$$\ll \frac{T}{L}(\log\log T)^3 \tag{6.9}$$

This establishes the upper bound in Corollary 1.5.

We conclude with a proof of Theorem 1.1. From (1.2) and (1.7) we have

$$N_h(t,L) = \int_0^\infty h\left((\xi-t)L\right)dN(\xi)$$
$$= \int_0^\infty h((\xi-t)L)\,dN_{\mathsf{smooth}}(\xi)$$
$$+\int_0^\infty h((\xi-t)L)\,dS(\xi)\,. \tag{6.10}$$

Using standard estimates for $N_{\mathsf{smooth}}(\xi)$ we have

$$N_h(t,L) \;=\; \frac{t}{6L} \;-\; \int_0^\infty L\,h'((\xi-t)L)\,S(\xi)\,d\xi \;+\; O(\log(1+|t|)) \tag{6.11}$$

Hence,

$$\begin{aligned}
&\left|N_h(t,L) \;-\; \frac{t}{6L}\right|^2 \\
&\qquad\ll\; (\log t)^2 \;+\; \left|\int_0^\infty L\,h'((\xi-t)L)\,S(\xi)\,d\xi\right|^2 \\
&\qquad\ll\; (\log t)^2 \;+\; \int_0^\infty |S(\xi)|^2\;|Lh'((\xi-t)L)|\;d\xi \\
&\qquad\qquad\cdot\int_0^\infty |Lh'((\xi-t)L)|\;d\xi \\
&\qquad\ll\; (\log t)^2 \;+\; \int_0^\infty |S(\xi)|^2\;|Lh'((\xi-t)L)|\;d\xi
\end{aligned} \tag{6.12}$$

Thus integrating w.r.t. t in [T,2T]

$$\begin{aligned}
&\int_T^{2T}\left|N_h(t,L) \;-\frac{t}{6L}\right|^2\,dt \\
&\qquad\ll\; T(\log T)^2 \;+\; \int_0^\infty |S(\xi)|^2 \int_T^{2T} |Lh'((\xi-t)L)|dt\,d\xi\,.
\end{aligned} \tag{6.13}$$

Now for $\xi \geq 4T$, $t\in[T,2T]$ or for $\xi \leqslant \frac{T}{2}$, $t\in[T,2T]$, $Lh'((\xi-t)L)$ is negligible.

It follows from (6.13) that

$$\int_T^{2T} |N_h(t,L) \;-\; \frac{t}{6L}|^2\,dt \;\ll\; T(\log T)^2 \;+\; \int_{T/2}^{4T} |S(\xi)|^2\,d\xi\,. \tag{6.14}$$

On the other hand, according to Corollary 1.5, the left-hand side of (6.14) is

$$\gg\; \frac{T^2}{L^2}\;\exp\left((\log\log T)^{5/17}\right).$$

This establishes Theorem 1.1.

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Figure 1.

Let

$$N^{+}_{\mathsf{smooth}}(\lambda) = \frac{\lambda}{24} - \frac{3}{4\pi}\sqrt{\lambda}\,\log\lambda + \frac{6+4\log\pi-\log 2}{4\pi}\sqrt{\lambda} - \frac{13}{144} + \frac{3}{32\pi}\,\frac{\log\lambda}{\sqrt{\lambda}}$$

and

$$N^{-}_{\mathsf{smooth}}(\lambda) = \frac{\lambda}{24} - \frac{1}{4\pi}\sqrt{\lambda}\,\log\lambda - \frac{3\log 2-2}{4\pi}\sqrt{\lambda} + \frac{23}{144} + \frac{1}{32\pi}\,\frac{\log\lambda}{\sqrt{\lambda}}$$

Denote by $\lambda_1^+ \leq \lambda_2^+ \leq \lambda_3^+ \leq \cdots$ the eigenvalues corresponding to even eigenfunctions on X and λ_j^- the ones corresponding to odd eigenfunctions.

Set

$$d_n^{\pm} = N^{\pm}_{\mathsf{smooth}}\left(\lambda_n^{\pm}\right) - n + \frac{1}{2}\,.$$

Below are the graphs of d_n^+ for $0 \leqslant t_n \leqslant 3000$ and d_n^- for $1500 \leqslant 3500$ as computed by Steil [St].

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