Restriction Theorems and Appendix 1 & 2

Letter to: Andrei Reznikov - (June, 2008)

Peter Sarnak

Dear Andrei,

Your lecture at Courant last week raised some interesting points and I looked briefly at one of them (see 4 below) as well as the connection to L-functions (Appendix 2).

1. In general "the convexity bound"is not a well defined concept though I think we might say that we recognize it when we see it.In the case of automorphic L-functions the notion is exact and I believe that everyone is satisfied with it. For the setting of restrictions to a submanifold of eigenfunctions on a Riemannian manifold, the general (and lovely I might add) results of Burq-Gerard and Tzvetkov [*Duke Math Jnl*, Vol. 138, 2007, 445-] generalizing the results from your paper ["Norms of geodesic restrictions .....representation theory" - Arxiv, 2004] are clearly what one should call the "convexity" bounds. They are sharp for the n-sphere and are also sharp in terms of applying to functions in the spectral projection in the range $|\sqrt{(-\Delta)} - t| < 1$ (what one might call a weak quasi-mode - see (*) below for a slight strengthening). That is they are really norm bounds for the projectors in L^p spaces and in the above range they are sharp for any manifold .So improving on these (with a power saving) in a rank 1 locally symmetric space case,without assuming something like Hecke eigenfunctions is very interesting. I don't recall any such known improvements for these kind of questions and since these are sharp, any exponent improvement appears to me to broaching the multiplicity issue.For example your results with Bernstein ["Subconvexity bounds for triple L-functions and representation theory" - Arxiv, 2006] on the size of integrals of triple products of eigenfunctions and Nalini Ananthraman's ["Entropy and localization of eigenfunction" *Annals of Math*, (to appear)] results on concentration, apply equally well for such projection operators or to weak quasi-modes.

2. In higher rank symmetric spaces for simultaneous eigenfunctions of the full ring of commuting differential operators, the convexity bounds are sharper than those of B-G-T. For the case of restriction to a point (or what is the same K-periods where K is the maximal compact subgroup of the isometry group) my letter to Cathleen Morawetz (see www.math.princeton.edu/sarnak) clarifies this point and

describes what the improved "convexity bound" is. It is sharp in a suitable joint spectral projection and is sharp on the corresponding globally compact Riemannian symmetric space. Generalizing the B-G-T analysis to this higher rank locally symmetric setting for other submanifolds (ie not points) would be interesting and I think would define what is meant by the geometric convexity bound in higher rank. As pointed out in my letter and as you pointed out in your lecture and is also evident in your work with Bernstein, the geometric notion of convex bound rarely agrees with the L-function notion of convex bound (when such period formula are available). Over the years it has become clear that the cases of L-infinity norms of eigenfunctions on negatively curved surfaces, ternary quadratic forms, QUE via Watson and Rankin Selberg, which were the instances of subconvexity for L-functions that drew me into this subject, are not typical.

3. My understanding from your lecture is the following. You are dealing with a quotient of the upper half plane and restrictions of eigenfunctions to closed geodesics. The above convexity bound for the L^2 norm for this restriction is $O(t^{1/4})$ [here t is $\sqrt{\lambda}$] and you can improve this in the exponent without any Hecke or arithmetic assumptions. This seems quite strong to me since your result cannot be valid for the projector on the spectral range in (1)-ie for a weak quasi-mode.You also mentioned that in the case that the surface is congruence and the eigenfunction is Hecke, then you Philippe and Akshay can show that the bound $O(t^{\epsilon})$ is valid.

4. To understand (3) above I looked at the case of the restriction to a fixed closed horocycle (eg in the modular group case). This allows one to diagnose where the issues are because these Fourier coefficients and their relation to Hecke operators, are very familiar (as compared to the special value of L-function formulae). You made some brief comments about this case in you paper ["norms ...theory 2004] and B-G-T gives the same sharp bound that is: the convexity bound for this L^2 restriction is $O(t^{1/6})$.

   In terms of Fourier coefficients, the proof boils down to estimation of the sums of absolute squares of Fourier coefficients and of sizes of the Bessel functions $K_{it}(ny)$. Since there is no cancellation involved this is mostly manageable. The exponent 1/6 comes from the behavior of K in in the transitional range when $|t-ny| < t^{(1/3)}$ and where the Airy function enters [I have learned to respect this feature having blundered with it in my L-infinity paper Iwaniec, our correction is at the end of my letter to Cathleen. This transion range is also the source of the finite analogue of Kuznetzov's 1/6 in cancellations in sums of Kloosterman sums -see my paper with Tsimerman at www.math.princeton.edu/sarnak].

   To improve on 1/6 in the fashion that you do in (3) above one needs to show that the coefficients don't put a large mass in the range: n lying in $[t - t^{(1/3)}$,

$t + t^{(1/3)}]$. Without any multiplicity hypothesis this seems problematic to me, especially as the 1/6 bound is sharp when dealing with the projector. So that improving this bound means that one has to exploit the difference between a weak quasi-mode and a genuine eigenfunction.

Now assume that the original eigenfunction is a Hecke eigenfunction. Then as you point out in ["norms ..."] one can use known bounds on Fourier coefficients to prove that the restriction of ϕ to the given closed horocycle has L^2 norm bounded by $t^{(\epsilon)}$. What you don't say and what I find is needed in order to make this deduction from the Fourier coefficient analysis,are the results of Kim and Shahidi on the $L(s, \phi, \mathrm{sym}^4)$ together with analytic techniques from the theory of L-functions giving uniform bounds on sums over the coefficients to the power 4. In fact one can go further if one assumes the Ramanujan Conjectures for Maass forms and also that the Rankin Selberg L-functions $L(s, \phi \times \phi)$, satisfy a subconvex bound in the t_j aspect. With these one can show that

the L^2 restrictions of ϕ to the horocycle are uniformly bounded.

It doesn't seem out of the question that one could drop one or both of these assumptions in deriving this result. This is a very strong form of QUE, it clearly implies that for fattened up box $[y_1, y_2] \times [0, 1]$, a quantum limit gives measure at most a fixed constant times the mass of the box. In fact it is clear that the L^2 norms of these restrictions should be converging to an explicit constant as the eigenvalue goes to infinity. I don't quite see how to deduce the last from a subconvex bound for the Rankin Selberg L-function . In fact in this situation one might conjecture that for each integer p,the integral of ϕ of $(\phi)^p$ over the fixed horocycle converges to the $p-$th moment of a Gaussian,as the eigenvalue goes to infinity. And hence that the restriction of ϕ to the closed horocycle becomes Gaussian as the eigenvalue goes to infinity. In fact Hejhal and Rackner [*exp. math.* vol. 1 (1992), 275-305] suggest as much and check this numerically in their numerical investigation of the Gaussian behavior of ϕ when considered as a function on the whole manifold. In this latter case, I am not sure if one can improve (assuming some higher rank subconvexity) my theorem with Watson that the L^4 norm of such a ϕ is at most t^ϵ [see BAMS Vol. 40 (2003) 441-478], to being uniformly bounded.For unitary Eisenstein Series (properly normalized) the uniform boundedness of L^4 norms was established by Spinu in his thesis (which I am afraid is still not prepared for publication).

5. It is instructive to examine the similar restriction problem for holomorphic forms of weight k (as is familiar the parameter k here plays the plays the role of t in the above). That is say for $X = SL(2, \mathbb{Z})\backslash\mathbb{H}, f(z)$ a holomorphic cusp form of weight k normalized so that $y^k.|f(z)|^2$ has mass 1 over X. Now consider the problem of

restriction of f to a closed horocycle C. The asymptotics as k goes to infinity of $y^k.exp(-y)$ localizes for y in an interval of length $\sqrt{k}$ about k and is quite different to the Bessel function localization. As a consequence the convexity bound for the L^2 restriction to C is $k^{(1/4)}$ unlike the Maass form case. This is sharp for weak-quasi modes which in this case means it is sharp for holomorphic modes (the projection onto the range $[k - 1/2, k + 1/2]$ consists of exactly of the holomorphic modular forms of weight k whose dimension is correspondingly $k/12$). If we assume that f is a Hecke form then as in (4) one can show that this L^2 restriction has norm at most k^ϵ. It is worth noting that since the corresponding range of summation of the squares of the Fourier coefficients that needs to be estimated in this case is shorter, ie $[k - \sqrt{(k)},\ k + \sqrt{(k)}]$, one needs to resort to the full force of Delignes' bounds for the coefficients in these cases, in order to make the k^ϵ estimate. Because of this short range issue it is not possible (as far as I can see) to deduce the uniform boundedness of the L^2 restriction to C, using subconvexity for the Rankin-Selberg L-functions $L(s, f \times f)$. Still I believe that the restriction of such Hecke-eigen holomorphic forms to closed horocycles have a similar behavior to the Maass forms.

6. In any case the analogue these sharper results and phenomenon for restrictions of Hecke eigenfunctions to closed horocycles should be valid for restrictions to closed geodesics as well. A proof of the uniform boundedness in L^2 of the restriction to a closed geodesic would establish a very strong and new form of non-scarring for the eigenfunctions.

7. The uniform bound of L^2 restrictions is even a delicate business for a flat torus. B-G-T ask if for the standard flat torus the L^2 restriction of an eigenfunction to a fixed curve is uniformly bounded. I looked at this for the case that curve is a geodesic segment and found after some analysis and reductions that this is equivalent to the question of whether there is a universal constant K such that for any arc of length $\sqrt{R}$ of any circle radius R centered at 0,the number of integer lattice points on the arc is bounded by K. Apparently this is not a trivial problem, it was raised in Cilleruelo and Cordoba [*Proc. Amer. Math. Soc.*, **115**, (1992), 899-905] where it shown that for acrs of length $R^{(\alpha)}$ with $\alpha < 1/2$ this is true. They suggested that for $\alpha = 1/2$ there is no uniform bound K but in more recent papers Granville and Cilleruelo ["Close lattice points on circles", to appear in the *Canadian Journal of Math*] conjecture the opposite is true. So it seems that even what to expect here is not clear.

Best regards,
Peter

Added:

1. Clarification of weak-quasi modes: In the setting of negatively curved manifolds the notion of weak quasi-mode should be strengthened a bit to accommodate the fact that using microlocal analysis one can analyze sums over the spectrum with

$$|t_j - t| < b\big/\log(t) \tag{*}$$

where b is a positive constant depending on the manifold. This smaller projection leads to the small improvements(i.e. logarithmic but not in the exponent) in bounds for eigenfunctions on such manifolds. For example the L-infinity convex bound of $t^{(1/2)}$ on such a surface can be improved to $t^{\frac{1}{2}}\big/(\log t)$ and this is sharp for this stronger notion of weak-quasi-mode. Similarly Nalini's work makes critical use of this smaller window.I checked that the convex bound $t^{(1/6)}$ in (5) above is still sharp (up to log factors)for quasi-modes in this smaller range and I expect that the same is true for the other restriction theorems.

2. Restrictions and periods in the Hecke case: Ichino showed me today his very recent work with Gan establishing a new case of Gross-Prasad, that is for $H = SO(4)$ periods in $G = SO(5)$ (both defined over $\mathbb{Q}$ say). We then looked for applications of this to restrictions. I expect the following is a good approximation to the Hecke (i.e. adelic but I restrict to fixed level Γ) story for restrictions for this and some other cases (there are also similar cases of restrictions to $SL((n-1)$ in $SL(n)$ associated with converse theorems of twisting by $GL(n-1)$).

   In general these period formulae take the shape: H is a subgroup of G (defined over $\mathbb{Q}$) with $H(\mathbb{R})$ intersect Gamma finite volume in $H(\mathbb{R})$ which we view as cycle C in $X = \Gamma\backslash G(\mathbb{R})/K$ (and I stick to everything invariant on the right under maximal compacts).

   The period formulae give the Integral over C of $|\phi \times \psi(h)|^2$ as being an explicit expression in terms of special values (Rankin Selberg) COMPLETED L-functions. Here ϕ is a form on X and ψ on C and both are L^2 normalized. Thus by Plancherel, the L^2 norm of the restriction to C of ϕ is explicitly a weighted nonnegative sum of special values of L-functions. The weighting is determined from the archimedian factors in the L-functions and is easily computed with Stirling. By breaking into pieces at appropriate scales [more than just dyadic depending on the archimedian weights in these formulae and here we abandon any attempt to get uniform bounds for the L^2 restriction but are satisfied with things up to $t^{(\epsilon)}$], we are left with establishing Lindelof on average over various ranges of the parameters. In some examples such a piece can consist of a very small ranges -thus essentially requiring (much as I found in the example (6) with holomorphic forms and pretty much needing the full full force of Deligne there)the full Lindeloff to get the desired sharp bound. In any case the following is clearly always true in these situations and in full generality:

Assuming Lindelof one can understand the L^2 restriction story that is: One gets the the sharp (up to ϵ) bound for the L^2 restriction of the (varying) form ϕ, to C.

In many cases the weightings are such that we can prove Lindelof unconditionally on average over the ranges in question (this is just like point (5) above where we get by without the full knowledge of Ramanujan for Maass forms) and hence one proves a t^ϵ bound for the restriction .

The examples based on such period formula that I know (with the new one from Ichino yesterday) are as follows :

(a) Take $G = SL(2)$, and $H = K$ (or the stabilizer of any CM point on the modular curve). In this case C is a point. Then viewing this as an $G = SO(3)$ and $H = SO(2)$ case, the above formulae is due to Waldpurger and the sum reduces to a single L-function and we find that Lindelof and the L-infinity conjecture for restriction (ie evaluation)at CM points, are equivalent. So this is the hardest case. I pointed to this analysis in my Schur lectures some time ago.So for this case there is no averaging and certainly $t^{(\epsilon)}$ is not something we can realistically contemplate proving. Also in this case the restriction is not uniformly bounded in $L^2(C)$. The method to show this that is discussed near the end of my letter to Cathleen was developed much further in the thesis of Milicevic [Princeton, 2006] who obtains rather sharp lower bounds for these special CM points.

(b) As above but H is $SO(1,1)$. If the $SO(1,1)$ is split over $\mathbb{Q}$ then C is a geodesic running from one cusp to another [in this case H intersect Γ is not finite volume in H but this should only spoil things logarithmically], and the period formula in question is just Hecke's integral formula (ϕ is a cusp form) against the character $y^s, re(s) = 0$. Plancherel converts the L^2 restriction to C to a weighted mean-square over $|v| << t$ of the $|L(1/2+iv, \phi)|^2$. The weight scales like the analytic conductor and one can show that the L^2 restriction is at most t^ϵ.

(c) As above but the SO(1,1) is not split over $\mathbb{Q}$. Then C is a closed geodesic. In this case the period formula is again that of Waldpurger. From your lecture this is the case that Philippe and Akshay and you have handled. I assume that this is done by establishing Lindelof on average over the ranges that present themselves.I look forward to seeing the details.

(3) G is $SO(4)$ and H is $SO(3)$. Depending on splitting over $\mathbb{Q}$ we get different cases.

(a) If G splits as an $SO(2,2)$ or $SL(2) \times SL(2)$ and H is $SL(2)$ embedded diagonally, then if ϕ is $f \times f$, with f and eigenfunction on $SL(2)$, the L^2 restriction of ϕ to H is the the L^4 norm of f!. The period formula in this case is Watson's formula and the above general method is is exactly the way that Watson and I established that the L^4 norm of f is at most t^ϵ. It is in the short ranges over which the corresponding triple product L-functions need to be averaged that we freely used the Ramanujan Conjectures for $GL(2)$ and the reason that I stated this theorem in my BAMS article as conditional on Ramanujan. At the time I felt one could probably get by without this and had planned to try do so when (if ever) I write this up.In Spinu's case of unitary Eisenstein series this wasn't an issue since these obey Ramanujan (but there is a deeper reason connected with averaging over families of L-function which factor as absolute squares that makes his case much easier).

(b) G is is $SO(3,1)/\mathbb{R}$ and H is the maximal compact subgroup. Then X is a hyperbolic three manifold and C is a point (an arithmetic "CM like point").This is the case that I discussed from various points of view in my letter to Cathleen. One of the view points is to use Jacquet-Lapid and Offen's formula which gives the period to L-function formula. The result is that ϕ restricted to C is zero if ϕ is not a base change and if it is then it is given by the value of a finite L-function at $s = 1$ times $t^{1/2}$. So in this case the L^2 restriction to C is not small and it is well understood.

(c) If G is $SO(3,1)/\mathbb{R}$ and H is $SO(2,1)$. In this case X is again a hyperbolic three manifold and C is totally geodesic closed hyperbolic surface in X. The period formula in this case has been established by Ichino "Trilinear forms and central values of triple product L-functions" to appear in *DMJ*. I expect (or should I say it would be nice) that the analysis of the averaging of L-functions used in (a) above will apply here (the average is again an $SL(2)$ spectral average and one can use Kuznetzov if at least H is split over $\mathbb{Q}$). This should yield (again under $GL(2)$ Ramanujan) that the L^2 restriction of ϕ on X to C is bounded by t^ϵ.

(4) $G = SO(5)$ and H is $SO(4)$. The period formula is the new Gross-Prasad formula that Ichino and Gan have just established. There are again many cases according to splittings but the one that attracts me (I like hyperbolic spaces!) is G equal to $SO(4,1)/\mathbb{R}$ and H is $SO(3,1)/\mathbb{R}$. Then C is is a totally geodesic closed hyperbolic three manifold in X. I would hope (this time expect is too strong) that one can do

the spectral L-function averages over this hyperbolic three manifold to establish again that the L^2 restriction to C is bounded by t^ϵ.

(5) For more general cases of the Gross-Prasad formula for $H = SO(n)$, $G = SO(n+1)$, $n > 4$ should they be proved, the true size of the restriction to the H cycle can be understood assuming Lindelof. However, to establish the sharp bound unconditionally will require at least averaging L-functions over families of forms in a genuinely higher rank group which is a separate challenge .