Tel:(609) 258-4229
Email: sarnak@math.princeton.edu

June, 2001

Professor Enrico Bombieri
Institute for Advanced Study
Einstein Drive
Princeton, New Jersey 08540

Dear Enrico:

Below is the symplectic pairing that I mentioned to you in Zurich. There is nothing deep about it or the analysis that goes with it. Still its existence is consistent with various themes. To put things in context, recall that the phenomenological and analytic results on the high zeroes of a given L-function and the low zeroes for families of L-functions, suggest that there is a natural spectral interpretation for the zeroes as well as a symmetry group associated with a family [Ka-Sa]. In particular for Dirichlet L-functions $L(s,\chi)$, $\chi^2 = 1$, the symmetry predicted in [Ka-Sa] is a symplectic one, ie $Sp(\infty)$. So we expect that in a suitable spectral interpretation, the linear transformation whose spectrum corresponds to the zeroes of $L(s,\chi)$, should preserve a symplectic form. It should be emphasized that this by itself does not put the zeroes on the line. Such a symplectic pairing is a symmetry which is central to understanding this family of L-functions. On the other hand, the existence of an invariant unitary (or hermitian) pairing for the operator, as suggested by Hilbert and Polya would of course put the zeroes on a line. However, I think the existence of the latter is not very likely. In the analogous function field settings there are spectral interpretations of the zeroes and invariant bilinear pairings due to Grothendieck. The known proofs of the Riemann Hypothesis (that is, the Weil Conjectures in this setting) do not proceed with any magic unitary structures but rather with families and their monodromy, high tensor power representations of the latter and positivity [De].

One can look for a symplectic pairing in the well-known spectral interpretation of the zeroes of $\zeta(2s)$ in the eigenvalue problem for $X = SL(2,\mathbb{Z})\backslash\mathbb{H}$. Indeed the resonances (or scattering frequencies) through the theory of Eisenstein series for X are at the zeroes of $\xi(2s)$, ξ being the completed zeta function of Riemann. Lax and Phillips have constructed an operator B (see [La-Ph]) whose spectrum consists of the Maass cusp forms on X together with the zeroes of $\xi(2s)$. The problem with finding a symplectic pairing for the part of the spectrum corresponding to $\xi(2s)$ is that I don't know of any geometric way of isolating this part of the spectrum of B. The space $L^2(X)$ as it stands is too big. Nevertheless, this spectral interpretation of the zeroes of $\xi(2s)$ is important since it can be used to give reasonable zerofree regions for $\zeta(s)$, see [Sa]. Moreover, this spectral proof of the nonvanishing of $\zeta(s)$ on $\Re(s) = 1$ extends to much more general L-functions where the method of Hadamard and de la Valleè Poisson does not work (at least with our present knowledge) [Sh].

Assuming the Riemann Hypothesis for $L(s, \chi)$, Connes [Co] gives a spectral interpretation of the zeroes. I recall his construction below. Like the spectral interpretation as resonances, Connes' space is defined very indirectly - as the annihilator of the complicated space of functions (it is very close to the space considered by Beurling [Be]). We can look for invariant pairings for his corresponding operator. Note that for an even dimensional space a necessary condition that a transformation A of determinant equal to 1, to preserve a standard symplectic or orthogonal pairing is that its eigenvalues (as a set) be invariant under $\lambda \longrightarrow \lambda^{-1}$. In fact, if the eigenvalues are also distinct, then this is also a sufficient condition. On the other hand, if A is not diagonalizable then there are other obstructions (besides the "functional equation" $\lambda \longrightarrow \lambda^{-1}$) for preserving such pairings.

The set up in [Co] is as follows: Let χ be a nontrivial Dirichlet character of conductor q and with $\chi(-1) = 1$ (one can easily include all χ). For $f \in S(\mathbb{R})$ and even and $x > 0$ set

$$\Theta_f(x) := \left(\frac{x}{\sqrt{q}}\right)^{1/2} \sum_{n=1}^{\infty} f\left(\frac{nx}{q}\right) \chi(n) . \tag{1}$$

According to Poisson Summation and Gauss sums we have

$$\Theta_f\left(\frac{1}{x}\right) = \Theta_{\hat{f}}(x) . \tag{2}$$

Hence $\Theta_f(x)$ is rapidly decreasing as $x \longrightarrow 0$ or $x \longrightarrow \infty$. Consider the vector space W of distributions D on $(0, \infty)$ (with respect to the multiplicative group) with suitable growth conditions at 0 and ∞ for which

$$D(\Theta_f) = \int_0^{\infty} D(x)\, \Theta_f(x)\, \frac{dx}{x} = 0 , \tag{3}$$

for all f as above.

For $y > 0$, let U_y be the translation on the space of distributions $U_y D(x) = D(yx)$. Clearly, U_y leaves the subspace W invariant and yields a representation of $\mathbb{R}^*_{>0}$. By (2) if $D \in W$ then so is $RD := D(1/x)$. Thus R acts as an involution on W. To see which characters (ie eigenvectors of U_y) x^s, $s \in \mathbb{C}$, of $\mathbb{R}^*$, are in W, consider

$$\int_0^{\infty} \Theta_f(x)\, x^s\, \frac{dx}{x} = q^{s/2}\, L(s + \frac{1}{2}, \chi) \int_0^{\infty} f(y)\, y^{s+\frac{1}{2}}\, \frac{dy}{y} . \tag{4}$$

Now $f \in S(\mathbb{R})$ and is even, hence

$$I = \int_0^\infty f(x)\, x^{s+\frac{1}{2}}\, \frac{dx}{x} = \int_0^1 f(x)\, x^{s+\frac{1}{2}}\, \frac{dx}{x} + g(s)\,,$$

where $g(s)$ is entire. Moreover, for $N \geq 0$

$$\begin{aligned} I &= \int_0^1 \sum_{n=0}^{N} a_{2n}\, x^{2n}\, x^{s+\frac{1}{2}}\, \frac{dx}{x} + \text{a holomophic function in } \Re(s) > -N+1\,, \\ &= \sum_{n=0}^{\infty} \frac{a_{2n}}{2n - \frac{1}{2} + s} + \text{hol in } \Re(s) > -N+1\,. \end{aligned} \tag{5}$$

Thus for general such f, I has simple poles at $s = \frac{1}{2} - 2n$.

According to (4) and (5) we have that $x^s(\Theta_f) = 0$ for all f iff

$$s = i\gamma \text{ where } \rho = \frac{1}{2} + i\gamma \text{ is a } \underline{\text{nontrivial}} \text{ zero of } L(s,\chi)\,. \tag{6}$$

If the multiplicity of zero of $L(s,\chi)$ at $\rho = \frac{1}{2} + i\gamma$ is $m_\gamma \geq 1$, then differentiating (4) $m_\gamma - 1$ times shows that

$$x^{i\gamma}\,, (\log x)\, x^{i\gamma}\,, \ldots\,, (\log x)^{m_\gamma - 1}\, x^{i\gamma} \tag{7}$$

are in W.

The involution R of W ensures that $x^{i\gamma} \in W$ iff $x^{-i\gamma} \in W$ (and similarly with multiplicities). Of special interest is $\gamma = 0$. We have from (2) that for j odd

$$\int_0^\infty (\log x)^j\, \Theta_f(x)\, \frac{dx}{x} = -\int_0^\infty (\log x)^j\, \Theta_{\widehat{f}}(x)\, \frac{dx}{x}\,. \tag{8}$$

Hence if $f = \widehat{f}$ and j is odd

$$\int_0^\infty (\log x)^j\, \Theta_f(x)\, \frac{dx}{x} = 0\,. \tag{9}$$

If $f = -\widehat{f}$ then

$$\int_0^\infty f(x)\, x^{1/2}\, \frac{dx}{x} = 0\,. \tag{10}$$

So from (4) we see that if $f = -\widehat{f}$ then

$$\int_0^\infty \Theta_f(x)\,(\log x)^j\,\frac{dx}{x} \text{ for } j \;=\; 0, 1, \ldots, m_0\,. \tag{11}$$

Combining (9) and (11) we see that

$$\begin{aligned} W_0 &= \text{span}\,\{1,\,\log x\,,(\log x)^2\,,\ldots\} \cap W \\ &= \text{span}\,\{1,\,\log x\,\ldots\,,(\log x)^{m_0-1}\} \end{aligned} \tag{12}$$

is even dimensional.

Hence m_0 is even (of course this also follows from the functional equation for $L(s,\chi)$).

In order to continue we need to specify the precise space of distributions that we are working with. To allow for zeroes ρ of $L(s,\chi)$ with $\Re(\rho) \neq \frac{1}{2}$ one needs to allow spaces of distributions which have exponential growth at infinity. This can be done and one can proceed as we do here, however to avoid such definitions we will assume the Riemann Hypothesis for $L(s,\chi)$ (anyway, this is not the issue as far as a symplectic pairing goes). This way we can work with the familiar tempered distributions. We change variable, setting $x = e^t$ so that our distributions $D(t)$ satisfy

$$\int_{-\infty}^\infty D(t)\,\Theta_f\,(e^t)\,dt \;=\; 0\,. \tag{13}$$

The group U_y now acts by translations $\tau \in \mathbb{R}$,

$$U_\tau\,D(t) \;=\; D(t\,+\,\tau)\,. \tag{14}$$

If now V is the space of such tempered distributions satisfying the annihilation condition (13) for all f (which is a topological vector space), then for $D \in V$, its Fourier transform $\widehat{D}(\xi)$ is supported in $\left\{\gamma\,|\,\xi\left(\frac{1}{2}\,+\,i\gamma,\chi\right) \;=\; 0\right\}$. Since $\widehat{D}$ is also tempered it is easy to describe $\widehat{D}$ and hence the space V. It consists of all tempered distributions D of the form

$$D(t) \;=\; \sum_\gamma \sum_{j=0}^{m_\gamma-1} a_{j,\gamma}\,(D)\,t^j\,e^{i\gamma t}\,. \tag{15}$$

The representation (15) is unique and the series converges as a tempered distribution ie.

$$\sum_{|\gamma|\,\leq T}\;\sum_{j=0}^{m_\gamma-1} |a_{j,\gamma}(D)| \;\ll\; T^A \tag{16}$$

for some A depending on D.

The action (14) on V gives a group of transformations whose spectrum consists of the numbers $e^{i\gamma\tau}$ with multiplicity m_γ. The subspaces V_γ of V given by

$$V_\gamma = \text{span}\left\{e^{i\gamma t},\, te^{i\gamma t}, \ldots, t^{m_\gamma - 1} e^{i\gamma t}\right\} \tag{17}$$

are U_τ invariant. The action taking the form

$$e^{i\gamma\tau} \begin{bmatrix} 1 & \tau & \tau^2 & \ldots & \tau^{m_\gamma - 1} \\ & 1 & 2\tau & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}, \tag{18}$$

in the apparent basis. Thus U_τ is not diagonalizable if $m_\gamma > 1$ for some γ. The span of the subspaces V_γ is dense in V. So the action U_τ on V, for what it is worth, gives a spectral interpretation of the nontrivial zeroes of $L(s, \chi)$. While one expects for these $L(s, \chi)$'s that all their zeroes are simple, there are more general L-functions (eg those of elliptic curves of rank bigger than 1) which have multiple zeroes. Thus the possibility of multiple zeroes especially at $s = \frac{1}{2}$ must be entertained and it is instructive to do so. In any case, since multiple zeroes mean that this action U_τ is not diagonalizable, we infer that there cannot be any direct unitarity that goes along with it.

There is however a symplectic pairing on V preserved by U_τ. It is borrowed from $\text{sym}^\nu \rho$, where ρ is the standard two dimensional representation of SL_2 (via (18) above) when ν is odd. We pair V_γ with $V_{-\gamma}$ for $\gamma > 0$ and separate the even dimensional space V_0.

For $D, E \in V$ set

$$[D, E] := \sum_{j=0}^{m_0 - 1} \frac{(-1)^j\, a_{j,0}(D)\, a_{m_0-1-j,0}(E)}{\binom{m_0 - 1}{j}} + \sum_{\gamma > 0} \gamma e^{-\gamma^2} \sum_{j=0}^{m_\gamma - 1} \frac{(-1)^j}{\binom{m_\gamma - 1}{j}} \cdot \tag{19}$$
$$\cdot \left(a_{j,\gamma}(D)\, a_{m_\gamma-1-j,-\gamma}(E) - a_{m_\gamma-1-j,-\gamma}(D)\, a_{j,\gamma}(E)\right).$$

There is nothing special about the factor $\gamma e^{-\gamma^2}$ it is put there for convergence.

The bilinear pairing $[\ ,\ ]$ on $V \times V$ is symplectic and U_τ invariant. That is

(i) $\quad [D, E] = -[E, D]$

(ii) It is nondegenerate: for $D \neq 0$ there is E s.t. $[D,E] \neq 0$.

(iii) $[U_\tau D, U_\tau E] \,=\, [D,E]$ for $\tau \in \mathbb{R}$.

The verification of these is straight forward. Note that if $m_0 > 0$ and being even, one checks that the transformations (18) cannot preserve a symmetric pairing. Thus the symplectic feature is intrinsic to this spectral interpretation of the zeroes of $L(s,\chi)$.

It would be of some interest to carry out the above adelically as in Connes' papers and also for other (say GL_2) L-functions, especially where for example an orthogonal rather than symplectic invariance is expected [Ka-Sa]. Another point is that it would be nice to define the pairing [,] directly without the Fourier Transform (ie without first diagonalizing to Jordan form). If we assume RH as we have done as well as that the zeroes of $L(s,\chi)$ are simple then such a definition is possible. Set $H(t) = e^{-t^2/2}$ then for D and E in V, $H * D(t)$ and $\frac{d}{dt}\,(H*E)\,(t)$ are almost periodic functions on $\mathbb{R}$. Up to a constant factor we have that

$$[D,E] \,=\, M\left((H*D)(t)\,\frac{d}{dt}\,(H*E)\,(t)\right). \tag{20}$$

Here for an almost periodic function $f(t)$ on $\mathbb{R}$, $M(f)$ is its mean-value given by

$$M(f) \,=\, \lim_{T\longrightarrow\infty} \frac{1}{2T}\int_{-T}^{T} f(x)\,dx\,.$$

With best regards,
Peter Sarnak

Added February, 2002:

The Hilbert-Polya idea that there is a naturally defined self-adjoint operator whose eigenvalues are simply related to the zeroes of an L-function seems far-fetched. However, Luo and I [Lu-Sa] have recently constructed a self-adjoint nonnegative operator A on

$$L_0^2(X) \,=\, \left\{\psi \in L^2(SL(2,\mathbb{Z})|\mathbb{H})|\int_X \psi(z)\,dv(z) \,=\, 0\right\}$$

whose eigenvalues are essentially the central critical values $L\left(\frac{1}{2},\phi\right)$ as ϕ varies over the (Hecke-eigen) Maass cusp forms for X. In particular, this gives a spectral proof that $L\left(\frac{1}{2},\phi\right) \geq 0$. The fact that $L\left(\frac{1}{2},\phi\right)$ cannot be negative (which is an immediate consequence of RH for $L(s,\phi)$) is known and was proven by theta function methods (see [Kat-Sa], [Wal]). The operator A comes from polarizing the quadratic form $B(\psi)$ on $L_0^2(X)$ which appears as the main term in the Shnirelman sums for the measures $\phi_j^2(z)\,dv(z)$, where ϕ_j is an orthonormal basis of Maass cusp forms for $L^2(X)$. Denote by λ_j the (Laplace) eigenvalue of ϕ_j. It is known, see page 688 in [Se] that

$$\sum_{\lambda_j \leq \lambda} 1 \sim \frac{\lambda}{12} \quad \text{as } \lambda \longrightarrow \infty\,.$$

The quadratic form $B(\psi)$ comes from the following: For $\psi \in L_0^2(X)$ fixed;

$$\sum_{\lambda_j \leq \lambda} |\langle \phi_j^2\,, \psi\rangle|^2 \sim B(\psi)\ \sqrt{\lambda}$$

as $\lambda \longrightarrow \infty$.

Incidentally the family of L-functions $L(s,\phi)$ as ϕ varies as above, has an orthogonal $O(\infty)$ symmetry in the sense of [Ka-Sa], see also [Ke-Sn].

References:

[Ka-Sa] Katz, N. and Sarnak, P., *Bulletin of the AMS*, **36**, (1999), 1-26.

[De] Deligne, P., *Publ. IHES*, **48**, (1974), 273-308.

[La-Ph] Lax, P. and Phillips, R., *Bulletin of the AMS*, **2**, (1980), 261-295.

[Sa] Sarnak, P., Letter to F. Shahidi and S. Gelbart, January, 2001.

[Sh] Shahidi, F., *Perspect Math*, **10**, (1990), 415-437, Academic Press.

[Co] Connes, *Selecta Math* (N.S.), **5**, (1999), No. 1, 29-106.

[Be] Beurling, A., *Proc. Nat. Acad. Sci.*, U.S.A., **41**, (1955), 312-314.

[Lu-Sa] Luo, W. and Sarnak, P., "Quantum ergodicity on $SL_2(\mathbb{Z})\backslash\mathbb{H}$ II" (in preparation).

[Kat-Sa] Katok, S. and Sarnak, P., *Israel Math. Jnl.*, **84**, (1993), 193-227.

[Wal] Waldpurger, L., *J. Math. Pures et Appl.*, **60**, (1981), 365-384.

[Se] Selberg, A., Collected Papers, Vol. I, (1989), Springer-Verlag.

[Ke-Sn] Keating, J. and Snaith, N., *Comm. Math. Phys.*, **214**, (2000), 91-110.

PS/gpp.