SELBERG TRACE FORMULA

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Ann Chapters 6 and 7
Eisenstein Series for Hyperbolic
manifolds.

CHAPTER 6

EISENSTEIN SERIES.

1. Motivation: In the case that S/Γ is not compact, but is never-the-less of finite volume, the spectrum of the corresponding invariant operators is not purely discrete. There will also be some continuous spectrum, and we will need to study this continuous spectrum quite carefully in order to eventually derive a trace formula. The key to the construction of the continuous spectrum in these cases is the Eisentein-Maass series. In this chapter we will develop the theory of the Eisenstein series and in particular their analytic continuation. As we shall see these series are of intrinsic interest in their own right. Our discussion will be limited to the case of hyperbolic spaces.

To motivate these series, consider the case of $\mathcal{H}^2$ and $\Gamma = PSL_2(Z)$ the classical modular group. The stabilizer of infinity is

$$\Gamma_\infty = \{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in Z \}.$$

The standard fundamental domain is $\mathfrak{F}$

Considering first the parabolic subgroup Γ_∞, it is clear that the functions y^s are Γ_∞ invariant and are also eigenfunctions of the Laplacian D.

$$Dy^s + s(1-s)y^s = 0. \tag{1.1}$$

These functions (y^s), are closely related to the spectrum of D on $\mathcal{H}/\Gamma$ functions, as the following construction of approximate eigenfunctions from the functions y^s, indicates.

Let ϵ, B be real parameters, and let ϕ be a fixed positive even function supported in $(-2,2)$, with say $\int_{-\infty}^{\infty} \phi(x)dx = 1$, and assume that $\epsilon \log B > 2$.

Define $u(z) = y^s \phi(\epsilon \log (y/B))$, then it is clear that u is Γ invariant, also

$$Du = y^2\{s(s-1)y^{s-2}\phi + 2s\, y^{s-1}\phi'(\epsilon \log y/B)_{\epsilon/y} + y^s[\phi''(\epsilon \log y/B)(\tfrac{\epsilon}{y})^2 - \phi'(\epsilon \log y/B)\tfrac{\epsilon}{y^2}]\}$$

$$Du - u = \epsilon y^s (2s-1)\phi'(\epsilon \log y/B) + y^s \epsilon^2 \phi''(\epsilon \log y/B).$$

Now if $s = 1/2 + it$, with $t \in \mathbb{R}$ then we see that

$$\|Du-u\|_2^2 = \int_0^\infty \left|(2s-1)\epsilon y^{\frac{1}{2}+it}\phi'(\epsilon \log y/B) + y^{\frac{1}{2}+it}\epsilon^2\phi''(\epsilon \log y/B)\right|^2 \frac{dy}{y^2}$$

$$= \epsilon \int_{-\infty}^{\infty} \left|(2s-1)\phi'(\epsilon x - \epsilon \log B) + \epsilon\, \phi''(\epsilon x - \epsilon \log B)\right|^2 dx$$

$$= \int_{-\infty}^{\infty} \left|(2s-1)\phi'(x) + \epsilon\phi''(x)\right|^2 dx .$$

On the other hand

$$\|u\|_2^2 = \int_0^\infty |\phi(\varepsilon \log y/B)|^2 \frac{dy}{y} = \frac{1}{\varepsilon} \int_{-\infty}^\infty |\phi|^2(x)\, dx.$$

and so

$$\|Du - s(s-1)u\| << \varepsilon \|u\| . \tag{1.2}$$

The last implies that $s(s-1) = -\frac{1}{4} - t^2$ is in the spectrum of D on $L^2(\mathcal{H}/\Gamma)$. The reason for this is that, for a self adjoint operator A, if λ_o is a distance ε_o from the spectrum of A, $\sigma(A)$, then the resolvent $R_{\lambda_o} = (\lambda_o - A)^{-1}$ is bounded by $1/\varepsilon_o$. For by the spectral theorem

$$A = \int_{\sigma(A)} \lambda\, d E_\lambda$$

$$R_{\lambda_o}(A) = \int_{\sigma(A)} \frac{1}{(\lambda_o - \lambda)}\, d E_\lambda$$

$$\|R_{\lambda_o}\| \leq \frac{1}{\varepsilon} .$$

But (1.2) says that $\|R_{(-1/4 - t^2)}\| \geq 1/\varepsilon$, $\forall \varepsilon$ therefore $-1/4 - t^2$ is in the spectrum.

The above arguments show that $\mathfrak{S}(D) \supseteq (-\infty, -1/4] \cup \{0\}$, where we have added $\{0\}$ since the constant function is clearly an eigenfunction. We will see later [ ] that indeed $\mathfrak{S}(D)$ on $L^2(\mathcal{H}/\Gamma)$, Γ the modular group, is exactly $\{0\} \cup (-\infty, -1/4]$. At this point we see that there is probably continuous spectrum and that the functions $y^{1/2+it}$ are usefull for the harmonic analysis of D in a cusp.

Section 1.2. We turn to the general situation. We have seen in a previous chapter that if H^{n+1}/Γ is of finite volume, then a fundamental domain for Γ consists of a compact part together with a finite number of cusps. To begin with, we assume that there is only one cusp, the general case will be dealt with later on. We assume that the cusp is at infinity, and that the stabilizer of ∞, Γ_∞, is simply a rank-n translational group of $\mathbb{R}^n$, i.e. a lattice L. So Γ_∞ acts by

$$(y,x) \rightarrow (y, x+\ell), \qquad \ell \in L .$$

Let $\mathfrak{F}_L$ be a fundamental domain for the lattice L in $\mathbb{R}^n$ and let

$$\mathfrak{F}_\infty = \{(y,x) : x \in \mathfrak{F}_L , \quad 0 < y < \infty\} .$$

Clearly $\mathfrak{F}_\infty$ is a fundamental domain for Γ_∞. Let $\mathfrak{F}$ denote a fundamental domain for Γ, $\mathfrak{F}$ may be chosen within $\mathfrak{F}_\infty$.

Consider as before the functions y^s. They are Γ_∞ invariant and satisfy

$$(1.3) \qquad Dy^s + s(n-s)y^s = 0 .$$

Of course y^s is not Γ invariant, so to make a function which is Γ invariant we average the function y^s - being already Γ_∞ invariant we need only (and can only!) average over cosets of Γ_∞ mod Γ.

Definition (1.4).

$$\text{Let}\quad E(\omega,s) = \sum_{\gamma\in\Gamma_\infty/\Gamma} y(\gamma\omega)^s$$

where $\omega \in \mathbb{H}^{n+1}$, $s \in \mathbb{C}$.

Formally the above series is Γ invariant, and since the γ's are isometries of the hyperbolic metric, and hence commute with D, we expect (since each $y(\gamma\omega)^s$ is eigenfunction of D with eigenvalue $s(n-s)$), that E will be too.

Proposition (1.5). The series 1.5 converges absolutely in $\mathrm{Re}(s) > n$ and uniformly on compact subsets.

Proof. Let $\sigma = \mathrm{Re}(s)$.

$$\Sigma\,|y^s(\gamma\omega)| \;\le\; \Sigma\, y^\sigma(\gamma\omega)\ ,$$

so we only consider the last sum. The idea is to compare the value of y^σ at ω, with the average value of y^σ in a small ball about w. We have

$$1.6 \qquad \int_{d(z,\omega)<\delta} y^\sigma\,\frac{dx\,dy}{y^{n+1}} = \frac{1}{c(\sigma,\delta)}\, y^\sigma(\omega) \qquad (z=(y,x))$$

for a suitable constant $c(\sigma,\delta) \neq 0$. The simplest way of seeing this is to observe that the left hand side of 1.6, is a point-pair operator applied to the function y^σ.

Now for a fixed w, or more generally if w lies in a compact subset of $\mathfrak{F}$, we may choose δ small enough so that the images under Γ_∞ / Γ of $B(w,\delta)$ are disjoint. Also keep in mind that the images of $\mathfrak{F}$ under suitable coset representatives of Γ_∞ / Γ fill exactly the strip $\mathfrak{F}_\infty$, so that

$$\sum_{\gamma \in \Gamma_\infty/\Gamma} y(\gamma\omega)^\sigma = y^\sigma + \sum_{\gamma \in \Gamma_\infty/\Gamma}{}' C(\sigma,\delta) \int_{d(z,\omega)} y^\sigma \frac{dx\,dy}{y^{n+1}}$$

$$\leq C(\sigma,\delta) \int_{\substack{y < A \\ x \in \mathfrak{F}_L}} y^\sigma \frac{dx\,dy}{y^{n+1}} + y^\sigma$$

where A depends on Γ only, and is chosen so that $\mathfrak{F}_L \times [A, \infty) \subset \mathfrak{F}$.

$$(1.6)' \quad \therefore \quad \Sigma\, |y(\gamma\omega)|^s \leq y^\sigma + \frac{C(\sigma,\delta)A^{\sigma-n}}{\sigma-n} \quad \text{if } \sigma > n.$$

Which clearly implies the absolute and uniform convergence claimed.

<u>Corollary 1.7</u>. For $\mathrm{Re}(s) > n$, $E(w,s)$ is Γ invariant, holomorphic in s and satisfies

$$DE(\omega,s) + S(n-s)\, E(\omega,s) = 0.$$

<u>Proof</u>. The Γ invariance is obvious. The functions $y(\gamma\omega)^s$ are all eigenfunctions of D with eigenvalues $s(n-s)$, and so are eigenfunctions of any integral operator of point pair invariant type. Since the series 1.4 converges

uniformly on compacta we have

$$K_o E(\cdot, s) = h(s) E(\cdot, s)$$

where K is point pair corresponding to k which is of compact support, and $h \leftrightarrow k$ is the Selberg correspondence. Choosing K smooth shows E is C^∞. Also $D_o K$ is a point pair integral operator therefore

$$D_o K(E(\cdot, s)) = \lambda(s) E(\cdot, s)$$

say where $\lambda \leftrightarrow D_z k(z, \omega)$. If we let k be an approximation to the identity $\lambda(s) \to -s(n-n)$ $D_o K(E) \to D_o E$ and, we learn that

$$D E(z, s) = -S(n-s) E(z, s).$$

<u>Proposition 1.8</u>. $E(\omega, s) = y^s + O(y^{n+1})$ as $y \to \infty$, $\text{Re}(s) > n$.

<u>Proof</u>. We need only estimate $C(\sigma, \delta)$ in (1.6). To do this we observe that at a height y, we need to choose a ball of radius about $1/y$ in the argument of 1.5. Thus from 1.6. The estimate in 1.8 is clear.

Actually we can do a lot better, we will see shortly by use of the Fourier expansion that

$$E(\omega, s) = y^s + O(y^{n-\sigma}).$$

§ 1.3 Fourier Expansions.

The function $E(\omega,s)$ as defined in the previous section satisfies

$$E(y, x+\ell, s) = E(y,x,s) \qquad \forall \ell \in L$$

since it is Γ_∞ invariant. Thus for each fixed y we may form its Fourier series development

$$E(\omega,s) = \sum_{\ell \in L^*} a_\ell(y)\, e(\langle x,\ell\rangle) \tag{1.9}$$

where L^* is the dual lattice to L, $e(\alpha) = e^{2\pi i\alpha}$, and $\langle x,\ell\rangle$ = $x_1\ell_1 + x_2\ell_2 \quad \ldots + x_n\ell_n$.

This separation of variables and the equation $DE(\omega,s)+s(n-s)E(\omega,s) = 0$ imply that the co-eff $a_\ell(y)$ satisfy the o.d.e.

$$y^2 a_\ell''(y) - (n-1)y\, a_\ell'(y) + s(n-s)\, a_\ell(y) - y^2 4\pi^2 |\ell|^2 a_\ell(y) = 0.$$

Thus if $b(y) = y^{-n/2} a_\ell(y)$ then

$$b'' + yb' + [-(\tfrac{n}{2})^2 / y^2 - 4\pi^2|\ell|^2]b = 0.$$

For $\ell \neq 0$:

The solution of the last equation is a Bessel function (see Appendix for its properties). The two independent solutions are the Bessel function growing exponentially at infinity and the one decaying exponentially at infinity. Since

$$|a_\ell(y)|^2 \leq \int_{\mathfrak{F}_L} |E(\omega,s)|^2 dx << y^\sigma$$

as we have seen, it follows that only the decaying solution can occure, i.e. we have for $\ell \neq 0$

$$(1.10) \qquad a_\ell(y) = a_\ell(s) y^{n/2} K_{s-n/2}(2\pi n |\ell| y)$$

for some function $a_\ell(s)$. $K_{s-n/2}$ is the Bessel function solution which decays exponentially at infinity

$$K_{s-n/2}(y) << e^{-cy} \quad \text{as} \quad y \to \infty \quad \text{for some} \quad c > 0.$$

On the other hand if $\ell = 0$, the linearity independent solutions are y^s and y^{n-s}. Since for large $\operatorname{Re}(s)$

$$\frac{1}{V(\mathfrak{F}_L)} \int_{\mathfrak{F}_L} E(\omega,s)dx = y^s + O(y^{n+1})$$

from 1.8 while we know

$$\frac{1}{V(\mathfrak{F}_L)} \int_{\mathfrak{F}_L} E(\omega,s)dx = \alpha y^s + \beta y^{n-s} = a_o(y)$$

from the o.d.e., we see that the zero'th co-efficient $a_o(y)$ must be of the form

$$(1.11) \qquad G_o(y) = y^s + \phi(s)\, y^{n-s}$$

for some function $\phi(s)$.

Returning to the co-efficients $G_\ell(y)$ in 1.10, we see from the behavior of the Bessel K-function at infinity (see Appendix) that

$$(1.11)' \qquad |G_\ell(y)| \le e^{-(2\pi|\ell|-\epsilon)(y-y_o)}|G_\ell(y_o)|$$

for $y > y_o$, y_o large enough and some $\epsilon > 0$.

As a corrolary we obtain a much stronger and more useful version of 1.8.

Corrolary (1.11)". For $Re(s) > n$

$$E(\omega, s) = y^s + \phi(s)\, y^{n-s} + g(\omega, s)$$

where $g(\omega,s) << e^{-cy}$ as $y \to \infty$ for c sufficiently small.

The function $\phi(s)$ will turn out to be of the utmost importance in what we do.

So far all that we have done is for $Re(s) > n$. In this region, in view of the y^s in the zero'th co-efficient there is no hope that $E(\omega,s)$ be anything near square integrable over $\mathfrak{F}$ w.r.t. $(dx\,dy)/y^{n+1}$. It becomes clear that one needs to analytically continue $E(\omega,s)$ to the left of $Re(s) = n$. In fact at least as far as $s = n/2 + it$, $t \in \mathbb{R}$.

The analytic continuation is a very important part of the theory and we will give two proofs of the continuation. One method, very much in the spirit of the book so far, was presented by Selberg in his lectures at Stanford 1980. The second method is an adoption by Colin deVerdiere of a method due to Lax and Phillips. However before embarking on these two proofs we pause to give some examples where the Eisenstein series may be computed exactly in terms of known special functions. These examples show that the analytic continuation is at least as deep as the continuation of the Zeta functions of number theory.

§ 1.4. Some examples.

1.12. Classical modular group. The case of $\mathbb{H}^2$ and $\Gamma = PSL(2, \mathbb{Z})$. There is only one cusp and so

$$E(z,s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \frac{y^s}{|cz+d|^{2s}}$$

which by the previous section is of the form

$$y^s + \phi(s) y^{1-s} + \sum_{n \neq 0} \alpha(n,s)\, y^{1/2} K_{s-1/2}(2\pi |n| y) e^{2\pi i n x}$$

for certain functions $\phi(s)$ and $\alpha(n,s)$. To compete $\phi(s)$ we consider

$$\int_0^1 E(z,s)dx = \int_0^1 \left(\sum_{\gamma \in \Gamma_\infty \setminus \Gamma} \frac{y^2}{|cz+d|^{2s}} \right) dx$$

$$= y^s + y^s \sum_{\substack{c>0 \\ (c,d)=1}} \int_0^1 \frac{dx}{|cz+d|^{2s}}$$

$$= y^s + y^s \sum_{\substack{c>0 \\ (c,d)=1}} \frac{1}{c^{2s}} \int_0^1 \frac{dx}{|z+d/c|^{2s}}$$

$$= y^s + y^s \sum_{\substack{c>0 \\ d \bmod c \\ (d,c)=1}} \frac{1}{c^{2s}} \sum_{q=-\infty}^{\infty} \int_0^1 \frac{dx}{|z+q+d/c|^{2s}}$$

$$= y^s + y^s \sum_{\substack{c>0 \\ (c,d)=1 \\ d \bmod c}} \frac{1}{c^{2s}} \int_{-\infty}^{\infty} \frac{dx}{|z+d/c|^{2s}}$$

$$= y^s + y^s \sum_{c>0} \frac{\Phi(c)}{c^{2s}} \int_{-\infty}^{\infty} \frac{dx}{(x^2+y^2)^s}$$

where $\Phi(e)$ is the Euler Φ function, i.e. $\Phi(c) = \#\{d \bmod c : (d,c)=1\}$

$$= y^s + y^{1-s} \left(\sum_{c>0} \frac{\Phi(c)}{c^{2s}} \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^s} \right)$$

$$= y^s + \frac{\zeta(2s-1)}{\zeta(2s)} \sqrt{\pi}\, \frac{\Gamma(s-1/2)}{\Gamma(s)}\, y^{1-s}$$

$\therefore$ for this example

$$\phi(s) = \frac{\zeta^*(2s-1)}{\zeta^*(2s)} \quad \text{where} \quad \zeta^*(s) = \pi^{-s/2}\,\Gamma(s/2)\zeta(s).$$

As an exercise we leave it to the reader to show that

$$E(z,s) = y^s + \frac{\zeta^*(2s-1)}{\zeta^*(2s)}\, y^{1-s} + \sum_{n=1}^{\infty} \frac{2\sigma_{s-1/2}(n)}{\zeta^*(2s)}\, y^{1/2} K_{s-1/2}(2\pi ny) \cos(nx\, 2\pi)$$

where

$$(1.13) \qquad \sigma_s(n) = \sum_{d|n} d^s .$$

Thus it is clear that $E(z,s)$ may be meromorphically continued to all of $\mathbb{C}$. On the other hand, the Eisenstein series - ζ function connection above may be exploited in the other direction, and we will have more to say about this later on.

1.14. Imaginary quadratic fields and the Bianchi groups. (A little knowledge of number theory is needed here), (see §). Nice examples for $\mathbb{H}^3$ are the Bianchi groups. Let $D > 0$ be square free, and let $Q(\sqrt{-D}) = k_D$ be the corresponding imaginary quadratic number field. Let $\mathcal{O}_D$ be the ring of integers of k_D. We consider the discrete subgroup of $SL_2(\mathbb{C})$,

$$\Gamma_D = SL_2(\mathcal{O}_D).$$

If $D \neq 1$ or 3 then $\Gamma_\infty = \{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \mid n \in \mathcal{O}\}$. Therefore $L = \mathcal{O}_D \subset \mathbb{R}^2 \cong \mathbb{C}$. We consider

$$E(\omega, s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} y(\gamma\omega)^s \ , \ \omega = (y, x_1, x_2) = (y, z) \ z = x_1 + i\, x_2 \ .$$

$$E(\omega, s) = y^s + \sum_{\substack{(c,d)=(1) \\ \text{mod} \pm I}} \frac{y^s}{(|cz+d|^2 + |c|^2 y^2)^s}$$

(using the action of $SL_2(\mathbb{C})$ on $\mathbb{H}^3$ see §),

$$= y^s + \frac{1}{2} \sum_{c \neq 0} \frac{y^2}{|c|^{2s}} \sum_{\substack{d \bmod c \\ (d,c)=1}} \sum_{m \in \mathcal{O}} \frac{1}{(|z + d/c + m|^2 + y^2)^s} \ .$$

We want the zero'th coefficient. Let $\mathfrak{F}_L$ be a fundamental domain for $\mathcal{O}$ in $\mathbb{C}$, then

$$\int_{\mathfrak{F}_L} E(y, z, s)\, dx_1\, dx_2 \ ,$$

$$= V(\mathfrak{F}_L) y^s + \frac{1}{2} \sum_{c \neq 0} \sum_{\substack{d \bmod c \\ (d,c)=1}} \frac{y^s}{|c|^{2s}} \int_{\mathbb{R}^2} \frac{dx_1\, dx_2}{(x_1^2 + x_2^2 + y^2)^s}$$

$$= V(\mathfrak{F}_L) y^s + \frac{1}{2} \sum_{c \neq 0} \sum_{\substack{d \bmod c \\ (d,c)=1}} \frac{y^s}{|c|^{2s}} \int_0^\infty \int_0^{2\pi} \frac{r\, d\theta\, dr}{(y^2 + r^2)^s}$$

$$= V(\mathfrak{F}_L) y^s + \frac{\pi}{2} \frac{1}{s-1} \sum_{c \neq 0} \sum_{\substack{d \bmod c \\ (d,c)=1}} \frac{1}{|c|^{2s}}$$

$$\therefore \quad E(\omega, s) = y^s + \left(\frac{\pi}{V(\mathfrak{F}_L) 2(s-1)} \sum_{c \neq 0} \frac{\Phi(c)}{|c|^{2s}} \right) y^{2-s}$$

+ nonzero Fourier coefficients

where Φ is the Euler function for $\mathfrak{O}$, i.e. $\Phi(\mathfrak{a}) = \#$ relatively residue classes of $\mathfrak{O}/\mathfrak{a}$ where $\mathfrak{a}$ is an ideal of $\mathfrak{O}$. We write the last as

$$E(\omega, s) = y^{s} + \phi_{11}(s) y^{2-s}$$

where the notation $\phi_{11}(s)$ is brought in, since in general there is more than just one inequivalent cusp at infinity, see later when we do the general case of a finite number of cusps.

$$\phi_{11}(s) = \left(\sum_{\substack{\mathfrak{a} \text{ principal} \\ \mathfrak{a} \neq 0}} \frac{\Phi(\mathfrak{a})}{N(\mathfrak{a})^{s}} \right) \frac{\pi}{V(\mathfrak{F}_{L})(s-1)}$$

where $N(\mathfrak{a})$ = norm of $\mathfrak{a}$. We continue to derive a more convenient form for $\phi_{11}(s)$. Let I be the ideal class group of $\mathfrak{O}_{D}$, $|I| = h$ = class number of $\mathfrak{O}_{D}$. Let $\psi_1, \psi_2 \cdots \psi_h$, with ψ_1 = identity charater, be the character group to I. Define the L-functions L_j by

$$L_j(s) = \sum_{\mathfrak{a} \neq 0} \frac{\psi_j(\mathfrak{a})}{N(\mathfrak{a})^{s}}$$

the sum being over the nonzero ideals of $\mathfrak{O}$. We see that

$$\phi_{11}(s) = \frac{\pi}{V(\mathfrak{F}_{L})(s-1)h} \sum_{j=1}^{h} \frac{L_j(s-1)}{L_j(s)} . \tag{1.15}$$

These $L \neq$ id. functions are entire [ ], and so again via number theory we see the continuation of $\phi_{11}(s)$. We will later use this $E(\omega, s)$, L function on $Q(\sqrt{-D})$ connection, to compute the volumes of $\mathcal{H}/\Gamma_{D}$.

§ 1.5 Further analysis of the Eisenstein series.

Lemma 1.16. Let $f \neq 0$, be in $L^2(\mathfrak{F})$ and is also a Γ invariant eigenfunction of Δ with eigenvalue $s(n-s)$, then $s(n-s) \in [0, \infty)$.

Proof. This follows from the fact that Δ is a positive self-adjoint operator on $L^2(\mathbb{H}^{n+1}/\Gamma)$. Indeed for smooth Γ automorphic functions of compact support in $\mathfrak{F}$, we may integrate by parts

$$(1.17) \qquad -\int_{\mathfrak{F}} (\Delta\phi)\bar{\psi}\, dV = \int_{\mathfrak{F}} (\nabla \phi, \nabla\psi) dV = -\int_{\mathfrak{F}} \phi(\overline{\Delta\psi})\, dV$$

where $\langle\ ,\ \rangle$ is the hyperbolic innerproduct, i.e.

$$\int_{\mathfrak{F}} \langle \nabla\phi, \nabla\psi\rangle dV = \int_{\mathfrak{F}} \left(\sum_{i=1}^{n} \left(\frac{\partial\varphi}{\partial x_i}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2 \right) y^2 \, \frac{dx_1 \ldots dx_n\, dy}{y^{n+1}} .$$

From 1.17 it is clear that the spectrum of Δ is real and positive.

Lemma 1.17. Let f be an automorphic eigenfunction of Δ with eigenvalue $s(n-s)$, and $\mathrm{Re}(s) > n$. Suppose also that $f(\omega) = O(y^m)$, $m < \infty$ as $y \to \infty$. Then $f(\omega) = \alpha E(\omega, s)$ for some α.

Proof. Being an eigenfunction and also Γ_∞ invariant, the function $f(\omega)$ may be expanded in a Fourier series, in a manner similar to what was done with the Eisenstein series. The polynomial bound shows that only the K-Bessel functions occur in the coefficients.

$$f(\omega) = a_o y^s + a'_o y^{n-s} + \text{non zero co-eff.}$$

$$\therefore \qquad f(\omega) - a_o E(\omega, s) = h(\omega) \in L^2(\mathfrak{F}),$$

and h has eigenvalue $s(n-s)$. By the previous lemma we conclude that $h \equiv 0$, since $\mathrm{Re}(s) > n$.

As in the compact case, we now form self-adjoint operators whose kernels are constructed via point pair invariants. Let $\Phi : \mathbb{R}^+ \to \mathbb{R}$ be smooth and of compact support. Correspondingly we have

$$(1.18) \qquad \left\{ \begin{aligned} k(\omega_1, \omega_2) &= \Phi\left(\frac{|\omega_1 - \omega_2|^2}{y_1 y_2} \right) \\ K(\omega_1, \omega_2) &= \sum_{\gamma \in \Gamma} k(\omega_1, \gamma\omega_2) . \end{aligned} \right.$$

The series in (1.18) clearly converges (being only a finite sum) and satisfies

$$(1.18)' \qquad K(\gamma\omega_1, \omega_2) = K(\omega_1, \omega_2) = K(\omega_2, \omega_1) .$$

Thus $K(\omega_1, \omega_2)$ is a Γ automorphic symetric kernel. It is also obviously locally bounded. We will now examine the behavior of K in the cusp. We find that unlike the compact case, K will not give rise to a compact operator, it will however be bounded and self-adjoint.

Behavior in the cusp.

Being locally bounded, we are only interested in the behavior of $K(\omega_1,\omega_2)$ as one of y_1, y_2 (or both) tend to infinity. (as usual $\omega_1 = (y_1,x_1)$, $\omega_2 = (y_2, x_2)$). It is clear that if y_1 is large enough (depending on our kernel k) say $y_1 \geq A$, then

$$(1.19) \qquad K(\omega_1,\omega_2) \sum_{\ell \in L} k(\omega_1,\ell\omega_2)$$

$$= \sum_{\ell \in L} \Phi\left(\frac{|x_1-x_2+\ell|^2}{y_1 y_2} + \frac{y_1}{y_2} + \frac{y_2}{y_1} - 2\right)$$

$$= \sum_{\ell \in L^*} \hat{\Phi}_{x_1,x_2,y_1,y_2}(\ell^*), \qquad L^* = \text{dual lattice to } L$$

by the Posson sum formula, i.e. where

$$\Phi_{x_1,x_2,y_1,y_2}(t) = \Phi\left(\frac{|x_1-x_2+t|^2}{y_1 y_2} + \frac{y_1}{y_2} + \frac{y_2}{y_1} - \alpha\right)$$

$$\therefore \quad \hat{\Phi}(\xi) = e^{2\pi i\langle \frac{x_1-x_2}{\sqrt{y_1 y_2}}, \xi\rangle} (y_1,y_2)^n \int_{\mathbb{R}^n} \Phi\left(|u|^2 + \frac{y_1}{y_2} + \frac{y_2}{y_1} - 2\right)\cdot e^{2\pi i\langle \sqrt{y_1,y_2}\ \xi,u\rangle} du.$$

It follows that uniformly in $x_1,x_2 \in \mathfrak{F}_L$ and y_1,y_2 that the above Fourier transform $\hat{\Phi}(\xi)$ decays rapidly in modulus as $|\xi| \to \infty$. Thus the sum over the dual lattice gives

$$(y_1y_2)^{n/2} \int_{\mathbb{R}^n} \Phi\left(|u|^2 + \frac{y_1}{y_2} + \frac{y_2}{y_1} - 2\right)du + O((y_1y_2)^{-N})$$

for any $N > 0$, and for $y_1 > A_1$. So we have

$$(1.20)\quad \begin{cases} K(\omega_1,\omega_2) = \int_{\mathbb{R}^n} k(\omega_1,\omega_2+t)dt + O((y_1y_2)^{-N}) \\ \quad = \begin{cases} 0 & \text{if } y_1/y_2 + y_2/y_1 \geq A_2 \\ (y_1y_2)^{n/2} \int_{\mathbb{R}^n} \Phi(|u|^2 + y_2/y_1 - 2)du \end{cases} \\ \qquad \text{for } y_1 > A_1 . \end{cases}$$

Thus as $y_1, y_2 \to \infty$ with y_1/y_2 bounded and small (near 1), $K(\omega_1,\omega_2) \sim y^n$. From this it is plain that $K \notin L^2(\mathfrak{F} \times \mathfrak{F})$. This is not surprising, as we have already seen that $[(\frac{n}{2})^2, \infty) \subset \sigma(\Delta) = \text{spectrum}(\Delta)$. So we do not have discrete spectrum, and so not every such K can be compact.

Lemma 1.21. K defines a bounded self adjoint operator of $L^2(\mathfrak{F}) \to L^2(\mathfrak{F})$.

Proof. The symmetry property of K that was remarked above, gives the self-adjointness. To see boundedness; let f be automorphic in $L^2(\mathfrak{F})$.

$$K_o f(\omega_1) = \int_{\mathfrak{F}} K(\omega_1,\omega_2)f(\omega_2)dV(\omega_2).$$

We are only interested in the L^2 boundedness of $K_o f$ as $y_1 \to \infty$, since locally K is bounded, even as a function. For y_1 large, we have from 1.20

$$|K_o f(\omega_1)| \leq C_2 y_1^{n/2} \int_{\mathfrak{F}_L} \int_{y_1/c_1}^{y_1 c_1} (y_2)^{n/2} |f(\omega_2)| \frac{dV(\omega_2)}{(y_2)^{n+1}}$$

c_1, c_2 depend on Γ and k only.

$$\therefore \int_{\mathfrak{F}_L} \int_A^\infty |K_o f|^2(\omega_1) \frac{dy_1 dx_1}{y_1^{n+1}} \leq \int_A^\infty \int_{\mathfrak{F}_L} |y_1^{n/2} \int_{\mathfrak{F}_L} \int_{y/c_1}^{yc_1} \frac{|f(\omega_2)|}{(y_2)^{(n+1)/2}} \frac{1}{y_2^{1/2}} dy_2|^2 \frac{dy_1 dx_1}{y_1^{n+1}} C_2$$

$$\leq C_2 \int_{\mathfrak{F}_L} \int_A^\infty \int_{\mathfrak{F}_L} (\int_{\mathfrak{F}_L} \int_{y_1/c}^{y_1 c} \frac{|f|^2 dy_2}{y_2^{n+1}})(\int_{y_1/c_1}^{y_1 c_1}) dx_2 \frac{dy_1 dx_1}{y_1}$$

$$\leq C_2' \int_{\mathfrak{F}_L} \int_A^\infty (\int_{\mathfrak{F}_L} \int_{y_1/c_1}^{y_1} \frac{|f|^2 dx_2 dy_2}{y_2^{n+1}}) \frac{dx_1 dy_1}{y_1}$$

$$\leq C_2'' \|f\|_2^2 .$$

Applying the kernel $K(\omega_1, \omega_2)$ to $E(\omega_2, s)$, (in $\mathrm{Re}(s) > n$)

$$\int_{\mathfrak{F}} K(\omega_1, \omega_2) E(\omega_2, s) dV(\omega_2) = \int_{H^{n+1}} k(\omega_1, \omega_2) E(\omega_2, s) dV(\omega_2)$$

$$= \hat{k}(s) E(\omega_1, s) \qquad (k \to \hat{k} \text{ Selberg transf.}) \text{ i.e.}$$

(1.22) $$K_o E(\omega_1, s) = \hat{k}(s) E(\omega_1, s).$$

The last of course is a pointwise identity rather than an L^2 statement. We now begin a series of modification of functions by "cut offs" which allow us to get L^2 vectors. Let

(1.23) $$\tilde{E}(\omega, s) = E(\omega, s) - \alpha(y)\, y^s \quad \text{for } \operatorname{Re}(s) > n,$$

where $\alpha(y) \in C^\infty(\mathbb{R})$, $\alpha \geq 0$ and

$$\alpha(y) = \begin{cases} 0 & \text{if } y \leq A \\ 1 & \text{if } y \geq A+1 \end{cases}, \quad A \text{ a large parameter.}$$

If A is large enough (depending on Γ) then 1.23 defines a Γ automorphic smooth function $\tilde{E}(\omega, s)$. From (1.11)" it is clear that $\tilde{E}(\omega, s) \in L^2(\mathbb{H}^{n+1}/\Gamma)$. For $\operatorname{Re}(s) > n$, clearly $\tilde{E}$ is holomorphic in s. Now

1.24 $$\begin{cases} (K_o \tilde{E})(\omega, s) = \hat{k}(s)\, E(\omega, s) - K_o\,(\alpha(y) y^s) \\ \qquad\qquad = \hat{k}(s)\, \tilde{E}(\omega, s) + G(\omega, s) \qquad \text{where} \\ G(\omega, s) = (K - \hat{k}(s))\, \tilde{E}(\omega, s) = K_o(\alpha(y) y^s) - \hat{k}(s) \alpha(y) y^s . \end{cases}$$

It follows that $G(\omega, s)$ is entire in s, and is actually supported in a compact subset (i.e. $y \leq A'''$) of $\mathfrak{F}$ independent of s. So $G(\omega, s)$

is in $L^2(\mathbb{H}^{n+1}/\Gamma)$, $\forall s$. Thus if $\hat{k}(s)$ is not in the spectrum of K,

$$(1.24)' \qquad G(\omega,s) = (K-\hat{k}(s))\tilde{E}(\omega,s)$$

has a unique solution $\tilde{E}(\omega,s)$, and (1.24)' will give an analytic continuation of $\tilde{E}$ to this region. The spectrum of K is real. Choosing $K = D_o K_1$, where K_1 is close to an approximation to the identity gives

$$K_o y^s = -s(n-s)\hat{K}_1(s)y^s$$

$K_1(s)$ near 1. So if $\mathrm{Re}(s) > \frac{n}{2}$, s not real then $s(n-s)$ avoid spec(K) and so we obtain the analytic continuation of $\tilde{E}(\omega,s)$ to this region. We have proven

<u>Theorem 1.25.</u> There is a holomorphic function $\tilde{E}(\omega,s)$ taking values in $L^2(\mathfrak{F})$ defined and holomorphic in $\mathrm{Re}(s) > n/2$, s not real and such that

$$E(\omega,s) = \alpha(y)y^s + \tilde{E}(\omega,s).$$

Thus E may be continued to this region. We now present Selberg's proof of the meromorphic continuation of E to all of $\mathbb{C}$.

To begin with we must modify our kernel by removing the part which is growing in the cusp. Let

$$\tilde{K}(\omega_1,\omega_2) = K(\omega_1,\omega_2) - \alpha(y_1)\int_{\mathbb{R}^n} k(\omega_1,\omega_2+t)dt$$

$\tilde{K}$ is Γ invariant and write it

(1.26) $$\tilde{K}(\omega_1,\omega_2) = K(\omega_1,\omega_2) - K_o(\omega_1,\omega_2).$$

In view of 1.20 $\tilde{K}(\omega_1,\omega_2) = O(y^{-n})$ as $y \to \infty$ and also $\tilde{K}(\omega_1,\omega_2) = 0$ if $y_1/y_2 + y_2/y_1 \geq A_2$ say. Note that $\tilde{K}$ is no longer self-adjoint. From the definition it is apparent that

(1.27) $$\int_{\mathfrak{F}_L} K_o(\omega_1,\omega_2)\, e(\langle m,x_1\rangle)dx_1 = 0 \quad \text{if} \quad m \neq 0$$

$\therefore$ In $\mathrm{Re}(s) > n$

(1.28) $$\tilde{K}_o(E(\omega,s)) = \hat{k}(s)E(\omega,s) - K_o(y^s + \phi(s)y^{n-s})$$

and

$$K_o(y^s) = \alpha(y)\int_{\mathfrak{F}_L}\int k(\omega_1,(t,y_2))\, y_2^s \frac{dt\, dy_2}{y_2^{n+1}} = \alpha(y)\,\hat{k}(s)y^s$$

(1.29) $$\therefore \quad (\tilde{K} - \hat{k}(s))\, E(\omega,s) = -\alpha(y)\,\hat{k}(s)[\, y^s + \phi(s)y^{n-s}] \,.$$

We saw by a previous remark that $\tilde{K}$ is a compact operator, so that 1.29 could be used to continue $E(\omega,s)$. However we do not know $\phi(s)$ nor are the various terms in L^2. To overcome these difficulties we consider the equation

(1.30) $$(\tilde{K} - k(s))\, E^*(\omega,s) = -\alpha(y)\,\hat{k}(s)\, y^s \,.$$

This is an equation for the unknown E^*. A further modification is needed to put the r.h.s. in L^2. Let

(1.30)' $$E^*(\omega,s) = \alpha(y)y^s + E^{**}(\omega,s)$$

The equation for E^{**} becomes

(1.31) $$(\tilde{K} - \hat{k}(s))\, E^{**}(\omega,s) = \tilde{K}_o(\alpha(y)y^s) = H(\omega,s).$$

The decay $\tilde{K}(\omega_1,\omega_2) = O(y^{-N})'$ implies that $H(\omega,s) \in L^2(\mathfrak{F})$. Thus (1.31) may be solved uniquely and meromorphically for $E^{**}(\omega,s)$. By varying k (e.g. making it an approximation to the identity, we learn that $E^{**}(\omega,s)$ has a meromorphic continuation to $\mathbb{C}$. Since $\tilde{K}$ is smooth (i.e. smooth kernel), $E^{**}(\omega,s)$ is smooth in ω. Thus by (1.30)' we have that $E^*(\omega,s)$ may be meromorphically continued to $\mathbb{C}$ and is smooth in ω. Since E^* satisfies (1.30) we have that for $\text{Re}(s) > n$;

(1.32) $$(\tilde{K}-\hat{k})[E^*(\omega,s) + \phi(s)\, E^*(\omega,n-s)] = -\,\alpha(y)\,\hat{k}(s)\,[y^s + \phi(s)y^{n-s}]$$

which is 1.29 with E replaced by $E^*(s) + \phi(s)\,E^*(n-s)$.

<u>Lemma 1.33</u>. For $\text{Re}(s) > n$

$$E(\omega,s) = E^*(\omega,s) + \phi(s)\,E^*(\omega, n-s)\,.$$

<u>Proof</u>. Both satisfy 1.29, and both differ from functions in $L^2(\mathfrak{F})$ by $\alpha(y)y^s$. Thus by the usual argument, the difference is annihilated by $\tilde{K}-\hat{k}(s)$. But for $\text{Re}(s) > n$ and s is some suitable open subset $\hat{k}(s)$ does not lie in the spectrum of $\tilde{K}$, whence the difference is zero identically.

Thus to meromorphically continue $E(\omega,s)$ we need only continue $\phi(s)$ as E^* has already been handled. Actually we can reduce the continuation of ϕ to that of E^*.

Lemma 1.34. Let $\mathrm{Re}(s) > n$, $s \notin \mathbb{R}$ and s not a pole of $E^*(\omega,s)$ or $E^*(\omega, n-s)$, then $E^*(\omega,s) + \lambda E^*(\omega, n-s)$ is an eigenfunction of Δ with eigenvalue $s(n-s)$ iff $\lambda = \phi(s)$.

Proof. We have just seen the one way implication. For the other direction, if $\lambda \neq \phi(s)$ satisfies the above properties then after a subtraction we get that $E^*(\omega,n-s)$ is again such an eigenfunction of Δ (non-zero), and $E^*(\omega,n-s) \in L^2$. But the usual lemma now tells us that $E^*(\omega,n-s) \equiv 0$. $\therefore$ $\lambda = \phi(s)$.

E^*, E^{**} are both smooth in ω and the equation

$$\Delta E^* + \lambda \Delta E^* = -s(n-s) E^*$$

leads to

$$\Delta E^{**}(\omega,s) + \lambda \Delta E^*(\omega,n-s) = s(n-s)[E^*(\omega,s) + \lambda E^*(\omega, n-s)] + H(\omega,s), \tag{1.35}$$

where $H(\omega,s)$ is holomorphic in s and is of compact support in ω. Also we know $DE^{**}(\omega,s)$ is meromorphic in s and in $L^2(\mathbb{H}^{n+1}/\Gamma)$ $\therefore$ 1.35 is of the form

$$u + \lambda v = 0, \; u, v \in L^2 . \tag{1.36}$$

By the previous lemma for s in a suitable open set (1.35) has a unique solution λ, and this λ is $\phi(s)$. Thus for s in this region, $v \neq 0$ and

$$(*) \qquad \phi(s) = \lambda = - \frac{(u,v)}{(v,v)} \ .$$

Now $u = u(\omega, s)$ and $v = v(\omega, s)$ are meromorphic in s (from 1.35 and their definition) and so (*) may be used to meromoprhically continue the function $\phi(s)$.

Theorem 1.37. $\phi(s)$ may be meromorphically continued to all of $\mathbb{C}$.

Theorem 1.38. $E(\omega, s)$ has a meromorphic continuation to $\mathbb{C}$, and satisfies

$$\Delta E(\omega, s) + s(n-s) E(\omega, s) = 0 .$$

This concludes Selberg's proof. We remark that one of it's strengths is that all that we have used is the theory of compact operators, and in principle at least this method should extend to higher rank.

§1.39. We now present another proof of the continuation of E. The proof makes use of differential rather than integral equations, but as the reader may see for him or herself there are similarities with the proof just presented. The proof in its present form is due to Colin De Verdiere [ ] which in turn uses ideas of Lax, Phillips [ ]. The proof is more sophisticated than the last, and requires some knowledge of functional analysis.

By this point the reader should be ready to consider the case of more than the cusp. We saw in a previous chapter [ ] that in the general case of a finite volume hyperbolic manifold the fundamental domain will have a finite number, say h, cusps. Let $k_1, k_2, \ldots k_h$, be a set of such h inequivalent cusp points - so $k_j \in \mathbb{R}^n \cup \{\infty\}$. Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_h$ be the stabilizers of these points - which we assume are rank n lattices. For any cusp k_j there is a rigid motion Θ_j of H^{n+1} taking k_j to ∞, under this, Γ_j is transformed into a rank n lattice of translations of $\mathbb{R}^n$. The corresponding coordinates are

$$\omega^{(j)} =$$

will be called normal co-ordinates for the j-th cusp. For each cusp we define an Eisenstein series as before.

Definition 1.40.

$$E_j(\omega, s) = \sum_{\gamma \in \Gamma_j/\Gamma} (y^j(\gamma\omega))^s, \quad \text{where } \omega^j = (y^j, x^j).$$

As in the case of one cusp, (prop$^{\underline{n}}$ 1.5, (1.11)") we obtain

1.41. $E_j(\omega,s)$ converges absolutely and uniformly for ω in compact subsets of $\mathfrak{F}$, for $Re(s) > n$. Furthermore in this region

$$\Delta E_j + s(n-s) E_j = 0.$$

On using the co-ordinates $\omega^{(j)}$, $E_i(\omega^{(j)},s)$ is periodic under the lattice at infinity L_j corresponding to the stabilizer Γ_j. The Fourier expansion as in 1.10 is of the form

1.42. $$E_i(\omega^{(j)},s) = a_o(i,y^{(j)},s) + \sum_{\ell \in L_j^*} c(\ell,s,i,j)(y^j)^{n/2} K_{s-n/2}(2\pi|\ell|ny^{(j)})$$

with

$$a_o(i,y^{(j)},s) = \delta_{ij}(y^j)^s + \phi_{ij}(s)(y^j)^{n-s} .$$

1.43. $$E_i(\omega^{(j)}, s) = \delta_{ij}(y^j)^s + \phi_{ij}(s)(y^j)^{n-s} + O(e^{-cy^j})$$

for $c > 0$, and as $y^j \to \infty$.

The last gives us the behavior of the i^{th} Eisenstein series in the j^{th} cusp.

The analogue of 1.17 is

1.44. Let f be an automorphic eigenfunction of Δ with eigenvalue $s(n-s)$, $\mathrm{Re}(s) > n$ and if

$$f(\omega^{(j)}) = O((y^j)^m), \quad m < \infty \qquad \text{for } j = 1, \ldots, h$$

then f is a linear combination (unique) of the Eisenstein series.

Proof. By using the Fourier expansion (for f) and 1.43 and the polynomial bound into the cusp, we see that $f(\omega^{(j)}) = \alpha_j (y^j)^s + L^2$ functions in the j^{th} cusp. From which

$$f(\omega) - \sum_{j=1}^{h} \alpha_j E_j(\omega, s) \in L^2(\mathfrak{F}) \Rightarrow f = \sum_{j=1}^{h} \alpha_j E_j ,$$

by the usual argument (1.16).

We turn to the meromorphic continuation of E_j and $\phi_{ij}(s)$.

Let ψ be a C^∞, Γ automorphic function, which for $a_1 < a_2$ large enough looks like

$$\psi(y^{(j)}) = \begin{cases} 0 & \text{for } y^{(j)} < a_1 \\ 1 & \text{for } y^{(j)} > a_2 \end{cases} .$$

Clearly for a_1 large enough such a function exists.

Let us be a little more precise about the operators we are talking about, since the proof is based on changing their domains. Δ is the Laplacian on $L^2(\mathcal{H}^{n+1}/\Gamma)$, which is a self adjoint operator with core domain (see) Γ automorphic smooth functions with compact support in $\mathfrak{F}$. Let R_λ be the resolvent of Δ, at λ, which is certainly bounded and holomorphic in λ, for λ outside of $[0, \infty]$, since $\Delta \geq 0$.

For a fixed cusp k_j, let $h_j(\omega)$ be the Γ automorphic function which looks like

$$h_j(\omega, s) = \begin{cases} \psi(\omega^j)(y^{(j)})^s & \text{in } j^{th} \text{ cusp} \\ 0 & \text{other cusp.} \end{cases}$$

Let

$$(1.45) \qquad H_j(\omega, s) = -(\Delta + s(n-s))(h_j(\omega, s)).$$

It is clear that $H_j(\cdot, s)$ is of a fixed (independent of s) compact support in $\mathfrak{F}$ for every s. That $H_j(\omega, s)$ is entire is also clear.

Now fix $a > a_2$, and define the Hilbert space $\mathcal{H}_a$ to be the closed subspace of $L^2(\mathcal{H}^{n+1}/\Gamma)$ for which the zeroth co-efficient of f in each cusp, call it $\hat{f}_j(0, y^{(j)})$ satisfies

$$\hat{f}_j(0, y^{(j)}) = 0 \quad \text{for } y^{(j)} \geq a.$$

$\mathcal{H}_a$ is the orthogonal complement of all functions $f(\omega)$ which are functions of y^j only for $y^j \geq a$, so. $\mathcal{H}_a$ is indeed a closed subspace of $\mathcal{H} = L^2(\mathcal{H}^{n+1}/\Gamma)$.

We now consider Δ_a which will be a self-adjoint operator, i.e. the Laplacian, but with domain, functions in $\mathcal{H}_a$. The easiest way to define Δ_a is to define it through a quadratic form. Let

$$(1.45) \qquad C(u,v) = \int_{\mathfrak{F}} \langle \nabla u, \nabla v \rangle \, dv + \int_{\mathfrak{F}} u \bar{v} \, dv .$$

If we let C_a be the form defined by 1.45 with domain

$$(1.46) \qquad \mathcal{D}_a = \mathcal{G} \cap \{f \in L^2, \ \nabla f \in L^2, \ f \text{ automorphic under } \Gamma\} .$$

$\mathcal{D}_a$ is dense in $\mathcal{H}_a$, and clearly C_a is a closed symmetric, positive quadratic form. Thus C_a gives rise to a unique self adjoint operator Δ_a on $\mathcal{H}_a$ (see for example). It is clear that locally away from the horospheres $y^{(j)} = a$, the action of Δ_a is that of Δ. Also functions of the form $f \in \mathcal{H}_a$ with

$$\Delta f = g + \sum_{j=1}^{h} \alpha_j T_j \qquad \text{(in the distributional sense)}$$

where

$$T_j(\psi) = \int_{\{y^i=a\} \cap \mathfrak{F}} \psi(\omega) dx, \qquad g \in L^2(H^{n+1}/\Gamma)$$

are in the domain of Δ_a. This follows since everything is happening on the zero'th Fourier coefficient and there the space $\mathcal{H}_a$ has zero'th coefficient is zero "at $y^{(j)} = a$". In fact for such an f

$$(1.47) \qquad \Delta_a f = g.$$

<u>Proposition 1.48</u>. The resolvent of Δ_a on $\mathcal{H}_a$ is compact and

$$(\Delta_a + s(n-s))^{-1}$$

is a meromorphic family of bounded operators.

<u>Proof</u>. We need only show that the C_a form embedds compactly into $\mathcal{H}_a$ [ ]. Or since $\mathcal{H}_a$ is closed that

$$B = \{u \in \mathcal{H}_a : C_a(u) \leq 1\}$$

is relatively compact in $L^2(\mathcal{H}^{n+1}/\Gamma)$. By the Rellich embedding theorem [ ], B is locally relatively compact. Thus all one needs to show is that

$$\int_{y^j \geq b} |u|^2 \frac{dx\,dy}{y^{n+1}} \to 0 \qquad \text{uniformly for } u \in B \text{ or } b \to \infty.$$

Now since $u(y,x)$ has it's zero Fourier coefficient on each horosphere $y = \text{const.} \geq a$, equal to zero we have

$$\int_{x \in \mathfrak{F}_{L_j}} |u(y^j, x^j)|^2 \, dx^j \leq C_j \int_{\mathfrak{F}_{L_j}} [(\frac{\partial u}{\partial x_1^j})^2 + \ldots + (\frac{\partial u}{\partial x_n^j})^2] \, dx^j$$

for a suitable constant C_j = Poincare construct for the torus $\mathbb{R}^n/L_j$.

$$\therefore \quad \sum_{j=1}^{h} \int_{y^{(j)} \geq b} |u|^2 \frac{dx\,dy}{y^{n-1}} \leq \tilde{C} \sum_{j=1}^{h} \int_{y \geq b} |\nabla u|^2 \frac{dx\,dy}{y^{n-1}} \leq \tilde{C}\, C(u)$$

$$\therefore \quad \sum_{j=1}^{h} \int_{y^j \geq b} |u|^2 \frac{dx\,dy}{y^{n+1}} \leq \frac{1}{b^2} \tilde{C}\, C(u) \qquad \text{as needed.} \qquad \square$$

We now define

$$(1.49) \qquad K_j(\omega, s) = h_j(\omega, s) + (\Delta_a + s(n-s))^{-1} H_j(\omega, s)$$

K_j is well defined as $\mathfrak{H}_j(\omega, s) \in \mathfrak{H}_a$.

It is clear that

$$(\Delta_a + s(n-s))[K_j(\omega, s) - h_j(\omega, s)] = H_j(\omega, s).$$

Thus locally away from $y^j = a$ we have

$$(\Delta + s(n-s))(K_j(\omega, s) - h_j(\omega, s)) = H_j(\omega, s) \quad \text{i.e.}$$

$$(1.50) \quad \begin{cases} (\Delta + s(n-s))\, K_j(\omega, s) = 0 \\ \qquad \text{for } y^j \neq a \end{cases}$$

1.50 is the key, since we have seen that E_j also satisfies 1.50 - so we should be able to relate K_j to E_j. (Of course 1.49 gives us the meromorphic continuation of K_j).

We must first examine the behavior of $K_j(\omega,s)$ for $y^j = a$. Now $K_j(\omega,s)$ is in the domain of Δ_a and is therefore continuous, we may expand it into a Fourier series in $y^{(j)}$ in the j^{th} cusp. $\hat{K}_j(0, y^{(j)}, s)$ will in view of 1.50, satisfy the usual zeroth coefficient ordinary differential equation for $y^{(j)} < a$ and $y^{(j)} > a$. $\hat{K}_j(0, y^{(j)}, s)$ is continuous in $y^{(j)}$ (though its derivative need not be) and so

$$(1.51) \qquad \hat{K}_j(0, y^{(j)}, s) = \begin{cases} A_j(s)(y^j)^s + B_j(s)(y^j)^{n-s}, & y^j < a \\ (y^j)^s, & y^j > a \end{cases}$$

for suitable $A_j(s)$, $B_j(s)$. Notice the second part of 1.51 comes from 1.49 since $\mathfrak{H}_j \in \mathfrak{H}_a$. Since $K_j(\omega,s)$ is meromorphic from 1.49 and 1.48, it follows that $A_j(s)$, $B_j(s)$ are meromorphic.

Now consider

$$(1.51) \qquad G_j(\omega,s) = K_j(\omega,s) + \chi_{(a,\infty)}(y^{(j)})$$

$$\times \; \{A_j(s)(y^j)^s + B_j(s)(y^j)^{n-s} - (y^j)^s\} .$$

It is apparent that $G_j(\omega,s)$ and it's first derivative are continuous accross $y^j = a$. Away from $y^j = a$

$$(1.52) \qquad (\Delta + s(n-s))\,G_j(\omega,s) = 0$$

and only the zero'th coefficient of $G_j(\omega,s)$ could be 'bad', since it has been fixed up, up to 1st derivative, 1.52 actually holds $\forall\omega$.

Now for $\mathrm{Re}(s) > n$ the behavior of G_j in jth cusp. is $A_j(s)(y^j)^s + L^2$, and in the ith cusp. $j \neq i$ it is L^2, we conclude by the usual lemma that

$$(1.53) \qquad G_j(\omega,s) \equiv A_j(s)\,E_j(\omega,s).$$

From 1.51 for example since $\hat{K}_j$ is continuous, $A_j(s) \not\equiv 0$, it is meromorphic, as is G_j. 1.53 then furnishes the meromorphic continuation of E_j. Once we have the continuation of $E_j(\omega,s)$ for $j = 1,\ldots,h$ it is clear that $\phi_{ij}(s)$ may be meromorphically continued. We have proven:

Theorem 1.54. For each $\omega \in \mathbb{H}^{n+1}$, $E_j(\omega,s)$ is meromorphic in $\mathbb{C}$, as is $\phi_{ij}(s)$, as if s is not a pole of $E_j(\omega,s)$

$$(\Delta + s(n-s))\,E_j(\omega,s) = 0.$$

Remark. A slightly more general type of Eisenstein series that will be used later is the following.

Let χ be a unitary one-dimensional character of Γ, trivial on Γ_∞. Define

$$E(\omega,s,\chi) = \sum_{\gamma\,\epsilon\,\Gamma_\infty/\Gamma} \chi(\gamma)(y(\gamma\omega))^s$$

then

$$E(\gamma\omega,s,\chi) = \chi(\gamma)\,E(\omega,s,\chi)$$

and $\Delta E + s(n-s)E = 0$. The meromorphic continuation is identical to what we did, except that the Hilbert space, is that of functions satisfying

$$f(\gamma\omega) = \chi(\gamma)f(\omega).$$

1.55 FURTHER PROPERTIES OF $E(\omega,s)$.

Though we have shown $\phi_{ij}(s)$ and $E(\omega,s)$ are meromorphic, we know very little else about them at this point (e.g. location of poles etc.). In this section we will prove the Maass - Selberg inner product formula and use it to prove some simple facts about $E(\omega,s)$.

PROPOSITION 1.56. The functions $E_j(\omega,s)$ or $\phi_{ij}(s)$ have no poles in $Re(s) > n/2$ except possibly for finitely many s_j's with $s_j\,\epsilon\,(n/2,n)$.

PROOF. If E_i has a pole of order ν at some point s_o, then

$$\lim_{s\to s_o} (s-s_o)^\nu E_i(\omega,s)$$

will be a nontrivial eigenfunction of Δ with eigenvalue $s_o(n-s_o)$. Since the Fourier expansion of this eigenfunction in the i-th cusp, will not contain the term $(y^i)^s$, it is clear that this eigenfunction will be square summable over $\mathfrak{F}$. The usual selfadjointness now implies that $s_o(n-s_o)$ is real. q.e.d. □

The pole at $s=n$.

At $s=n$, $E_j(\omega,s)$ has a pole and its residue is easily calculated. To see this, we will show shortly (see) that if $E(\omega,s)$ has a pole in $(n/2, n)$ it must be a simple pole. Now at $s=n$, if there is a pole, the residue will be an L^2 eigenfunction with eigenvalue 0. Thus it is an L^2 harmonic function, which must therefore reduce to a constant, say c_j (if there is no pole then $c_j=0$). Consider in the jth cusp

$$(1.57)\qquad \int_{\mathfrak{F}}[E_j(\omega^{(j)},s)-(y^j)^s]\,\frac{dx^j\,dy^j}{(y^j)^{n+1}} = \int_{S_j-\mathfrak{F}}(y^j)^s\,\frac{dx^j\,dy^j}{(y^j)^{n+1}}$$

where

$$S_j = \{(y^j,x^j) : 0 < y^j < \infty,\ \ x^j \in \mathfrak{F}_{L_j}\}$$

$\therefore$ l.h.s of 1.57.
$$= \int_{\mathfrak{F}_{L_j}}\int_0^{A(x^j)}(y^j)^s\,\frac{dy^j}{(y^j)^{n+1}}\,dx^j$$

where $A(x^j)$ is the height function as in figure (1.3). So (1.57)

$$= \int_{\mathfrak{F}_{L_j}}\frac{A(x^j)^{s-n}}{s-n}\,dx^j$$

$\therefore$ Res(of 1.57) at $s = n = V(\mathfrak{F}_{L_j})$.

On the other hand from 1.57 directly $E_j(\omega^{(j)}, s) - (y^j)^s$ is uniformly in L^2 over $\mathfrak{F}$, and has residue C_j at $s = n$.

$$\therefore \qquad C_j V(\mathfrak{F}) = V(\mathfrak{F}_{L_j}).$$

Theorem 1.58. $E_j(\omega, s)$ has a simple pole at $s = n$, the residue being the constant function $V(\mathfrak{F}_{L_j})/V(\mathfrak{F})$.

Thus 1.58 may be used to compute volumes of fundamental domains for some cases such as the Bianchi groups introduced earlier. We defer these and other applications of the Eisenstein series to another chapter.

PROPOSITION 1.59.

$$E_i(\omega, n-s) = \sum_{j=1}^{h} \phi_{ij}(n-s) E_j(\omega, s).$$

In matrix notation if $E(\omega, s) = (E_1(\omega, s), \ldots, E_h(\omega, s))^h$ and if $\Phi(s) = (\phi_{ij}(s))$.

$$E(\omega, n-s) = \Phi(n-s) E(\omega, s).$$

Proof. $\sum_{j=1}^{h} \phi_{ij}(n-s) E_j(\omega, s)$, has its zeroth coefficient in the ℓ^{th} cusp equal to

$$\sum_{j=1}^{h} \phi_{ij}(n-s)\delta_{j\ell}(y^{\ell})^{s} + \sum_{j=1}^{h} \phi_{ij}(n-s)\phi_{i\ell}(s)(y^{\ell})^{n-s} .$$

On the other hand $E_i(\omega, s)$ has zeroth coefficient

$$\delta_{i\ell}(y^{j})^{n-s} + \phi_{i\ell}(s)(y^{\ell})^{s} .$$

It follows that for $\mathrm{Re}(s) > n$

$$\sum_{j=1}^{h} \phi_{ij}(n-s)\,E_j(\omega, s) - E_i(\omega, s) \in L^2(\mathfrak{H}^{n+1}/\Gamma),$$

and has eigenvalue $s(n-s) \Rightarrow$ (as usual) the result.

Corollary 1.60. $\Phi(s)$ satisfies the functional equation

$$\Phi(n-s)\,\Phi(s) = I .$$

We now turn to the inner product formulas. These will give us more insight into $E(\omega, s)$ and also are vital in using $E(\omega, s)$ to produce continuous spectrum.

As is by now customary, we must begin by truncating the Eisenstein series (to get back to L^2 quantities).

Definition 1.61. Let A be a large parameter, we define

$$\tilde{E}_{i,A}(\omega,s) = \begin{cases} E_i(\omega,s) - (\delta_{ij}(y^j)^s + \phi_{ij}(y^j)^{n-s} & \text{if } y^{(j)} > A \\ E_i(\omega,s) & \text{otherwise.} \end{cases}$$

It is clear from 1.43 that $\tilde{E}_{i,A}$ is square summable over $\mathfrak{F}$.

1.62. Maass - Selberg Relation. For $s_1 \neq \bar{s}_2$, $s_1 + \bar{s}_2 \neq n$

$$\int_{\mathfrak{F}} \tilde{E}_{i,A}(\omega,s_1)\,\tilde{E}_{j,A}(\omega,s_2)\,dV(\omega)$$

$$= \frac{\delta_{ij}\,A^{s_1+\bar{s}_2-n} - A^{n-(s_1+\bar{s}_2)} \sum_{k=1}^{h} \phi_{ik}(s_1)\,\overline{\phi_{jk}(s_2)}}{s_1+\bar{s}_2-n}$$

$$+ \frac{\overline{\phi_{ji}}(s_2)\,A^{s_1-\bar{s}_2} - \phi_{ij}(s_1)\,A^{\bar{s}_2-s_1}}{s_1-\bar{s}_2}\,.$$

In this formula we are assuming, as we will from now on as far as the general theoretical aspect of Eisenstein series is concerned, that the cusps k_j, and transformations $\mathfrak{S}_j$ are normalized so that $V(\mathfrak{F}_{L_j}) = 1$.

Proof. The quantity on the left of the identity may be split as

$$\int_{\mathfrak{F}_A} \tilde{E}_i(\omega,s_1)\,\tilde{E}_j(\omega,s_2)\,dV(\omega) + \sum_{\ell=1}^{h} \int_{y^\ell \geq A} \tilde{E}_i\,\bar{\tilde{E}}_j\,dV(\omega)\,.$$

For each of these integrals, $\tilde{E}_i$ and $\tilde{E}_j$ are genuine eigenfunctions of Δ, and we may apply Green's formula, (non Euclidian version).

$$(1.63)' \quad \int_S (u\,\Delta v - v\,\Delta u)dV = \int_{\partial S} (u\,\partial_n v - v\,\partial_n u)d\sigma$$

with $S = \mathfrak{F}_A$ or $S = \{\omega : y^{(\ell)} \geq A\}$, for some $\ell = 1, \ldots, n$.

The boundary terms will come from the sets $y^j = A$, $x \in L_j$, which for the volume integrals we have

$$(1.63) \quad \int \Delta\, \tilde{E}_i(\omega, s_1)\, \overline{\tilde{E}_j(\omega, s_2)}\, dV(\omega) = -\, s_1(n-s_1) \int \overline{\tilde{E}_i(\omega, s_1)}\; dV(\omega).$$

Also, since the only alternation to E_i, that is made when going to $\tilde{E}_i$ is on the zero'th Fourier coefficients in each cusp - all other boundary terms will cancel. Thus 1.63 and (1.63)' will yield an expression for the required inner product in terms of a boundary integral on $y^j = A$ of the zeroth coefficients. When these are evaluated, keeping in mind that the volume element on ∂S is $\frac{dx}{y^n}$, and $y \frac{\partial}{\partial y}$ is the unit normal vector to ∂S (all in a typical cusp), one obtains 1.62. We leave it to the reader to supply these details.

Corollary 1.64. $\phi_{ij}(s) = \phi_{ji}(s)$.

Proof. For $s_1, s_2 > n$ and real, we see from 1.62 that as $s_2 \to s_1$ all the terms except possibly the last remain bounded. But then this means the last must be bounded too and hence $\overline{\phi_{ij}}(s_1) = \phi_{ji}(s_1)$. However for s real $\phi_{ij}(s)$

is real, which is apparent from the series defining $E(\omega,s)$. Thus $\phi_{ij}(s) = \phi_{ji}(s) \quad \forall\, s \in \mathbb{R}$, but then this is so far all s, by analytic continuation.

Corollary 1.65. $\Phi(s)$ is holomorphic on $\mathrm{Re}(s) = \frac{n}{2}$ and is in fact a unitary matrix on this line.

Proof. We have seen that $\Phi(s)\,\Phi(n-s) = I$. Now 1.64 shows that $\Phi(\bar{s}) = \overline{\Phi(s)}$, so $\Phi(\frac{n}{2} - it) = \overline{\Phi(\frac{n}{2} + it)} = \overline{\Phi^{tr}}(\frac{n}{2} + it)$ by 1.64. This holds at all points of holomorphy of Φ along the line $\frac{n}{2} + it$, but it trivially also guaranties holomorphy on the whole line. The equation says precisely that Φ is unitary along the line.

Corollary 1.66. If s_o is a point of holomorphy of Φ then it is also a point of holomorphy of E.

Proof. We have seen that E is meromorphic in $\mathbb{C}$ and in particular at s_o. Thus

$$E(\omega, s) = \frac{f_k(\omega)}{(s-s_o)^k} + \dots + f_o(\omega) + \text{regular} \dots .$$

If there is a genuine singular part of E at s_o, then the L^2 norm of $\tilde{E}_A(o,s)$ would tend to infinity as $s \to s_o$. An inspection of 1.62, with $s_1 = \bar{s}_2 = s_o \notin R$, gives the left hand side $\to \infty$, which r.h.s. is finite since Φ is holomorphic at s_o.

If $s_o \in \mathbb{R}$, say $s_o = \sigma$, then taking a limit in 1.62 gives

$$(1.67) \qquad \int_{\mathfrak{F}} \tilde{E}(\omega,\sigma)\,\tilde{E}(\omega,\sigma)^{tr}\,dV(\omega) = 2\Phi(\sigma) - \frac{d\Phi}{d\sigma} + \frac{IA^{2\sigma-n} - A^{n-2\sigma}\,\Phi^2(\sigma)}{2\sigma - n} .$$

Thus we have the same conclusion, since even if $\sigma = n/2$, $\Phi^2(\frac{n}{2}) = I$ so that the right hand side is still bounded.

Corollary 1.66 and 1.65 are vital to the theory since they guarantee that $E(\omega,s)$ is holomorphic on $s = n/2 + it$ and we will need all of these functions to construct the continuous spectrum.

Corollary 1.68. The poles of Φ in $(\frac{n}{2}, n)$ are simple.

Proof. Taking the trace of equation 1.67 yields a positive quantity on the left hand side, while if Φ had a higher order pole the r.h.s. clearly becomes negative.

Theorem 1.69. The residues of $E_i(\omega,s)$ at the poles of Φ in $(\frac{n}{2}, n)$ are L^2 eigenfunctions of Δ (some may be zero).

Proof. This is clear from the Fourier expansion since the residues will not involve the $(y^j)^s$ terms.

Definition. Let $\phi(s) = \det \Phi(s)$.

The function $\phi(s)$ will be of utmost importance as it measures the quantity of continuous spectrum and it will appear as an important term in the trace formula.

Theorem. $\Phi(s)$ and $\phi(s)$ are bounded in $\mathrm{Re}(s) \geq \frac{n}{2}$, $\mathrm{Im}(s) \geq 1$.

Proof. The inner product formula with $s_1 = s_2$ yields

$$0 \leq \mathrm{Trace}\left[\frac{IA^{2\sigma-n} - A^{\sigma-2n}\,\Phi(s)\,\Phi^*(s)}{2\sigma-n} + \frac{\overline{\Phi(s)}\,A^{2it} - \Phi(s)A^{-2it}}{2it} \right].$$

In the region in question $2\sigma - n$, is positive so since $\Phi\Phi^*$ is the self-adjoint, the above gives an upper bound for $\|\Phi\Phi^*\|$ in $\frac{n}{2} \leq \sigma \leq n+1$ to the right of $n+1$ it's trivially bounded. To end this chapter we continue with the examples in 1.14.