ANALYSIS OF THE HETEROGENEOUS MULTISCALE METHOD FOR ELLIPTIC HOMOGENIZATION PROBLEMS

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Abstract. A comprehensive analysis is presented for the heterogeneous multiscale method (HMM for short) applied to various elliptic homogenization problems. These problems can be either linear or nonlinear, with deterministic or random coefficients. In most cases considered, optimal estimates are proved for the error between the HMM solutions and the homogenized solutions. Strategies for retrieving the microstructural information from the HMM solutions are discussed and analyzed.

Contents

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1. Introduction and the main results

1.1. General methodology. Consider the classical elliptic problem

$$\begin{cases} -\operatorname{div}\Big(a^{\varepsilon}(x)\nabla u^{\varepsilon}(x)\Big) = f(x), & x \in D \subset \mathbb{R}^d, \\ u^{\varepsilon}(x) = 0, & x \in \partial D. \end{cases} \tag{1.1}$$

Here ε is a small parameter that signifies explicitly the multiscale nature of the coefficients $a^{\varepsilon}(x)$. There already exist several classical multiscale methodologies for the numerical solution of this elliptic problem, the most well-known among which is the multigrid technique [9]. These classical multiscale methods are designed to resolve the details of fine scale problem (1.1) and are applicable for general problems, i.e., no special assumptions are required for the coefficient $a^{\varepsilon}(x)$. In contrast modern multiscale methods are designed specifically for recovering partial information about u^{ε} at a sublinear cost, i.e., the total cost grows sublinearly with the cost of solving the fine scale problem [19]. This is only possible by exploring the special features that $a^{\varepsilon}(x)$ might have, such as scale separation. The simplest example is when

$$a^{\varepsilon}(x) = a\Big(x, \frac{x}{\varepsilon}\Big), \tag{1.2}$$

where $a(x,y)$ can either be periodic in y, in which case we assume the period to be $I = [-\frac{1}{2}, \frac{1}{2}]^d$, or random but stationary under shifts in y, for each fixed $x \in D$. In both cases, it has been shown that

$$\|u^{\varepsilon}(x) - U(x)\|_{L^2(D)} \to 0, \tag{1.3}$$

where $U(x)$ is the solution of a homogenized equation:

$$\begin{cases} -\operatorname{div}\Big(\mathcal{A}(x)\nabla U(x)\Big) = f(x), & x \in D, \\ U(x) = 0, & x \in \partial D. \end{cases} \tag{1.4}$$

The homogenized coefficient $\mathcal{A}(x)$ can be obtained from the solutions of the so called cell problem. In general, there are no explicit formulas for $\mathcal{A}(x)$, except in one dimension.

Several numerical methods have been developed to deal specifically with the case when $a(x,y)$ is periodic in y. [3, 8, 14] propose to solve the homogenized equations as well as the equations for the cell problem. Schwab et al. [26, 35] use multiscale test functions of the form $\varphi(x, x/\varepsilon)$ where $\varphi(x,y)$ is periodic in y to extract the leading order behavior of $u^{\varepsilon}(x)$, extending an idea that was used analytically in the work of [2, 16, 30, 40] for the homogenization problems. These methods have the feature that their cost is independent of ε, hence sublinear as $\varepsilon \to 0$, but so far they are restricted to the periodic homogenization problem.

1.2. **Heterogeneous multiscale methods.** HMM [17, 18, 19] is a general methodology for designing sublinear algorithms by exploiting scale separation and other special features of the problem. It consists of two components: selection of a macroscopic solver, and estimating the missing macroscale data by solving locally the fine scale problem.

For (1.1) the macroscopic solver can be chosen as a conventional $\mathcal{P}_k$ finite element method on a triangulation of element size H which should resolve the macroscale features of $a^{\varepsilon}(x)$. The missing data is the effective stiffness matrix at this scale. This is obtained as follows. Assuming that the effective coefficient at this scale is $\mathcal{A}_H(x)$, we could evaluate the quadratic form

$$A_H(V,V) = \int_D \nabla V(x) \cdot \mathcal{A}_H(x) \nabla V(x)\, dx \tag{1.5}$$

by numerical quadrature if we know $\mathcal{A}_H(x)$ explicitly: For any $V \in X_H$, the finite element space,

$$A_H(V,V) \simeq \sum_{K \in \mathcal{T}_H} |K| \sum_{x_\ell \in K} \omega_\ell \Big(\nabla V \cdot \mathcal{A}_H \nabla V \Big)(x_\ell), \tag{1.6}$$

where x_ℓ and ω_ℓ are the quadrature points and weights in K. In the absence of explicit knowledge of $\mathcal{A}_H(x)$, we approximate $\Big(\nabla V \cdot \mathcal{A}_H \nabla V \Big)(x_\ell)$ by solving the problem:

$$\begin{cases} -\operatorname{div}\Big(a^{\varepsilon}(x) \nabla v_\ell^{\varepsilon}(x) \Big) = 0, & \text{in } I_\delta(x_\ell), \\ v_\ell^{\varepsilon}(x) = V_\ell(x), & \text{on } \partial I_\delta(x_\ell), \end{cases} \tag{1.7}$$

where $I_\delta(x_\ell)$ is a cube of size δ centered at x_ℓ, and V_ℓ is the linear approximation of V at x_ℓ. We then let

$$\Big(\nabla V \cdot \mathcal{A}_H \nabla V \Big)(x_\ell) = \frac{1}{\delta^d} \int_{I_\delta(x_\ell)} \nabla v_\ell^{\varepsilon}(x) \cdot a^{\varepsilon}(x) \nabla v_\ell^{\varepsilon}(x)\, dx. \tag{1.8}$$

In order to reduce the effect of the imposed boundary condition on $\partial I_\delta(x_\ell)$, we may replace (1.8) with

$$\Big(\nabla V \cdot \mathcal{A}_H \nabla V \Big)(x_\ell) = \frac{1}{(\delta')^d} \int_{I_{\delta'}(x_\ell)} \nabla v_\ell^{\varepsilon}(x) \cdot a^{\varepsilon}(x) \nabla v_\ell^{\varepsilon}(x)\, dx, \tag{1.8'}$$

where $\delta' < \delta$. For example, we may choose $\delta' = \delta/2$. In (1.7), we used Dirichlet boundary condition. Other boundary conditions are possible. In the case when $a^{\varepsilon}(x) = a(x, x/\varepsilon)$ and $a(x,y)$ is periodic in y, one can take $I_\delta(x_\ell)$ to be $x_\ell + \varepsilon I$, i.e., $\delta = \varepsilon$ and use the boundary condition that $v_\ell^{\varepsilon}(x) - V_\ell(x)$ is periodic on I_δ.

So far the algorithm is completely general. The savings compared with solving the full fine scale problem comes from the fact that we can choose $I_\delta(x_\ell)$ to be smaller than K. The size of $I_\delta(x_\ell)$ is determined by many factors, including the accuracy and cost requirement, the degree of scale separation, and the microstructure in $a^\varepsilon(x)$. One purpose for the error estimates that we present below is to give guidelines on how to select $I_\delta(x_\ell)$. In the special case when $a^\varepsilon(x) = a(x, x/\varepsilon)$ and $a(x, y)$ is periodic in y, we can simply choose $I_\delta(x_\ell)$ to be $x_\ell + \varepsilon I$, i.e., $\delta = \varepsilon$. If $a(x, y)$ is random, then δ should be a few times larger than the local correlation length. In the former case, the total cost is independent of ε. In the latter case, the total cost depends only weakly on ε (see [27]).

The final problem is to solve

$$\min_{V \in X_H} \frac{1}{2} A_H(V, V) - (f, V). \tag{1.9}$$

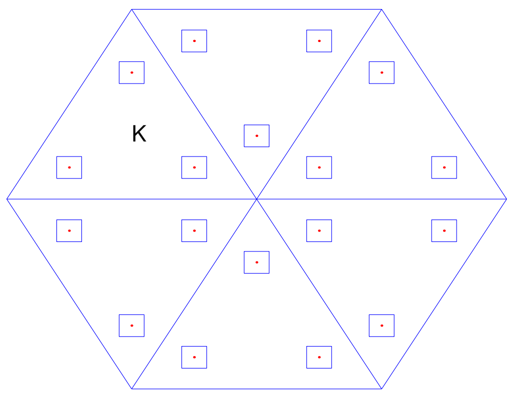

FIGURE 1. Illustration of HMM for solving (1.1). The dots are the quadrature points. The little squares are the microcell $I_\delta(x_\ell)$.

To summarize, HMM has the following features:

(1) It gives a framework that allows us to maximally take advantage of the scale separation. The coefficient a^ε need not to be the form $a(x, x/\varepsilon)$. For periodic homogenization problems, the cost of HMM is comparable to the special techniques discussed in [3, 8, 14, 26, 31]. However HMM is also applicable for random problems and for problems whose coefficients $a^\varepsilon(x)$ does not has the structure of $a(x, x/\varepsilon)$.
(2) For problems without scale separation, HMM becomes a fine scale solver by choosing H that resolves the fine scales and letting $\mathcal{A}_H(x) = a^\varepsilon(x)$.

Some related ideas exist in the literature. Durlofsky [15] proposed an up-scaling method, which solves directly some local problems to obtain the effective coefficients [37]. Oden et al [31] proposed a method that aims at recovering the oscillation in ∇u^ε locally by solving a local problem with some given approximation to the macroscopic state U_0 as the boundray condition. Their method is sometimes used in HMM to recover the microstructural information.

The numerical performance of HMM including comparison with other methods is discussed in [27].

The analysis in this paper is restricted to the case of periodic and random homogenization problems. This is mainly due to the fact that the analytical behavior of $u^\varepsilon(x)$ is best understood in these cases. The strategy of the analysis is however not restricted to these cases, and can be extended to other cases as long as some analytical results on $u^\varepsilon(x)$ are available.

We will always assume that $a^\varepsilon(x)$ is smooth, symmetric and uniformly elliptic:

$$\lambda I \le a^\varepsilon(x) \le \Lambda I$$

for some $\lambda, \Lambda > 0$. We will use summation conventional rule. Standard notations for Sobolev spaces are used (see [1]). For the quadrature formula, we will assume the following accuracy conditions: kth-order numerical quadrature scheme [11]:

$$\tag{1.10} ⨍_K p(x) = \sum_{\ell=1}^{L} \omega_\ell p(x_\ell) \quad \forall p(x) \in \mathcal{P}_{2k-2}.$$

Here $\omega_\ell > 0$, for $\ell = 1, \cdots, L$.

1.3. **Main results.**

Our main results for the linear problem are as follows.

Theorem 1.1. *Denote by U_0 and U_{HMM} the solutions of* (1.4) *and the HMM solution, respectively. Let*

$$e(\mathrm{HMM}) = \max_{\substack{x_\ell \in K \\ K \in \mathcal{T}_H}} \|\mathcal{A}(x_\ell) - \mathcal{A}_H(x_\ell)\|,$$

and $\|\cdot\|$ is the Euclidean norm. If (1.10) *holds, then there exists a constant C independent of ε, δ and H, such that*

$$\tag{1.11} \|U_0 - U_{\mathrm{HMM}}\|_1 \le C\Big(H^k + e(\mathrm{HMM})\Big).$$

If (1.10) *holds for $k \ge 2$ and the quadrature scheme* (1.10) *is exact for $\mathcal{P}_1$ if $k = 1$, then*

$$\tag{1.12} \|U_0 - U_{\mathrm{HMM}}\|_0 \le C\Big(H^{k+1} + e(\mathrm{HMM})\Big).$$

If there exits a constant C_0 *such that* $e(\mathrm{HMM}) < C_0$*, then there exists a constant* H_0 *such that for all* $H \le H_0$*, we have*

$$\|U_0 - U_{\mathrm{HMM}}\|_{1,\infty} \le C\Big(H^k + e(\mathrm{HMM})\Big)|\ln H|. \tag{1.13}$$

At this stage, no assumption on the form of $a^\varepsilon(x)$ is necessary. U_0 can be the solution of an arbitrary macroscopic equation with the same right-hand side. Of course for U_{HMM} to converge to U_0, i.e., $e(\mathrm{HMM}) \to 0$, U_0 must be chosen on the solution of an effective homogenized equation, which we now assume exist. To obtain quantitative estimates on $e(\mathrm{HMM})$, we must make more specific assumptions on the form of $a^\varepsilon(x)$.

In the special case of homogenization problems, $e(\mathrm{HMM})$ can be estimated as follows.

Theorem 1.2. *For the periodic homogenization problem, we have*

$$e(\mathrm{HMM}) \le \begin{cases} C\delta & \text{if } \delta \text{ is an integer multiple of } \varepsilon, \\ C\Big(\dfrac{\varepsilon}{\delta} + \delta\Big) & \text{if } \delta \text{ is not an integer multiple of } \varepsilon. \end{cases}$$

In the first case, we replace the boundary condition in (1.7) by a periodic boundary condition: $v_\ell^\varepsilon - V_\ell$ is periodic with period I_δ.

Theorem 1.3. *For the random homogenization problem, assuming that* $(\mathbf{A})$ *in § 3.2 holds (see* [39]*), then we have*

$$\mathbb{E}e(\mathrm{HMM}) \le \begin{cases} C\Big(\dfrac{\varepsilon}{\delta}\Big)^{\kappa}, & d = 3 \\ \textit{remains open}, & d = 2 \\ C\Big(\dfrac{\varepsilon}{\delta}\Big)^{1/2}, & d = 1 \end{cases}$$

where

$$\kappa = \frac{6 - 12\gamma}{25 - 8\gamma}$$

for any $0 < \gamma < 1/2$*.*

The probabilistic set-up will be given in §3.2. To prove this result, we assume that (1.8') is used with $\delta' = \delta/2$.

1.4. **Recovering the miscrostructural information.** In many cases, the microstructure information in $u^\varepsilon(x)$ is very important. U_{HMM} by itself does not give this information. However, this information can be recovered using a simple post-processing technique. For the general case, having U_{HMM}, one can obtain locally the microstructural information using an idea in [31]. Assume that we are interested in recovering u^ε and ∇u^ε only in the subdomain $\Omega \subset D$. Consider the following auxiliary problem:

$$\begin{cases} -\operatorname{div}\Big(a^\varepsilon(x)\nabla\overline{u}^{\,\varepsilon}(x)\Big) = f(x), & x\in\Omega_\eta,\\ \overline{u}^{\,\varepsilon}(x) = U_{\mathrm{HMM}}(x), & x\in\partial\Omega_\eta, \end{cases} \tag{1.14}$$

where Ω_η satisfies $\Omega\subset\Omega_\eta\subset D$ and $\operatorname{dist}(\partial\Omega,\partial\Omega_\eta)=\eta$. We then have

Theorem 1.4. *We have*

$$\Big(⨍_\Omega |\nabla(u^\varepsilon-\overline{u}^{\,\varepsilon})|^2\,dx\Big)^{1/2} \le \frac{C}{\eta}\Big(\|U_0-U_{\mathrm{HMM}}\|_{L^\infty(\Omega_\eta)}+\|u^\varepsilon-U_0\|_{L^\infty(\Omega_\eta)}\Big). \tag{1.15}$$

For the random problem, the last term was estimated in [39].

For the periodic homogenization problem, in particular, consider the case when $k=1$. Let $\overline{u}^{\,\varepsilon}$ be defined as follows:

(1) $\Big(\overline{u}^{\,\varepsilon}-U_{\mathrm{HMM}}\Big)|_K$ is periodic with period εI.

(2) $\overline{u}^{\,\varepsilon}(x)=v^\varepsilon(x)$ on $I_{\delta/2}$, where $v^\varepsilon(x)$ is the solution of (1.7) with $V_\ell(x)=U_{\mathrm{HMM}}(x)$.

If the period is known a-priori, the period extension in the second step can be made over I_ε. For this case, we can prove

Theorem 1.5. *Let $\overline{u}^{\,\varepsilon}$ be defined as above, then*

$$\Big(\sum_{K\in\mathcal{T}_H}\|u^\varepsilon-\overline{u}^{\,\varepsilon}\|^2_{1,K}\Big)^{1/2}\le C\Big(\sqrt{\varepsilon}+H\Big). \tag{1.16}$$

For the general case, we have

$$\Big(\sum_{K\in\mathcal{T}_H}\|u^\varepsilon-\overline{u}^{\,\varepsilon}\|^2_{1,K}\Big)^{1/2}\le C\Big(\sqrt{\varepsilon}+\frac{\varepsilon}{\delta}+\delta+H\Big).$$

This can be proved by combining the proofs of Lemma 3.2 and Theorem 1.5. We leave the details to the interested readers.

Similar results with some modification hold for nonlinear problems. The details are given in §5.

2. Generalities

We prove Theorem 1.1. We will let $U_H=U_{\mathrm{HMM}}$ for short.

Obviously, U_H is the numerical solution associated with the quadratic form A_H, and U_0 is the exact solution associated with the quadratic form A, which is defined for any $V\in H_0^1(D)$ as

$$A(V,V)=\int_D \nabla V(x)\cdot\mathcal{A}(x)\nabla V(x)\,dx.$$

To estimate $U_0 - U_H$, we view $A_H(V,V)$ as an approximation to $A(V,V)$, and use *Strang's first lemma* [10]. We begin with the following observation, which gives stability.

On $I_\delta(x_\ell)$, notice that $v_\ell^\varepsilon = V_\ell$ on the edges of $I_\delta(x_\ell)$, using the fact that ∇V_ℓ is a constant in $I_\delta(x_\ell)$, an integration by parts leads to

$$\int\limits_{I_\delta(x_\ell)} \nabla(v_\ell^\varepsilon - V_\ell)(x)\nabla V_\ell(x)\,dx = 0,$$

which implies

$$\int\limits_{I_\delta(x_\ell)} |\nabla v_\ell^\varepsilon(x)|^2\,dx = \int\limits_{I_\delta(x_\ell)} |\nabla V_\ell(x)|^2\,dx + \int\limits_{I_\delta(x_\ell)} |\nabla(v_\ell^\varepsilon - V_\ell)(x)|^2\,dx. \tag{2.1}$$

Multiplying (1.7) by $v_\ell^\varepsilon(x) - V_\ell(x)$ and integrating by parts we obtain

$$\begin{aligned}\int\limits_{I_\delta(x_\ell)} \nabla v_\ell^\varepsilon(x)\cdot a^\varepsilon(x)\nabla v_\ell^\varepsilon(x)\,dx \;+ &\int\limits_{I_\delta(x_\ell)} \nabla(v_\ell^\varepsilon - V_\ell)(x)\cdot a^\varepsilon(x)\nabla(v_\ell^\varepsilon - V_\ell)(x)\,dx\\ = &\int\limits_{I_\delta(x_\ell)} \nabla V_\ell(x)\cdot a^\varepsilon(x)\nabla V_\ell(x)\,dx. \end{aligned}\tag{2.2}$$

Using (2.1) and (1.10), for any $V \in X_H$, we have

$$\begin{aligned} A_H(V,V) &\ge \lambda \sum_{K\in\mathcal{T}_H} |K| \sum_{x_\ell\in K} \omega_\ell ⨍_{I_\delta(x_\ell)} |\nabla V_\ell(x)|^2\,dx\\ &= \lambda \sum_{K\in\mathcal{T}_H} |K| \sum_{x_\ell\in K} \omega_\ell |\nabla V(x_\ell)|^2\\ &= \lambda \|\nabla V\|_0^2. \end{aligned}\tag{2.3}$$

Similarly, for any $V, W \in X_H$, in light of (2.2) and (1.10), we obtain

$$\begin{aligned} |A_H(V,W)| &\le \Lambda \sum_{K\in\mathcal{T}_H}\sum_{x_\ell\in K} \frac{|K|}{|I_\delta(x_\ell)|}\omega_\ell \|\nabla v_\ell^\varepsilon\|_{0,I_\delta(x_\ell)}\|\nabla w_\ell^\varepsilon\|_{0,I_\delta(x_\ell)}\\ &\le \frac{\Lambda^2}{\lambda}\sum_{K\in\mathcal{T}_H}\sum_{x_\ell\in K} \frac{|K|}{|I_\delta(x_\ell)|}\omega_\ell \|\nabla V_\ell\|_{0,I_\delta(x_\ell)}\|\nabla W_\ell\|_{0,I_\delta(x_\ell)}\\ &= \frac{\Lambda^2}{\lambda}\sum_{K\in\mathcal{T}_H}\int\limits_K |\nabla V(x)\nabla W(x)|\,dx\\ &\le \frac{\Lambda^2}{\lambda}\|\nabla V\|_0 \|\nabla W\|_0. \end{aligned}\tag{2.4}$$

The existence and the uniqueness of the solutions to (1.9) follows from (2.3) and (2.4) via Lax-Milgram lemma.

Remark 2.1. *In view of* (2.3) *and* (2.4)*, we have*

$$\lambda I \le \mathcal{A}_H(x) \le \frac{\Lambda^2}{\lambda} I.$$

This is consistent with the bounds on the effective tensor for general elliptic problem (1.1) *in the sense of $H-$convergence* [29].

To streamline the proof of Theorem 1.1, we introduce the following auxiliary bilinear form $\hat{A}_H$.

$$\hat{A}_H(V,W) = \sum_{K\in\mathcal{T}_H} \hat{A}_K(V,W) \quad \text{with} \quad \hat{A}_K(V,W) = |K| \sum_{x_\ell\in K} \omega_\ell (\nabla W \cdot \mathcal{A}\nabla V)(x_\ell).$$

Classical results on numerical integration [11, Theorem 7] yields that for any $V,W \in X_H$,

$$\left| \hat{A}_K(V,W) - \int_K \nabla W \cdot \mathcal{A}\nabla V\,dx \right| \le CH^m \|V\|_{m,K} \|\nabla W\|_{0,K} \quad 1\le m\le k. \tag{2.5}$$

Moreover, for any $V,W \in X_H$, if $\|V\|_{k+1}$ and $\|W\|_2$ are bounded, we have [11, Theorem 8],

$$|\hat{A}_H(V,W) - A(V,W)| \le CH^{k+1} \|V\|_{k+1} \|W\|_2. \tag{2.6}$$

Proof for Theorem 1.1. Invoking the *first Strang lemma* [10, Theorem 4.1.1], we have

$$\|U_0 - U_H\|_1 \le C \inf_{V\in X_H} \Big(\|U_0 - V\|_1 + \sup_{W\in X_H} \frac{|A_H(V,W) - A(V,W)|}{\|W\|_1} \Big).$$

Let Π be the standard Lagrange interpolate operator [10], $V = \Pi U_0$, we have

$$\inf_{V\in X_H} \|U_0 - V\|_1 \le \|U_0 - \Pi U_0\|_1 \le CH^k. \tag{2.7}$$

It remains to estimate $|A_H(V,W) - A(V,W)|$ for $V = \Pi U_0$ and $W \in X_H$. In view of (2.5), we get

$$\begin{aligned} |A_H(V,W) - A(V,W)| &\le |A_H(V,W) - \hat{A}_H(V,W)| + |\hat{A}_H(V,W) - A(V,W)| \\ &\le \Big(e(\text{HMM}) \|\nabla V\|_0 + CH^k \|V\|_k \Big) \|\nabla W\|_0. \end{aligned} \tag{2.8}$$

This gives (1.11)

To get the L^2 estimate, we use the Aubin-Nitsche argument [10]. To this end, consider the following auxiliary problem: Find $w \in H_0^1(D)$ such that

$$A(v,w) = (U_0 - U_H, v) \quad \forall v \in H_0^1(D). \tag{2.9}$$

Standard regularity result reads [23]

$$\|w\|_2 \le C \|U_0 - U_H\|_0. \tag{2.10}$$

Putting $v = U_0 - U_H$ into the right-hand side of (2.9), we obtain

$$\begin{aligned} \|U_0 - U_H\|_0^2 &= A(U_0 - U_H, w - \mathit{\Pi} w) + \Big(A_H(U_H, \mathit{\Pi} w) - A(U_H, \mathit{\Pi} w)\Big) \\ &= A(U_0 - U_H, w - \mathit{\Pi} w) \\ &\quad + \Big(A_H(U_H - \mathit{\Pi} U_0, \mathit{\Pi} w) - A(U_H - \mathit{\Pi} U_0, \mathit{\Pi} w)\Big) \\ &\quad + \Big(A_H(\mathit{\Pi} U_0, \mathit{\Pi} w) - A(\mathit{\Pi} U_0, \mathit{\Pi} w)\Big). \end{aligned} \tag{2.11}$$

Using (2.8) with $k = 1$, we bound the first two terms at the right-hand side of the above identities as

$$|A(U_0 - U_H, w - \mathit{\Pi} w)| \le C\|U_0 - U_H\|_1 \|w - \mathit{\Pi} w\|_1 \le CH\|U_0 - U_H\|_1 \|w\|_2,$$

and

$$|A_H(U_H - \mathit{\Pi} U_0, \mathit{\Pi} w) - A(U_H - \mathit{\Pi} U_0, \mathit{\Pi} w)| \le \Big(e(\mathrm{HMM}) + CH\Big)\|U_0 - U_H\|_1 \|\mathit{\Pi} w\|_1.$$

The last term at the right-hand side of (2.11) may be decomposed into

$$\begin{aligned} A_H(\mathit{\Pi} U_0, \mathit{\Pi} w) - A(\mathit{\Pi} U_0, \mathit{\Pi} w) &= \Big(A_H(\mathit{\Pi} U_0, \mathit{\Pi} w) - \hat{A}_H(\mathit{\Pi} U_0, \mathit{\Pi} w)\Big) \\ &\quad + \Big(\hat{A}_H(\mathit{\Pi} U_0, \mathit{\Pi} w) - A(\mathit{\Pi} U_0, \mathit{\Pi} w)\Big). \end{aligned}$$

In view of (2.6), it follows that

$$|\hat{A}_H(\mathit{\Pi} U_0, \mathit{\Pi} w) - A(\mathit{\Pi} U_0, \mathit{\Pi} w)| \le CH^{k+1}\|U_0\|_{k+1}\|w\|_2.$$

By definition, we get

$$|A_H(\mathit{\Pi} U_0, \mathit{\Pi} w) - \hat{A}_H(\mathit{\Pi} U_0, \mathit{\Pi} w)| \le e(\mathrm{HMM})\|\nabla \mathit{\Pi} U_0\|_0 \|w\|_2.$$

Combining the above estimates and using (2.10) leads to (4.1).

It remains to prove (1.13). For a given $z \in D$, we define the Green's function $G^z \in H_0^1(D)$ and the discrete Green's function $G_H^z \in X_H$ as

$$\begin{aligned} A(G^z, V) &= \partial V(z) \quad \forall V \in H_0^1(D), \\ A(G_H^z, V) &= \partial V(z) \quad \forall V \in X_H. \end{aligned} \tag{2.12}$$

Here $\partial V(z)$ is a generic notation for $\partial_{z_j} V(z)$. It is well-known [33]

$$\|G^z - G_H^z\|_{1,1} \le C \quad \text{and} \quad \|G_H^z\|_{1,1} \le C|\ln H|. \tag{2.13}$$

Using the definition of G^z and G^z_H, a simple manipulation gives

$$\begin{aligned}\partial(U_0-U_H)(z) &= A(G^z,U_0-\mathit{\Pi}U_0)+A(G^z,\mathit{\Pi}U_0-U_H)\\ &= A(G^z-G^z_H,U_0-\mathit{\Pi}U_0)+A(G^z_H,U_0-U_H)\\ &= A(G^z-G^z_H,U_0-\mathit{\Pi}U_0)+A_H(U_H,G^z_H)-A(U_H,G^z_H)\\ &= A(G^z-G^z_H,U_0-\mathit{\Pi}U_0)+\Big(A_H(\mathit{\Pi}U_0,G^z_H)-A(\mathit{\Pi}U_0,G^z_H)\Big)\\ &\quad+\Big(A_H(U_H-\mathit{\Pi}U_0,G^z_H)-A(U_H-\mathit{\Pi}U_0,G^z_H)\Big).\end{aligned}$$

Using (2.13), we obtain

$$\begin{aligned}\|U_0-U_H\|_{1,\infty}\le C\|U_0-\mathit{\Pi}U_0\|_{1,\infty}&+|A(\mathit{\Pi}U_0,G^z_H)-A_H(\mathit{\Pi}U_0,G^z_H)|\\ &+|A(U_H-\mathit{\Pi}U_0,G^z_H)-A_H(U_H-\mathit{\Pi}U_0,G^z_H)|.\end{aligned}$$

Using (2.8) once again, we get

$$\begin{aligned}|A(\mathit{\Pi}U_0,G^z_H)-A_H(\mathit{\Pi}U_0,G^z_H)| &\le \Big(e(\mathrm{HMM})+CH^k\Big)\sum_{K\in\mathcal{T}_H}\|\mathit{\Pi}U_0\|_{k,K}\|\nabla G^z_H\|_{0,K}\\ &\le C\Big(e(\mathrm{HMM})+H^k\Big)\sum_{K\in\mathcal{T}_H}\|\mathit{\Pi}U_0\|_{k,\infty,K}\|\nabla G^z_H\|_{0,1,K}\\ &\le C\Big(e(\mathrm{HMM})+H^k\Big)|\ln H|\,\|U_0\|_{k+1,\infty},\end{aligned}$$

where we have used the *inverse inequality* [10] on each element.

Similarly, we have

$$\begin{aligned}&|A(U_H-\mathit{\Pi}U_0,G^z_H)-A_H(U_H-\mathit{\Pi}U_0,G^z_H)|\\ &\le\Big(e(\mathrm{HMM})+CH\Big)\sum_{K\in\mathcal{T}_H}\|U_H-\mathit{\Pi}\|_{1,K}\|\nabla G^z_H\|_{0,K}\\ &\le C\Big(e(\mathrm{HMM})+H\Big)\sum_{K\in\mathcal{T}_H}\|U_H-\mathit{\Pi}U_0\|_{1,\infty,K}\|\nabla G^z_H\|_{0,1,K}\\ &\le C\Big(e(\mathrm{HMM})+H\Big)|\ln H|\|U_0-U_H\|_{1,\infty}\\ &\quad+C\Big(e(\mathrm{HMM})+H)\Big)|\ln H|\,H^k\|U_0\|_{k+1,\infty}.\end{aligned}$$

A combination of the above three yields

$$\begin{aligned}\|U_0-U_H\|_{1,\infty}\le CH^k&+C\Big(e(\mathrm{HMM})+H\Big)|\ln H|\,\|U_0-U_H\|_{1,\infty}\\ &+C\Big(e(\mathrm{HMM})+H^k\Big)|\ln H|\,\|U_0\|_{k+1,\infty}.\end{aligned}$$

Let $C_0=\frac{1}{2C}$, then if $e(\mathrm{HMM})<C_0$, there exits a constant H_0 such that for all $H\le H_0$,

$$C\Big(e(\mathrm{HMM})+H\Big)|\ln H|\le 2Ce(\mathrm{HMM})<1,$$

we thus obtain (1.13) and completes the proof. □

Combining the proof for L^2 and $W^{1,\infty}$ estimates in the above lemma, using the Green's function defined in [36], we obtain

Remark 2.2. *Under the same condition for $W^{1,\infty}$ estimate in Theorem 1.1, we have*

$$\|U_0 - U_H\|_{L^\infty} \le C\Big(e(\mathrm{HMM}) + H^{k+1}\Big)|\ln H|^2.$$

3. Estimating e(HMM)

In this section, we estimate e(HMM) for various cases.

3.1. Problems with locally periodic coefficients.

We assume that $a^\varepsilon(x) = a(x, x/\varepsilon)$, a^ε is smooth in x and periodic in y with period I. Define $\kappa = \lfloor \delta/\varepsilon \rfloor$, and we introduce $\hat{V}_\ell$ as

$$\hat{V}_\ell(x) = V_\ell(x) + \varepsilon\chi^k\Big(x_\ell, \frac{x}{\varepsilon}\Big)\frac{\partial V_\ell}{\partial x_k}(x), \tag{3.1}$$

where $\{\chi^j\}_{j=1}^d$ is defined as: For $j = 1, \cdots, d$, χ^j is periodic in y with period I and satisfy

$$-\frac{\partial}{\partial y_i}\Big(a_{ik}\frac{\partial \chi^j}{\partial y_k}\Big)(x,y) = \frac{\partial}{\partial y_i}a_{ij}(x,y) \quad \text{in } I, \qquad \int_I \chi^j(x,y)\,dy = 0. \tag{3.2}$$

A direct calculation leads to

$$\begin{aligned}\nabla W(x_\ell)\cdot\mathcal{A}(x_\ell)\nabla V(x_\ell) &= \nabla W_\ell(x)\cdot\mathcal{A}(x_\ell)\nabla V_\ell(x)\\ &= \fint_{I_{\kappa\varepsilon}(x_\ell)} \nabla W_\ell\cdot a\Big(x_\ell, \frac{x}{\varepsilon}\Big)\nabla \hat{V}_\ell\,dx.\end{aligned} \tag{3.3}$$

We first establish some estimates on the solution of the cell problem (1.7). $I_\delta(x_\ell)$ will be replaced by I_δ if there is no risk of confusion.

Lemma 3.1. *There exists a constant C independent of ε and δ such that for each ℓ,*

$$\|\nabla(v_\ell^\varepsilon - \hat{V}_\ell)\|_{0,I_\delta} \le C\Big(\Big(\frac{\varepsilon}{\delta}\Big)^{1/2} + \delta\Big)\|\nabla V_\ell\|_{0,I_\delta}, \tag{3.4}$$

and

$$\|\nabla v_\ell^\varepsilon\|_{0,I_\delta\setminus I_{\kappa\varepsilon}} \le C\Big(\Big(\frac{\varepsilon}{\delta}\Big)^{1/2} + \delta\Big)\|\nabla V_\ell\|_{0,I_\delta}. \tag{3.5}$$

Proof. Denote by $\hat{v}_\ell^\varepsilon$ the solution of (1.7) with the coefficients $a^\varepsilon(x)$ replaced by $a(x_\ell, x/\varepsilon)$. Obviously, we still have

$$\|\nabla V_\ell\|_{0,I_\delta} \le \|\nabla \hat{v}_\ell^\varepsilon\|_{0,I_\delta} \le \sqrt{\frac{\Lambda}{\lambda}}\|\nabla V_\ell\|_{0,I_\delta}. \tag{3.6}$$

In view of the definitions of v_ℓ^ε and $\hat{v}_\ell^\varepsilon$, we get

$$\begin{aligned}\lambda\|\nabla(v_\ell^\varepsilon-\hat{v}_\ell^\varepsilon)\|_{0,I_\delta}^2 &\le \int_{I_\delta}\nabla(v_\ell^\varepsilon-\hat{v}_\ell^\varepsilon)\cdot a\Big(x,\frac{x}{\varepsilon}\Big)\nabla(v_\ell^\varepsilon-\hat{v}_\ell^\varepsilon)\,dx\\ &=\int_{I_\delta}\nabla(v_\ell^\varepsilon-\hat{v}_\ell^\varepsilon)\cdot\Big[a\Big(x_\ell,\frac{x}{\varepsilon}\Big)-a\Big(x,\frac{x}{\varepsilon}\Big)\Big]\nabla\hat{v}_\ell^\varepsilon\,dx\\ &\le C\delta\|\nabla\hat{v}_\ell^\varepsilon\|_{0,I_\delta}\|\nabla(v_\ell^\varepsilon-\hat{v}_\ell^\varepsilon)\|_{0,I_\delta},\end{aligned}$$

which together with (3.6) gives

$$\|\nabla(v_\ell^\varepsilon-\hat{v}_\ell^\varepsilon)\|_{0,I_\delta}\le C\delta\|\nabla V_\ell\|_{0,I_\delta}. \tag{3.7}$$

Define $\theta_\ell^\varepsilon=\hat{v}_\ell^\varepsilon-\hat{V}_\ell$, which obviously satisfies

$$\begin{cases}-\operatorname{div}\Big(a\Big(x_\ell,\dfrac{x}{\varepsilon}\Big)\nabla\theta_\ell^\varepsilon(x)\Big)=0, & \text{in } I_\delta(x_\ell),\\ \qquad\qquad\theta_\ell^\varepsilon(x)=-\varepsilon\chi^k\Big(x_\ell,\dfrac{x}{\varepsilon}\Big)\dfrac{\partial V_\ell}{\partial x_k}(x), & \text{on } \partial I_\delta(x_\ell).\end{cases} \tag{3.8}$$

It is seen that θ_ℓ^ε is just the "boundary layer correction" for the cell problem (1.7) [6]. Rescaling the problem by introducing $x'=x/\delta$, we get a problem over I with $\varepsilon'=\varepsilon/\delta$. Invoking a well-known result for this rescaled homogenization problem [41, (1.51) in §1.4], we obtain

$$\|\nabla\theta_\ell^\varepsilon\|_{0,I_\delta}\le C\Big(\frac{\varepsilon}{\delta}\Big)^{1/2}\|\nabla V_\ell\|_{0,I_\delta}, \tag{3.9}$$

which together with (5.24) leads to (3.4).

A straightforward calculation gives

$$\|\nabla\hat{V}_\ell\|_{0,I_\delta\backslash I_{\kappa\varepsilon}}\le C\Big(\frac{\varepsilon}{\delta}\Big)^{1/2}\|\nabla V_\ell\|_{0,I_\delta}. \tag{3.10}$$

Using (3.4) we have

$$\begin{aligned}\|\nabla v_\ell^\varepsilon\|_{0,I_\delta\backslash I_{\kappa\varepsilon}} &\le \|\nabla\hat{V}_\ell\|_{0,I_\delta\backslash I_{\kappa\varepsilon}}+\|\nabla(v_\ell^\varepsilon-\hat{V}_\ell)\|_{0,I_\delta\backslash I_{\kappa\varepsilon}}\\ &\le \|\nabla\hat{V}_\ell\|_{0,I_\delta\backslash I_{\kappa\varepsilon}}+\|\nabla(v_\ell^\varepsilon-\hat{V}_\ell)\|_{0,I_\delta}\\ &\le C\Big(\Big(\frac{\varepsilon}{\delta}\Big)^{1/2}+\delta\Big)\|\nabla V_\ell\|_{0,I_\delta}.\end{aligned}$$

This gives (3.5). □

Theorem 3.2.

$$e(\mathrm{HMM})\le C\Big(\frac{\varepsilon}{\delta}+\delta\Big). \tag{3.11}$$

Proof. In view of (3.3), we have

$$\begin{aligned}\nabla W(x_\ell)\cdot(\mathcal{A}_H-\mathcal{A})(x_\ell)\nabla V(x_\ell)=&\fint_{I_\delta}\nabla w_\ell^\varepsilon\cdot a\Big(x,\frac{x}{\varepsilon}\Big)\nabla v_\ell^\varepsilon\,dx\\&-\fint_{I_{\kappa\varepsilon}}\nabla W_\ell\cdot a\Big(x_\ell,\frac{x}{\varepsilon}\Big)\nabla\hat{V}_\ell\,dx,\end{aligned}$$

with

$$\fint_{I_\delta}\nabla w_\ell^\varepsilon\cdot a\Big(x,\frac{x}{\varepsilon}\Big)\nabla v_\ell^\varepsilon\,dx=I_1+\cdots+I_4,$$

where

$$\begin{aligned}I_1&=\fint_{I_\delta}\nabla w_\ell^\varepsilon\cdot\Big[a\Big(x,\frac{x}{\varepsilon}\Big)-a\Big(x_\ell,\frac{x}{\varepsilon}\Big)\Big]\nabla v_\ell^\varepsilon\,dx,\\I_2&=\fint_{I_\delta}\nabla w_\ell^\varepsilon\cdot a\Big(x_\ell,\frac{x}{\varepsilon}\Big)\nabla(v_\ell^\varepsilon-\hat{v}_\ell^\varepsilon)\,dx,\\I_3&=\fint_{I_\delta}\nabla w_\ell^\varepsilon\cdot a\Big(x_\ell,\frac{x}{\varepsilon}\Big)\nabla(\hat{v}_\ell^\varepsilon-\hat{V}_\ell)\,dx,\\I_4&=\fint_{I_\delta}\nabla w_\ell^\varepsilon\cdot a\Big(x_\ell,\frac{x}{\varepsilon}\Big)\nabla\hat{V}_\ell\,dx.\end{aligned}$$

Using (2.4), we have

$$|I_1|\le C\delta^{1-d}\|\nabla v_\ell^\varepsilon\|_{0,I_\delta}\|\nabla w_\ell^\varepsilon\|_{0,I_\delta}\le C\delta^{1-d}\|\nabla V_\ell\|_{0,I_\delta}\|\nabla W_\ell\|_{0,I_\delta}=C\delta|\nabla V_\ell|\,|\nabla W_\ell|.$$

Using (5.24) and (2.4), we bound I_2 as

$$|I_2|\le\delta^{-d}\|\nabla(v_\ell^\varepsilon-\hat{v}_\ell^\varepsilon)\|_{0,I_\delta}\|\nabla w_\ell^\varepsilon\|_{0,I_\delta}\le C\delta^{1-d}\|\nabla V_\ell\|_{0,I_\delta}\|\nabla W_\ell\|_{0,I_\delta}=C\delta|\nabla V_\ell|\,|\nabla W_\ell|.$$

In view of the symmetry of a^ε, I_3 can be rewritten as

$$I_3=\fint_{I_\delta}\nabla\theta_\ell^\varepsilon\cdot a\Big(x,\frac{x}{\varepsilon}\Big)\nabla w_\ell^\varepsilon\,dx+\fint_{I_\delta}\nabla\theta_\ell^\varepsilon\cdot\Big[a\Big(x_\ell,\frac{x}{\varepsilon}\Big)-a\Big(x,\frac{x}{\varepsilon}\Big)\Big]\nabla w_\ell^\varepsilon\,dx.$$

The first term at the right-hand side of the above identity can be further decomposed into

$$\begin{aligned}\fint_{I_\delta}\nabla\theta_\ell^\varepsilon\cdot a\Big(x,\frac{x}{\varepsilon}\Big)\nabla\theta_\ell^\varepsilon\,dx=&\fint_{I_\delta}\nabla\Big(\theta_\ell^\varepsilon+\varepsilon\chi^k\Big(x_\ell,\frac{x}{\varepsilon}\Big)\frac{\partial V_\ell}{\partial x_k}(1-\rho^\varepsilon)\Big)\cdot a\Big(x,\frac{x}{\varepsilon}\Big)\nabla w_\ell^\varepsilon\,dx\\&-\fint_{I_\delta}\nabla\Big(\varepsilon\chi^k\Big(x_\ell,\frac{x}{\varepsilon}\Big)\frac{\partial V_\ell}{\partial x_k}(1-\rho^\varepsilon)\Big)\cdot a\Big(x,\frac{x}{\varepsilon}\Big)\nabla w_\ell^\varepsilon\,dx.\end{aligned}$$

where $\rho^\varepsilon(x)\in\mathcal{C}_0^\infty(I_\delta),|\nabla\rho^\varepsilon|\le C/\varepsilon$, and

$$\rho^\varepsilon(x)=\begin{cases}1&\operatorname{dist}(x,\partial I_\delta)\ge2\varepsilon,\\0&\operatorname{dist}(x,\partial I_\delta)\le\varepsilon.\end{cases}\tag{3.12}$$

An integration by parts makes the first term on the right-hand side of I_3 vanish due to (1.7) and $\theta_\ell^\varepsilon + \varepsilon\chi^k\Big(x_\ell, \frac{x}{\varepsilon}\Big)\frac{\partial V_\ell}{\partial x_k}(1-\rho^\varepsilon) \in H_0^1(I_\delta)$. We expand I_3 into

$$
\begin{aligned}
I_3 = &-\fint_{I_\delta} a_{ij}\Big(x, \frac{x}{\varepsilon}\Big)\frac{\partial w_\ell^\varepsilon}{\partial x_i}\frac{\partial \chi^k}{\partial y_j}\frac{\partial V_\ell}{\partial x_k}(1-\rho^\varepsilon)\,dx + \varepsilon\fint_{I_\delta} a_{ij}\Big(x, \frac{x}{\varepsilon}\Big)\frac{\partial w_\ell^\varepsilon}{\partial x_i}\chi^k\frac{\partial V_\ell}{\partial x_k}\frac{\partial \rho^\varepsilon}{\partial x_j}\,dx\\
&+\fint_{I_\delta}\nabla\theta_\ell^\varepsilon\cdot\Big[a\Big(x_\ell, \frac{x}{\varepsilon}\Big) - a\Big(x, \frac{x}{\varepsilon}\Big)\Big]\nabla w_\ell^\varepsilon\,dx.
\end{aligned}
$$

Invoking (3.5), we bound I_3 as

$$
\begin{aligned}
|I_3| &\le C\frac{1}{\delta^d}\|\nabla w_\ell^\varepsilon\|_{0,I_\delta\setminus I_{(k-1)\varepsilon}}\|\nabla V_\ell\|_{0,I_\delta\setminus I_{(k-1)\varepsilon}} + C\Big(\frac{\varepsilon}{\delta}\Big)^{1/2}\delta^{1-d}\|\nabla V_\ell\|_{0,I_\delta}\|\nabla W_\ell\|_{0,I_\delta}\\
&\le C\Big(\frac{\varepsilon}{\delta} + \delta^2 + (\varepsilon\delta)^{1/2}\Big)|\nabla W_\ell|\,|\nabla V_\ell|.
\end{aligned}
$$

Using (3.2) and integrating by parts give

$$
I_4 = \fint_{I_\delta}\nabla W_\ell\cdot a\Big(x_\ell, \frac{x}{\varepsilon}\Big)\nabla \hat{V}_\ell\,dx.
$$

So we have

$$
\begin{aligned}
I_4 - \fint_{I_{\kappa\varepsilon}}\nabla W_\ell\cdot a\Big(x_\ell, \frac{x}{\varepsilon}\Big)\nabla\hat{V}_\ell\,dx &= \Big(\frac{1}{\delta^d} - \frac{1}{|\kappa\varepsilon|^d}\Big)\int_{I_{\kappa\varepsilon}}\nabla W_\ell\cdot a\Big(x_\ell, \frac{x}{\varepsilon}\Big)\nabla\hat{V}_\ell\,dx\\
&\quad+\frac{1}{\delta^d}\int_{I_\delta\setminus I_{\kappa\varepsilon}}\nabla W_\ell\cdot a\Big(x_\ell, \frac{x}{\varepsilon}\Big)\nabla\hat{V}_\ell\,dx.
\end{aligned}
$$

Invoking (3.10), we get

$$
\begin{aligned}
\left|\frac{1}{\delta^d} - \frac{1}{|\kappa\varepsilon|^d}\right|\left|\int_{I_{\kappa\varepsilon}}\nabla W_\ell\cdot a\Big(x_\ell, \frac{x}{\varepsilon}\Big)\nabla\hat{V}_\ell\,dx\right| &\le C\frac{\varepsilon}{\delta}\delta^{-d}\|\nabla\hat{V}_\ell\|_{0,I_{\kappa\varepsilon}}\|\nabla W_\ell\|_{0,I_{\kappa\varepsilon}}\\
&\le C\frac{\varepsilon}{\delta}\delta^{-d}\|\nabla\hat{V}_\ell\|_{0,I_\delta}\|\nabla W_\ell\|_{0,I_\delta}\\
&\le C\frac{\varepsilon}{\delta}|\nabla V_\ell||\nabla W_\ell|,
\end{aligned}
$$

and

$$
\begin{aligned}
\delta^{-d}\left|\int_{I_\delta\setminus I_{\kappa\varepsilon}}\nabla W_\ell\cdot a\Big(x_\ell, \frac{x}{\varepsilon}\Big)\nabla\hat{V}_\ell\,dx\right| &\le C\frac{1}{\delta^d}\|\nabla\hat{V}_\ell\|_{0,I_\delta\setminus I_{\kappa\varepsilon}}\|\nabla W_\ell\|_{0,I_\delta\setminus I_{\kappa\varepsilon}}\\
&\le C\Big(\frac{\varepsilon}{\delta} + \delta^2\Big)|\nabla V_\ell||\nabla W_\ell|,
\end{aligned}
$$

Combining the estimates for I_1, I_2, I_3 and the above two estimates give the desired result (3.11). $\square$

If we know the exact local period of the coefficients a-priori, we do not need to solve the problem over I_δ. It is enough to solve the cell problem (1.7) over one period I_ε with periodic boundary condition. In this case (5.24) is valid with δ replaced by ε and $\hat{v}_\ell^\varepsilon(x) = \hat{V}_\ell(x)$ for $1 \le \ell \le L$, so a direct calculation yields

$$\nabla W(x_\ell) \cdot (\mathcal{A}_H - \mathcal{A})(x_\ell) \nabla V(x_\ell) = \fint_{I_\varepsilon} \nabla w_\ell^\varepsilon \cdot \Big[a\Big(x, \frac{x}{\varepsilon}\Big) - a\Big(x_\ell, \frac{x}{\varepsilon}\Big)\Big] \nabla v_\ell^\varepsilon \, dx + \fint_{I_\varepsilon} \nabla w_\ell^\varepsilon \cdot a\Big(x_\ell, \frac{x}{\varepsilon}\Big) \nabla (v_\ell^\varepsilon - \hat{v}_\ell^\varepsilon) \, dx.$$

Invoking (3.3) once again, we get

$$e(\mathrm{HMM}) \le C\varepsilon.$$

Similarly, if δ is an integer multiple of ε, and the periodic boundary condition is used, then we have

$$e(\mathrm{HMM}) \le C\delta.$$

Remark 3.3. *An explicit expression for v_ℓ^ε is available in one dimension, from which we may show that the bound for $e(\mathrm{HMM})$ is sharp.*

3.2. **Problems with random coefficients.** Here we estimate $e(\mathrm{HMM})$ for the random case. Our basic strategy follows that of [39].

Denote by $(\Omega, \mathcal{F}, P)$ a probability space and let $a(y, \omega) = \Big(a_{ij}(y, \omega)\Big)$ be a random field, whose statistics is invariant under integer shifts, and satisfies a uniform ellipticity condition that there exist constants λ and Λ such that

$$\lambda |\xi|^2 \le \sum_{i,j=1}^d a_{ij}(y, \omega) \xi_i \xi_j \le \Lambda |\xi|^2,$$

for almost all $y \in \mathbb{R}^d$ and $\omega \in \Omega$. For $j = 1, \cdots, d$, denote by $\varphi_j(y, \omega)$ the solutions of the cell problem:

$$\mathcal{L}_y \varphi_j \colon = -\operatorname{div}_y \Big(a_{ij}(y, \omega) \nabla_y \varphi_j\Big) = \operatorname{div}_y (a_{ij} \cdot e_j). \tag{3.13}$$

where $\{e_j\}_{j=1}^d$ are the canonic basis in $\mathbb{R}^d$. $\nabla \varphi_j$ is required to be stationary under integer shift. The existence of φ_j is proved in [32, 25]. In general φ_j is not stationary. The homogenization coefficient $\mathcal{A}$ [32, 25] is given by

$$\mathcal{A} = \langle\, a(I + \nabla \varphi) \,\rangle.$$

Here and in the following, we use the notation

$$\langle\, f \,\rangle = \mathbb{E} \int_{[0,1]^d} f(y) \, dy,$$

and

$$[f;m] = \frac{1}{m^d} \int\limits_{[0,m]^d} f(y)\, dy.$$

As in [39], we will consider the following auxiliary problem:

$$\mathcal{L}u + \rho u = f, \tag{3.14}$$

for any $\rho > 0$, where f is of the form

$$f = \sum_{j=1}^{d} D_j g_j + h,$$

with $g_j, h \in \rho_G$ which is defined as

$$\rho_G\colon = \{\psi \mid \langle\, |\psi|^2 \,\rangle \leq G^2\},$$

and ψ a stationary random field, whose statistics is stationary with respect to integer shifts.

The solution of (3.14) can be expressed with the help of a diffusion process η generated by the operator $\mathcal{L}$.

For each fixed realization of $\{a(y,\cdot)\}$, denote by η_x the diffusion process generated by $\mathcal{L}_1$, and M_x the expectation with respect to η_x. Let

$$\Gamma(\tau) = \int_0^{\tau} f(\eta(t)) e^{-\rho t}\, dt,$$

then it is well-known [21] that the solution of (3.14) is given by

$$u(x) = M_x \Gamma(\infty).$$

The following results are either standard or proved in [39, 25].

Lemma 3.4. *If u is the solution of* (3.14)*, then there exists constants independent of ρ such that*

$$\langle\, |\nabla u|^2 \,\rangle + \rho \langle\, u^2 \,\rangle \leq C\Big(\langle\, g^2 \,\rangle + \frac{1}{\rho} \langle\, h^2 \,\rangle \Big). \tag{3.15}$$

$$\langle\, (M_x \Gamma(\infty))^2 \,\rangle^{1/2} \leq C \frac{G^2}{\rho}. \tag{3.16}$$

$$\langle\, M_x (\Gamma(\infty) - \Gamma(s))^2 \,\rangle \leq C \frac{G^2}{\rho} e^{-2s\rho}. \tag{3.17}$$

Because of the lowest order term ρu, the Green's function associated with the operator $\mathcal{L}_1 + \rho I$ decays exponentially with rate $\mathcal{O}(\sqrt{\rho})$. To make this statement precise, we define the norm $\|x\|\colon = \max_i |x_i|$, and

$$Q_\rho\colon = \Big\{ x \in \mathbb{R}^d \mid \|x\| \leq \Big(\frac{1}{\rho}\Big)^{\frac{1}{2}(\ln \frac{1}{\rho})^{1/2}} \Big\}.$$

Let τ be the first exit time of Q_ρ starting at $x \in Q_\rho$. Denote by $\hat{\varphi}_\rho(x) = M_x \Gamma(\tau)$.

Lemma 3.5. [39] *If ρ is sufficiently small, then*

$$\mathbb{E}\int\limits_{\|x\|\le 10}|\varphi_\rho(x)-\hat{\varphi}_\rho(x)|^2\,dx\le CG^2e^{-C|\ln(1/\rho)|^2}. \tag{3.18}$$

$$\mathbb{E}\int\limits_{\|x\|\le 1}|\nabla\varphi_\rho(x)-\nabla\hat{\varphi}_\rho(x)|^2\,dx\le CG^2e^{-C|\ln(1/\rho)|^2}. \tag{3.19}$$

To prepare for the discussion on the consequence of the mixing condition, we mention

Lemma 3.6. [39] *Let $\{a_{ij},g_j\}$ and $\{\tilde{a}_{ij},\tilde{g}_j\}$ be two sets of data such that $\{a_{ij}(y),g_j(y)\}=\{\tilde{a}_{ij}(y),\tilde{g}_j(y)\}$ for $y\notin B$, where B is a domain in $\mathbb{R}^d$, and let φ_ρ and $\hat{\varphi}_\rho$ be the solutions of* (3.14) *associated with $\{a_{ij},g_j\}$ and $\{\tilde{a}_{ij},\tilde{g}_j\}$, respectively (with $h=0$), then*

$$\int\limits_{\mathbb{R}^d}|\varphi_\rho(x)-\tilde{\varphi}_\rho(x)|^2\,dx\le\frac{C}{\rho}\int\limits_{\mathbb{R}^d}(G^2+|\nabla\varphi_\rho(x)|^2I_B(x)\,dx, \tag{3.20}$$

where I_B is the indicator function of the domain B.

Now we introduce the crucial mixing condition. Let B be a domain in $\mathbb{R}^d$. Denote by $\mathcal{F}(B)$ the $\sigma-$algebra generated by $\{a_{ij}(y,\omega),y\in B\}$. Let ξ,η be two random variables that are measurable with respect to $\mathcal{F}(B_1)$ and $\mathcal{F}(B_2)$, respectively, then

$$\frac{|\mathbb{E}\xi\eta-\mathbb{E}\xi\mathbb{E}\eta|}{(\mathbb{E}\xi^2)^{1/2}(\mathbb{E}\eta^2)^{1/2}}\le e^{-\lambda q}, \tag{$\mathbf{A}$}$$

where $q=\operatorname{dist}(B_1,B_2)$.

Lemma 3.7. *Under the condition* ($\mathbf{A}$), *we have*

$$\mathbb{E}[\varphi_\rho;m]^2\le C\Big(\frac{G^2}{\rho}\Big(\frac{|\ln(1/\rho)|^2}{\rho^{1/2}m}\Big)^d+e^{-c(\ln(1/\rho))^2}\Big). \tag{3.21}$$

Proof. For $\ell=(\ell_1,\cdots,\ell_d)\in\mathbb{Z}^d\cap[0,m)^d\colon=\mathbb{Z}^d_m$. Denote by I^ℓ the cube of size 1 centered at $\ell+\frac12=(\ell_1+\frac12,\cdots,\ell_d+\frac12)$, and let $\varphi^\ell=\int_{I^\ell}\varphi(x)\,dx$, then

$$[\varphi;m]=\frac{1}{m^d}\sum_{\ell\in\mathbb{Z}^d_m}\varphi^\ell.$$

We first estimate $\mathbb{E}\varphi^\ell\varphi^k$. If $|\ell-k|\le C\rho^{-1/2}|\ln(1/\rho)|^2$, then

$$\mathbb{E}\varphi^\ell\varphi^k\le\langle\,\varphi^2_\rho(x)\,\rangle^{1/2}\langle\,\varphi^2_\rho(x)\,\rangle^{1/2}\le C\frac{G^2}{\rho}.$$

If $|\ell-k|\ge C\rho^{-1/2}|\ln(1/\rho)|^2$, then let $B_1\colon=\ell+B(\rho^{-1/2}|\ln(1/\rho)|^2)$ and $B_2\colon=k+B(\rho^{-1/2}|\ln(1/\rho)|^2)$, where $B(s)$ is a ball of size s in the norm $\|\cdot\|$. Denote by $\tilde{\varphi}_1(x)$ the solution of (3.14) in which the coefficient $\Big(a_{ij}(y,\omega)\Big)$ is modified in $\mathbb{R}^d\backslash B_1$ such that it is measurable with respect to $\mathcal{F}(B_1)$, and similarly $\tilde{\varphi}_2(x)$ the solution of (3.14) in which the coefficient $\Big(a_{ij}(y,\omega)\Big)$ is modified in $\mathbb{R}^d\backslash B_2$ such that it is measurable with respect to $\mathcal{F}(B_2)$. The modified coefficients $\Big(\tilde{a}_{ij}(y,\omega)\Big)$ should still satisfies the condition on a_{ij} listed in the beginning of this subsection. Using $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$, we can similarly define $\tilde{\varphi}_1^\ell$ and $\tilde{\varphi}_2^\ell$. Using (3.18), we have

$$\begin{aligned}\mathbb{E}(\tilde{\varphi}_1^\ell-\tilde{\varphi}^\ell)^2 &\le CG^2e^{-C|\ln(1/\rho)|^2},\\ \mathbb{E}(\tilde{\varphi}_1^k-\tilde{\varphi}^k)^2 &\le CG^2e^{-C|\ln(1/\rho)|^2}.\end{aligned}$$

Since

$$\begin{aligned}\mathbb{E}(\varphi^\ell\varphi^k) &= \mathbb{E}(\tilde{\varphi}^\ell\tilde{\varphi}^k)+\mathbb{E}(\varphi^\ell-\tilde{\varphi}_1^\ell)\tilde{\varphi}_2^k+\mathbb{E}\tilde{\varphi}_1^\ell(\varphi^k-\tilde{\varphi}_2^k)\\ &\quad+\mathbb{E}(\varphi^\ell-\tilde{\varphi}_1^\ell)(\varphi^k-\tilde{\varphi}_2^k),\end{aligned}$$

and

$$\begin{aligned}|\mathbb{E}\tilde{\varphi}_1^\ell|,|\mathbb{E}\tilde{\varphi}_2^k| &\le CGe^{-C|\ln\frac{1}{\rho}|^2},\\ |\mathbb{E}\tilde{\varphi}_1^\ell\tilde{\varphi}_2^k| &\le C\Big(\frac{G^2}{\rho}e^{-C|\ell-k|}+G^2e^{-C|\ln\frac{1}{\rho}|^2}\Big).\end{aligned}$$

We thus have

$$|\mathbb{E}\varphi^\ell\varphi^k| \le C\Big(\frac{G^2}{\rho}e^{-C|\ell-k|}+G^2e^{-C|\ln\frac{1}{\rho}|^2}\Big).$$

Hence

$$\begin{aligned}\mathbb{E}[\varphi_\rho;m]^2 &= \frac{1}{m^{2d}}\sum_{\ell,k\in\mathbb{Z}_m^d}\varphi^\ell\varphi^k\\ &= \frac{1}{m^{2d}}\Big(\sum_{|\ell-k|\le\rho^{-1/2}|\ln(1/\rho)|^2}\mathbb{E}\varphi^\ell\varphi^k+\sum_{|\ell-k|>\rho^{-1/2}|\ln(1/\rho)|^2}|\ln(1/\rho)|^2\Big)\mathbb{E}\varphi^\ell\varphi^k\\ &\le \frac{1}{m^{2d}}\Big(\frac{G^2}{\rho}m^d\Big(\rho^{-1/2}|\ln(1/\rho)|^2\Big)^d+m^{2d}G^2e^{-C|\ln(1/\rho)|^2}\\ &\quad\times\frac{G^2}{\rho}\sum_{|\ell-k|\ge\rho^{-1/2}|\ln(1/\rho)|^2}e^{-C|\ell-k|}\Big)\\ &\le C\frac{G^2}{\rho}\Big(\frac{\ln(1/\rho)}{\rho^{1/2}m}\Big)^d+e^{-C|\ln(1/\rho)|^2}.\end{aligned}$$

□

Proceeding along the same line as in [39, Theorem 2.1], using condition (**A**), we have

Lemma 3.8. *For any $0 < \gamma < 1/2$, under condition* $(\mathbf{A})$*, there exists a constant C such that*

$$|\mathcal{A} - \langle\, a(I + \nabla\chi_\rho)\,\rangle| \le C\rho^{\frac{d-2-2\gamma}{4+d}}, \tag{3.22}$$

where $\chi_\rho = (\chi_{1,\rho}, \cdots, \chi_{d,\rho})$*, and* $\chi_{k,\rho}$ *is the solution of* (3.13) *with* $g = (a_{k1}, \cdots, a_{kd})$.

Now we are ready to estimate $e(\mathrm{HMM})$. Define $m = \frac{\delta}{2\varepsilon}$ and denote by φ_j^m the solution of (3.13) on $I_m = [0, m]^d$ with the boundary condition that $\varphi_j^m(y) = 0$ on ∂I_m, and let $\varphi^m = (\varphi_1^m, \cdots, \varphi_d^m)$, then

$$e(\mathrm{HMM}) = |\mathcal{A} - [a(I + \nabla\varphi^m)]_m|.$$

Define $\varphi_\rho, \varphi_\rho^m$ similarly. We have

$$e(\mathrm{HMM}) \le E_1 + E_2 + E_3,$$

where

$$\begin{aligned}
E_1 &= |\mathcal{A} - [a(I + \nabla\varphi_\rho)]_m|,\\
E_2 &= |[a(I + \nabla\varphi_\rho) - a(I + \nabla\varphi_\rho^m)]_m|,\\
E_3 &= |[a\nabla(\varphi_\rho^m - \varphi^m)]_m|.
\end{aligned}$$

Obviously,

$$E_1 = |\mathcal{A} - \langle\, a(I + \nabla\varphi_\rho)\,\rangle + [\tilde{\psi}]_m|,$$

with $\tilde{\psi} = \langle\, a(I + \nabla\varphi_\rho)\,\rangle - a(I + \nabla\varphi_\rho)$. It was proved in [39, Lemma 2.5] that

$$\mathbb{E}|[\tilde{\psi}]_m| \le \Big(\mathbb{E}[\tilde{\psi}]_m^2\Big)^{1/2} \le C\Big(\frac{(\ln\frac{1}{\rho})^2}{\rho^{1/2}m}\Big)^{d/2}.$$

The above inequality together with Lemma 3.8 gives

$$\mathbb{E}E_1 \le C\Big(G\rho^{\frac{d-2-2\gamma}{4+d}} + \Big(\frac{(\ln\frac{1}{\rho})^2}{\rho^{1/2}m}\Big)^{d/2}\Big).$$

To estimate E_2, denote by τ_m the first exist time for the domain I_{2m}, then $\varphi_\rho^{2m} = M_x\Gamma(\tau_m)$. For any $s > 0$,

$$\begin{aligned}
|\varphi_\rho - \varphi_\rho^{2m}| &= |M_x(\Gamma(\infty) - \Gamma(\tau_m))|\\
&\le M_x\{|\Gamma(\infty)| + |\Gamma(\tau_m)|; \tau_m \le s\}\\
&\quad + M_x\{e^{-s\rho}M_{\rho(s)}|\Gamma(\infty) - \Gamma(\tau_m)|; \tau_m > s\}\\
&\le C\Big(M_x((\Gamma(\infty)^2 + \Gamma(\tau_m)^2)\Big)^{1/2}\{P_x(\tau_m \le s)^{1/2} + e^{-s\rho}\},
\end{aligned}$$

since

$$P\{\tau_m \le s\} \le e^{-Cm^2/s},$$

we get

$$\mathbb{E}[|\varphi_\rho - \varphi_\rho^{2m}|^2]_{2m} \le C\frac{G^2}{\rho}\Big(e^{-Cm^2/s} + e^{-s\rho}\Big)^2.$$

Optimizing in s, we get

$$\mathbb{E}[|\varphi_\rho - \varphi_\rho^{2m}|^2]_{2m} \leq C\frac{G^2}{\rho}e^{-Cm^2\rho}.$$

Using standard interior estimates, we have

$$\mathbb{E}E_2 \leq C\mathbb{E}[|\nabla(\varphi_\rho - \varphi_\rho^{m})|^2]_m \leq \frac{C}{m}\mathbb{E}[|\varphi_\rho - \varphi_\rho^{2m}|^2]_{2m} \leq \frac{CG^2}{m\rho}e^{-Cm^2\rho}.$$

As for E_3, proceeding along the same line as in the estimate of E_1, we get

$$\mathbb{E}E_3 \leq C\Big(G^2\rho^{\frac{d-2-2\gamma}{4+d}} + \Big(\frac{|\ln\frac{1}{\rho}|^2}{\rho^{1/2}m}\Big)^{d/2}\Big).$$

To sum up, we have

$$\mathbb{E}e(\text{HMM}) \leq C\Big(G^2\rho^{\frac{d-2-2\gamma}{4+d}} + \Big(\frac{|\ln\frac{1}{\rho}|^2}{\rho^{1/2}m}\Big)^{d/2} + \frac{CG^2}{m\rho}e^{-Cm^2\rho}\Big).$$

Optimizing in ρ with respect to the first two terms, we get $\rho_0 = m^{-\frac{2d}{d+4\alpha}}$. Hence

$$\mathbb{E}e(\text{HMM}) \leq C\Big(\frac{|\ln m|^d}{m^\kappa} + \frac{CG^2}{m\rho_0}e^{-Cm^2\rho_0}\Big) \leq C\frac{|\ln m|^d}{m^\kappa}, \tag{3.23}$$

with

$$\kappa = \frac{d/2}{1 + \dfrac{d(d+4)/4}{d-2-2\gamma}}, \qquad 0 < \gamma < \frac{1}{2}.$$

Obviously, the $|\ln m|^d$ factor in (3.23) can be absorbed into the factor $m^{-\kappa}$. This inequality together with Theorem 1.2 completes the proof of Theorem 1.1 for the random case.

Remark 3.9. *The estimate* (3.23) *is unlikely to be optimal. In* $1-d$*, a direct calculation gives:*

$$\mathbb{E}e(\text{HMM}) \leq C\Big(\frac{\varepsilon}{\delta}\Big)^{1/2},$$

whereas the estimate (3.23) *does not apply for* $d = 1$ *and* $d = 2$*. We may use the techniques in* [13] *to derive improved bounds for* $e(\text{HMM})$ *if the magnitude of the oscillation in the coefficients* $\Big(a_{ij}^\varepsilon\Big)$ *is sufficiently small,.*

4. Reconstruction and compression

4.1. **Reconstruction procedure.** It is seen from Theorem 1.1 that the HMM solution U_H is indeed a good approximation of the macroscopic solution U_0. Next we consider how u^ε can be approximated. We will restrict to $k = 1$.

Proof of Theorem 1.4. Subtracting (1.1) from (1.14), we obtain

$$\begin{cases} -\operatorname{div}\Big(a^{\varepsilon}(x)\nabla(\overline{u}^{\,\varepsilon}-u^{\varepsilon})(x)\Big)=0, & x\in\Omega_\eta,\\ \qquad\qquad \overline{u}^{\,\varepsilon}(x)-u^{\varepsilon}(x)=U_H(x)-u^{\varepsilon}(x), & x\in\partial\Omega_\eta. \end{cases}$$

In view of the classic interior estimate for elliptic equations [23], we have

$$\|\nabla(\overline{u}^{\,\varepsilon}-u^{\varepsilon})\|_{0,\Omega}\le\frac{C}{\eta}\|\overline{u}^{\,\varepsilon}-u^{\varepsilon}\|_{0,\Omega_\eta}.$$

By Hopf maximum principle, we get

$$\begin{aligned}\frac{1}{\eta^2}\fint_{\Omega_\eta}|(\overline{u}^{\,\varepsilon}-u^{\varepsilon})(x)|^2\,dx&\le\frac{C}{\eta^2}\|\overline{u}^{\,\varepsilon}-u^{\varepsilon}\|_{L^\infty(\Omega_\eta)}\le\frac{C}{\eta^2}\|u^{\varepsilon}-U_H\|_{L^\infty(\partial\Omega_\eta)}\\ &\le\frac{C}{\eta^2}\Big(\|U_0-U_H\|_{L^\infty(\Omega_\eta)}+\|u^{\varepsilon}-U_0\|_{L^\infty(\Omega_\eta)}\Big).\end{aligned}$$

A combination of the above two implies (1.4). □

Proof of Theorem 1.5. Denote by $\hat{u}^{\,\varepsilon}$ the solution of (1.7) with $V_\ell=U_H$ and the coefficients $a^{\varepsilon}(x)$ replaced by $a(x_K,x/\varepsilon)$, where x_K is the barycenter of K. In light of (5.24), we have

$$\|\nabla(\overline{u}^{\,\varepsilon}-\hat{u}^{\,\varepsilon})\|_{0,I_\varepsilon(x_K)}\le C\varepsilon\|\nabla U_H\|_{0,I_\varepsilon(x_K)}.$$

Due to the construction of $\overline{u}^{\,\varepsilon}$ and the expression of $\hat{u}^{\,\varepsilon}$ (3.1), we have for any cell $I_\varepsilon(x_1)$, where x_1 is an arbitrary point in K,

$$\|\nabla(\overline{u}^{\,\varepsilon}-\hat{u}^{\,\varepsilon})\|_{0,I_\varepsilon(x_1)}=\|\nabla(\overline{u}^{\,\varepsilon}-\hat{u}^{\,\varepsilon})\|_{0,I_\varepsilon(x_K)}.$$

Since K is compact, we may find a finite covering such that $K\subset\cup_{x\in K}I_\varepsilon(x)$, so combining the above two estimates and notice that $|\cup_{x\in K}I_\varepsilon(x)\,|\le C|K|$, we get

$$\|\nabla(\overline{u}^{\,\varepsilon}-\hat{u}^{\,\varepsilon})\|_{0,K}\le C\varepsilon\|\nabla U_H\|_{0,K}. \tag{4.1}$$

Invoking (3.1) once again, we have

$$\frac{\partial\hat{u}^{\,\varepsilon}}{\partial x_i}=\frac{\partial U_H}{\partial x_i}+\frac{\partial\chi^k}{\partial y_i}\Big(x_K,\frac{x}{\varepsilon}\Big)\frac{\partial U_H}{\partial x_k},$$

where x_K is the barycenter of K. Define the $1-$order approximation of u^{ε} as

$$u_1^{\varepsilon}(x)=U_0+\varepsilon\chi^k\Big(x,\frac{x}{\varepsilon}\Big)\frac{\partial U_0}{\partial x_k},$$

where $\{\chi^k\}_{k=1}^d$ is the solutions of (3.2). Obviously,

$$\frac{\partial u_1^{\varepsilon}}{\partial x_i}=\frac{\partial U_0}{\partial x_i}+\Big(\varepsilon\frac{\partial\chi^k}{\partial x_i}+\frac{\partial\chi^k}{\partial y_i}\Big)\Big(x,\frac{x}{\varepsilon}\Big)\frac{\partial U_0}{\partial x_i}+\varepsilon\chi^k\Big(x,\frac{x}{\varepsilon}\Big)\frac{\partial^2U_0}{\partial x_i\partial x_k}.$$

A combination of the above leads to

$$\begin{aligned}\|\nabla(\hat{u}^{\varepsilon}-u_1^{\varepsilon})\|_{0,K} &\le C\|\nabla(U_H-U_0)\|_{0,K}+C\varepsilon|U_0|_{1,K}\\ &\quad+\Big\|\Big(\frac{\partial\chi^k}{\partial y_i}\Big(x,\frac{x}{\varepsilon}\Big)-\frac{\partial\chi^k}{\partial y_i}\Big(x_K,\frac{x}{\varepsilon}\Big)\Big)\frac{\partial U_0}{\partial x_k}\Big\|_{0,K}+C\varepsilon|U_0|_{2,K}.\\ &\le C|U_0-U_H|_{1,K}+C(\varepsilon+H)\|U_0\|_{2,K}.\end{aligned}$$

Summing up for all $K\in\mathcal{T}_H$ and using Theorem 1.1 for $k=1$ and the fact that $I_\delta=I_\varepsilon$, we get

$$\Big(\sum_{K\in\mathcal{T}_H}\|\hat{u}^{\varepsilon}-u_1^{\varepsilon}\|_{1,K}^2\Big)^{1/2}\le C(\varepsilon+H),$$

which together with the classical estimate for $u^\varepsilon-u_1^\varepsilon$ [41, 28, 6], i.e.,

$$\|u^\varepsilon-u_1^\varepsilon\|_1\le C\sqrt{\varepsilon},$$

gives

$$\begin{aligned}\Big(\sum_{K\in\mathcal{T}_H}\|u^\varepsilon-\hat{u}^{\varepsilon}\|_{1,K}^2\Big)^{1/2} &\le \|u^\varepsilon-u_1^\varepsilon\|_1+\Big(\sum_{K\in\mathcal{T}_H}\|\hat{u}^{\varepsilon}-u_1^{\varepsilon}\|_{1,K}^2\Big)^{1/2}\\ &\le C(\sqrt{\varepsilon}+H).\end{aligned}$$

A combination of this estimate and (4.1) gives the desired result. ☐

Corollary 4.1.

$$\|\overline{u}^{\,\varepsilon}-u^\varepsilon\|_0\le C(\varepsilon+H^2). \tag{4.2}$$

Proof. Since $\int_{I_\varepsilon}(\overline{u}^{\,\varepsilon}-\hat{u}^{\varepsilon})(x)\,dx=0$, so application of the *Poincaré inequality* gives

$$\|\overline{u}^{\,\varepsilon}-\hat{u}^{\varepsilon}\|_{0,I_\varepsilon(x_K)}\le C\varepsilon\|\nabla(\overline{u}^{\,\varepsilon}-\hat{u}^{\varepsilon})\|_{0,I_\varepsilon(x_K)}\le C\varepsilon^2\|\nabla U_H\|_{0,I_\varepsilon(x_K)}.$$

As before for any $I_\varepsilon(x_1)$ we have

$$\|\overline{u}^{\,\varepsilon}-\hat{u}^{\varepsilon}\|_{0,I_\varepsilon(x_1)}=\|\overline{u}^{\,\varepsilon}-\hat{u}^{\varepsilon}\|_{0,I_\varepsilon(x_K)}.$$

So

$$\|\overline{u}^{\,\varepsilon}-\hat{u}^{\varepsilon}\|_{0,K}\le C\varepsilon^2\|\nabla U_H\|_{0,K}. \tag{4.3}$$

On each element K, we have

$$\|\hat{u}^{\varepsilon}-U_H\|_{0,K}\le C\varepsilon\|\nabla U_H\|_{0,K}.$$

A combination of the above two and summing up for all $K\in\mathcal{T}_H$, we get

$$\|\overline{u}^{\,\varepsilon}-U_H\|_0\le C\varepsilon\|\nabla U_H\|_0\le C\varepsilon,$$

which together with

$$\|u^\varepsilon-U_H\|_0\le\|u^\varepsilon-U_0\|_0+\|U_0-U_H\|_0\le C(\varepsilon+H^2).$$

leads to (4.2), where we have used the estimate for U_0 [41, 28, 6], i.e.,

$$\|u^\varepsilon - U_0\|_0 \le C\varepsilon.$$

□

4.2. **Compression operator.** Compression operator (denoted by Q) maps the micro variables to the macro variables [17]. It plays an important role in the general framework of HMM, even though for the present problem, HMM can be formulated without explicitly specifying the compression operator beforehand. Typically the compression operator is some spatial/temporal averaging, or projection to some slow manifolds. It is of interest to consider the error bound for $Qu^\varepsilon - U_H$. We first list some natural properties of the *compression operator*.

- For any $\phi \in X$, $Q\phi \in X_H$.
- There exists a constant C such that

$$\|Q\phi\|_0 \le C\|\phi\|_0.$$

- If $\phi \in H^2(\Omega) \cap H_0^1(\Omega)$, then

$$\|Q\phi - \phi\|_0 \le CH^2|\phi|_2.$$

Theorem 4.2. *Assuming that Q satisfies all three requirements, then*

$$\|Qu^\varepsilon - U_H\|_0 \le C(\varepsilon + H^2). \tag{4.4}$$

Moreover, if $\mathcal{T}_H$ is quasi-uniform*, then*

$$\|Qu^\varepsilon - U_H\|_1 \le C\Big(\frac{\varepsilon}{H} + H\Big). \tag{4.5}$$

Proof. We decompose $Qu^\varepsilon - U_H$ into

$$Qu^\varepsilon - U_H = Q(u^\varepsilon - U_0) + (QU_0 - U_0) + (U_0 - U_H). \tag{4.6}$$

Using the fact that Q is L^2 bounded, we obtain

$$\|Q(u^\varepsilon - U_0)\|_0 \le C\|u^\varepsilon - U_0\|_0 \le C\varepsilon.$$

In view of the second property of Q, we have

$$\|QU_0 - U_0\|_0 \le CH^2.$$

A combination of (4.1) and the first estimate in Theorem 1.2 gives

$$\|U_0 - U_H\|_0 \le C(\varepsilon + H^2).$$

A combination of these three implies (4.4), which together with the *inverse inequality* [10, 3.2.28] leads to (4.5). □

It remains to gives some examples of the *compression operator* Q. The following two types operators meet all three requirements.

- The L^2-projection operator onto X_H.

- The Clément-type interpolation operator [12].

Remark 4.3. *Notice that in one-dimension, the standard Lagrange interpolant does not meet the second requirement. However, it is still possible to derive* (4.5) *via other approach. Moreover, a careful study of one dimensional examples shows that the term* ε/H *in* (4.5) *is sharp.*

5. Nonlinear homogenization problems

5.1. **Algorithms and main results.** We consider the following nonlinear problem which has been discussed in [7].

$$\begin{cases} -\operatorname{div}\Big(a^{\varepsilon}\big(x,u^{\varepsilon}(x)\big)\nabla u^{\varepsilon}(x)\Big) = f(x) & \text{in } D,\\ u^{\varepsilon}(x)=0 & \text{on } \partial D. \end{cases} \tag{5.1}$$

In this section, we define $X\colon = W_0^{1,p}(D)$ with $p>1$ and X_H is defined as the $\mathcal{P}_k$ finite element subspace of X.

We assume that $a^{\varepsilon}(x,u^{\varepsilon})$ satisfies

$$\lambda|\xi|^2 \le a_{ij}^{\varepsilon}\xi_i\xi_j \le \Lambda|\xi|^2, \quad \forall \xi\in\mathbb{R}^d,$$

with $0<\lambda\le\Lambda$. Moreover, we assume that $a^{\varepsilon}(x,z)$ is equi-continuous in z uniformly with respect to x and ε. Under these conditions, it is proved in [7] that u^{ε} converges weakly in X to the solution of the following problem:

$$\begin{cases} \mathcal{L}U_0\colon = -\operatorname{div}\Big(\mathcal{A}\big(x,U_0(x)\big)\nabla U_0(x)\Big) = f(x) & \text{in } D,\\ U_0(x)=0 & \text{on } \partial D. \end{cases}$$

If we let

$$A(v,w) = \big(\mathcal{A}(x,v)\nabla v,\nabla w\big) \quad \forall v,w\in X,$$

then

$$A(U_0,v) = (f,v) \quad \forall v\in X', \tag{5.2}$$

where X' is the dual space of X. The tensor $\mathcal{A}(x,z)$ is also equi-continuous in z uniformly with respect to x and satisfies

$$\lambda|\xi|^2 \le \mathcal{A}_{ij}\xi_i\xi_j \le \frac{\Lambda^2}{\lambda}|\xi|^2, \quad \forall\xi\in\mathbb{R}^d.$$

The linearized operator of $\mathcal{L}$ at U_0 is defined for any $v\in H_0^1(D)$ by

$$\mathcal{L}_{\mathrm{lin}}(U_0)v = -\operatorname{div}\Big(\mathcal{A}(x,U_0)\nabla v + \mathcal{A}_p(x,U_0)\nabla U_0\, v\Big),$$

where $\mathcal{A}_p(x,z)\colon = \nabla_z\mathcal{A}(x,z)$. $\mathcal{L}_{\mathrm{lin}}$ induces a bilinear form through

$$\hat{A}(u;v,w)\colon = \Big(\mathcal{A}(x,u)\nabla v,\nabla w\Big) + \big(\mathcal{A}_p(x,u)\nabla u\, v,\nabla w\big) \quad \forall (v,w)\in H_0^1(D)\times H_0^1(D).$$

Our basic assumption is that the linearized operator $\mathcal{L}_{\rm lin}$ is an isomorphism from $H_0^1(D)$ to $H^{-1}(D)$, so U_0 must be an isolated solution of (5.2).

To formulate HMM, for each quadrature points x_ℓ, define v_ℓ^ε to be the solutions of:

$$
\begin{cases} -\operatorname{div}\Big(a^\varepsilon\big(x, v_\ell^\varepsilon\big)\nabla v_\ell^\varepsilon(x)\Big) = 0 & x \in I_\delta(x_\ell), \\ v_\ell^\varepsilon(x) = V_\ell(x) & x \in \partial I_\delta(x_\ell). \end{cases} \tag{5.3}
$$

w_ℓ^ε can be defined similarly.

For any $V, W \in X_H$, define

$$
\nabla W(x_\ell)\cdot \mathcal{A}_H\big(x_\ell, V(x_\ell)\big)\nabla V(x_\ell) = \frac{1}{\delta^d}\int_{I_\delta(x_\ell)} \nabla w_\ell^\varepsilon(x)\cdot a^\varepsilon\big(x, v_\ell^\varepsilon(x)\big)\nabla v_\ell^\varepsilon(x)\,dx,
$$

and

$$
A_H(V, W) := \sum_{K\in\mathcal{T}_H} |K| \sum_{x_\ell\in K} \omega_\ell \nabla W(x_\ell)\cdot\mathcal{A}_H\big(x_\ell, V(x_\ell)\big)\nabla V(x_\ell).
$$

The HMM solution is given by the problem:

Problem 5.1. *Find $U_H \in X_H$ such that*

$$
A_H(U_H, V) = (f, V) \quad \forall V \in X_H. \tag{5.4}
$$

It is observed that the solution of (5.3) still satisfies (2.3), (2.4) and

$$
\|\nabla V_\ell\|_{0,I_\delta} \le \|\nabla v_\ell^\varepsilon\|_{0,I_\delta} \le \sqrt{\frac{\Lambda}{\lambda}}\|\nabla V_\ell\|_{0,I_\delta}. \tag{5.5}
$$

For any $v, v_H, w \in X$, define

$$
R(v, v_H, w) := A(v_H, w) - A(v, w) - \hat{A}(v; v_H - v, w). \tag{5.6}
$$

Here R satisfies for $e_H := v - v_H$ and $\frac{1}{p}+\frac{1}{q} = 1,\ \ p, q \ge 1$,

$$
|R(v, v_H, w)| \le C(M)(\|e_H\|_{0,2p}^2 + \|e_H\nabla e_H\|_{0,p})\|\nabla w\|_{0,q} \tag{5.7}
$$

with any v and v_H satisfying $\|v\|_{1,\infty} + \|v_H\|_{1,\infty} \le M$ (see [38, Lemma 3.1] for a similar result). Therefore we have

Lemma 5.2. *$U_H \in X_H$ is the solution of Problem 5.1 if and only if*

$$
\hat{A}(U_0; U_0 - U_H, V) = R(U_0, U_H, V) + A_H(U_H, V) - A(U_H, V) \quad \forall V \in X_H. \tag{5.8}
$$

For any $V, W \in X_H$, define

$$
E(V, W) := \nabla W(x_\ell)\cdot(\mathcal{A}_H - \mathcal{A})(x_\ell, V(x_\ell))\nabla V(x_\ell). \tag{5.9}
$$

Define $e(\text{HMM})$ as

$$
e(\text{HMM}) = \max_{\substack{x_\ell\in K, K\in\mathcal{T}_H \\ V,W\in X_H\cap W^{1,\infty}(D)}} \frac{|E(V, W)|}{\|\nabla V_\ell\|\,\|\nabla W_\ell\|}. \tag{5.10}
$$

The existence and uniqueness of the solution of Problem 5.1 are proved in the following lemma.

Lemma 5.3. *Assuming that* $U_0 \in W^{2,p}(D)$ *with* $p > d$*, then there exists a constant* $H_0 > 0$ *such that for* $0 < H \le H_0$*, Problem 5.1 has a solution* U_H *satisfying*

$$
\begin{aligned}
\|U_H - P_H U_0\|_{1,\infty} &\le e(\mathrm{HMM})^{1/2} + H^{1-d/p}, \\
\|U_0 - U_H\|_{1,\infty} &\le C\Big(e(\mathrm{HMM})^{1/2} + H^{1-d/p}\Big),
\end{aligned}
\tag{5.11}
$$

where $P_H U_0$ *is defined as*

$$
\hat{A}(U_0; P_H U_0, V) = \hat{A}(U_0; U_0, V) \quad \forall V \in X_H. \tag{5.12}
$$

Moreover, if there exists a constant η *with* $0 < \eta < 1$ *such that*

$$
\sum_{K \in \mathcal{T}_H} \sum_{x_\ell \in K} \omega_\ell |E(V_1, W) - E(V_2, W)| \le \eta \|V_1 - V_2\|_1 \|W\|_1 \tag{5.13}
$$

for any V_1, V_2 *and* $W \in X_H \cap W^{1,\infty}(D)$*, then there exists a constant* $H_0 > 0$ *such that for* $0 < H \le H_0$*, the HMM solution* U_H *is unique.*

Proof. Notice that $\mathcal{L}_{\text{lin}}$ is an isomorphism from $H_0^1(D)$ to $H_0^1(D)$ and using Schatz's argument [34], we infer that there exists a constant $H_0 > 0$ such that for $0 < H \le H_0$,

$$
\sup_{W \in X_H} \frac{\hat{A}(U_0; V, W)}{\|W\|_1} \ge C\|V\|_1 \quad \forall V \in X_H. \tag{5.14}
$$

So there is a unique solution $P_H U_0$ satisfying (5.12) and

$$
\|U_0 - P_H U_0\|_{1,\infty} \le C H^{1-d/p}. \tag{5.15}
$$

Define a nonlinear mapping $T\colon\ X_H \to X_H$ by

$$
\hat{A}(U_0; T(V), W) = \hat{A}(U_0; U_0, W) - R(U_0, V, W) + A(V, W) - A_H(V, W),
$$

for any $W \in X_H$. Obviously T is continuous due to (5.14) and (5.7).

Define the set

$$
B\colon = \{\, V \in X_H \mid \|V - P_H U_0\|_{1,\infty} \le e(\mathrm{HMM})^{1/2} + H^{1-d/p} \,\}.
$$

We claim that there exists a constant $H_0 > 0$ such that for all $0 < H \le H_0$, $T(B) \subset B$. Notice that

$$
\hat{A}(U_0; T(V) - P_H U_0, W) = -R(U_0, V, W) + A(V, W) - A_H(V, W).
$$

Taking $W = G_H^z$ as defined in $(2.12)_2$, we obtain

$$\begin{aligned}
\|T(V) - P_H U_0\|_{1,\infty} &\le C|\ln H|\, \|U_0 - V\|_{1,\infty}^2 + C\Big(e(\mathrm{HMM}) + H\Big)|\ln H| \\
&\le C\Big(\|U_0 - P_H U_0\|_{1,\infty}^2 + \|P_H U_0 - V\|_{1,\infty}^2 + e(\mathrm{HMM}) + H\Big)|\ln H| \\
&\le C\Big(e(\mathrm{HMM}) + H^{2-2d/p} + H\Big)|\ln H| \\
&\le e(\mathrm{HMM})^{1/2} + H^{1-d/p}.
\end{aligned}$$

An application of Brouwer's fixed point theorem gives the existence of a $U_H \in B$ such that $T(U_H) = U_H$. By definition, U_H satisfies $(5.11)_1$. An application of the triangle inequality and (5.19) yield $(5.11)_2$.

Suppose that U_H and $\hat{U}_H$ are both solutions of (5.4). Using (5.14) we obtain

$$\begin{aligned}
C\|U_H - \hat{U}_H\|_1 &\le \sup_{W \in X_H} \frac{\int_0^1 \hat{A}(U_H^t; U_H - \hat{U}_H, W)\, dt}{\|W\|_1} \\
&\le \sup_{W \in X_H} \frac{|A(U_H, W) - A(\hat{U}_H, W)|}{\|W\|_1},
\end{aligned}$$

where $U_H^t = (1-t)\hat{U}_H + tU_H$. Notice that

$$\begin{aligned}
A(U_H, W) - A(\hat{U}_H, W) &= \Big(A(U_H, W) - A_H(U_H, W)\Big) \\
&\quad - \Big(A(\hat{U}_H, W) - A_H(\hat{U}_H, W)\Big).
\end{aligned}$$

In view of (5.13), we finally have

$$\|U_H - \hat{U}_H\|_1 \le (\eta + CH)\|U_H - \hat{U}_H\|_1.$$

Hence, for $\eta < 1$, there exists a constant $H_0 > 0$ such that for $0 < H \le H_0$,

$$\eta + CH \le \eta + CH_0 \le \frac{1}{2}(1 + \eta) < 1,$$

this implies $U_H = \hat{U}_H$. Therefore the HMM solution is unique. □

Based on the above lemma, we can prove a nonlinear analog of Theorem 1.1.

Theorem 5.4. *Let U_0 and U_H be solutions of* (5.2) *and* (5.4)*, respectively. Assuming that $U_0 \in W^{k+1,\infty}$, then there exist C_0 $(0 < C_0 < 1)$ and η_0 such that if*

$$e(\mathrm{HMM}) < C_0, \qquad \|U_0 - U_H\|_{1,\infty} \le \eta_0, \tag{5.16}$$

then there exists a constant H_0 such that for $0 < H < H_0$, we have

$$\|U_0 - U_H\|_1 \le C\Big(H^k + e(\mathrm{HMM})\Big), \tag{5.17}$$

$$\|U_0 - U_H\|_{1,\infty} \le C\Big(H^k + e(\mathrm{HMM})\Big)|\ln H|. \tag{5.18}$$

Proof. Note that $U_0 \in W^{k+1,\infty}$ and from (5.14), we have

$$\|U_0 - P_H U_0\|_1 \le CH^k, \qquad \|U_0 - P_H U_0\|_{1,\infty} \le CH^k. \tag{5.19}$$

Using (5.8) with $V = P_H U_0 - U_H$, invoking (5.14) and (2.5) we obtain

$$\begin{aligned}\|P_H U_0 - U_H\|_1 \le C\|U_0 - U_H\|_{1,4}^2 + \Big(e(\mathrm{HMM}) + CH\Big)\|P_H U_0 - U_H\|_1 \\ + C\Big(H^k + e(\mathrm{HMM})\Big).\end{aligned}$$

If $e(\mathrm{HMM}) < C_1 < 1$, then there exists a constant H_0 such that for $0 < H < H_0$,

$$e(\mathrm{HMM}) + CH \le C_1 + CH_0 \le \frac{1}{2}(1 + C_1) < 1,$$

which implies

$$\|P_H U_0 - U_H\|_1 \le C\|U_0 - U_H\|_{1,4}^2 + C\Big(H^k + e(\mathrm{HMM})\Big).$$

Using the interpolation inequality we have

$$\|U_0 - U_H\|_{1,4}^2 \le \|U_0 - U_H\|_1 \|U_0 - U_H\|_{1,\infty}.$$

If $\|U_0 - U_H\|_{1,\infty} \le \eta_1 = \frac{1}{2C}$, then combining the above two inequalities leads to (5.17).

Let $V = G_H^z$ in (5.8) and using (2.13) we obtain

$$\begin{aligned}\|P_H U_0 - U_H\|_{1,\infty} \le C\Big(e(\mathrm{HMM}) + H\Big)\|P_H U_0 - U_H\|_{1,\infty} \\ + C\Big(\|U_0 - U_H\|_{1,\infty}^2 + e(\mathrm{HMM}) + H^k\Big)|\ln H|.\end{aligned}$$

If $e(\mathrm{HMM}) < \frac{1}{2C}$, then there exists a constant H_0 such that for $0 < H < H_0$,

$$C\Big(e(\mathrm{HMM}) + H\Big) < \frac{1}{2} + CH_0 < 1,$$

we thus have

$$\|P_H U_0 - U_H\|_{1,\infty} \le C\Big(\|U_0 - U_H\|_{1,\infty}^2 + e(\mathrm{HMM}) + H^k\Big)|\ln H|.$$

If $\|U_0 - U_H\|_{1,\infty} \le \eta_2 = \frac{1}{2C|\ln H|}$, then we have

$$\begin{aligned}\|U_0 - U_H\|_{1,\infty} \le \|U_0 - P_H U_0\|_{1,\infty} + C\eta_2|\ln H|\, \|U_0 - U_H\|_{1,\infty} \\ + C\Big(H^k + e(\mathrm{HMM})\Big)|\ln H|,\end{aligned}$$

which implies

$$\|U_0 - U_H\|_{1,\infty} \le \|U_0 - P_H U_0\|_{1,\infty} + C\Big(H^k + e(\mathrm{HMM})\Big)|\ln H|,$$

this inequality together with the standard interpolation results gives (5.18).

To sum up, define $C_0 = \min\big(C_1, 1/(2C)\big)$ and $\eta_0 = \min(\eta_1, \eta_2)$, and if (5.16) holds for such η_0, we get (5.17) and (5.18). $\square$

5.2. **Estimating e(HMM).** It remains to estimate e(HMM) and verify the assumptions (5.13) and (5.16). We assume that $a^\varepsilon(x,u^\varepsilon) = \Bigl(a_{ij}(x,x/\varepsilon,u^\varepsilon)\Bigr)$, and for $1 \le i,j \le d$, the coefficients $a^\varepsilon_{ij}(x,y,z)$ are smooth in x,z and periodic in y with period I. These types of problems have been considered in [6, 7, 22]; among others. The homogenized coefficient $\mathcal{A} = \Bigl(\mathcal{A}_{ij}(x,p)\Bigr)$ is given for any $p \in \mathbb{R}$ by

$$\mathcal{A}_{ij}(x,p) = ⨍_I \Bigl(a_{ij} + a_{ik}\frac{\partial \chi^j}{\partial y_k}\Bigr)(x,y,p)\,dy,$$

where $\{\chi^k\}_{k=1}^d$ is defined for any $p \in \mathbb{R}$ by

$$-\frac{\partial}{\partial y_i}\Bigl(a_{ij}\frac{\partial \chi^k}{\partial y_k}\Bigr)(x,y,p) = \frac{\partial}{\partial y_i} a_{ij}(x,y,p), \tag{5.20}$$

with periodic boundary condition in y and $\int_I \chi^k(x,y,p)\,dy = 0$. It is clear that $\mathcal{A}(x,p)$ is also smooth in x and p.

To simplify the presentation, we will show how to estimate e(HMM) when (5.3) is slightly changed into

$$\begin{cases} -\operatorname{div}\Bigl(a^\varepsilon\bigl(x, V(x_\ell)\bigr)\nabla v^\varepsilon_\ell(x)\Bigr) = 0 & x \in I_\delta(x_\ell), \\ v^\varepsilon_\ell(x) = V_\ell(x) & x \in \partial I_\delta(x_\ell). \end{cases} \tag{5.21}$$

If $\delta = \varepsilon$, we replace the Dirichlet boundary condition in (5.21) by the periodic boundary condition, i.e., $v^\varepsilon_\ell(x) - V_\ell(x)$ is periodic on $\partial I_\varepsilon(x_\ell)$.

Theorem 5.5. *In general,* (5.13) *and* (5.16) *hold if δ and ε/δ are sufficiently small, and*

$$e(\text{HMM}) \le C\Bigl(\Bigl(\frac{\varepsilon}{\delta}\Bigr)^{1/2} + \delta\Bigr). \tag{5.22}$$

If $\delta = \varepsilon$, then (5.13) *and* (5.16) *hold if ε is sufficiently small,*

$$e(\text{HMM}) \le C\varepsilon. \tag{5.23}$$

In what follows, we concentrate on the case when δ is not an integer multiple of ε. The other case will be commented on.

Let us first fix more notations. Denote by $\hat{v}^\varepsilon_\ell$ the solutions of (5.21) with the coefficients $a(x,x/\varepsilon,V(x_\ell))$ replaced by $a(x_\ell,x/\varepsilon,V(x_\ell))$. Similarly we define w^ε_ℓ to be the solutions of (5.21) with V replaced by $W \in X_H$. $\hat{w}^\varepsilon_\ell$ can be defined in the same way. w^ε_ℓ and $\hat{w}^\varepsilon_\ell$ can be viewed as the perturbation of v^ε_ℓ and $\hat{v}^\varepsilon_\ell$, respectively.

Observed that $\hat{v}^\varepsilon_\ell$ and $\hat{w}^\varepsilon_\ell$ also satisfy (5.5), and

$$\|\nabla(v^\varepsilon_\ell - \hat{v}^\varepsilon_\ell)\|_{0,I_\delta} \le C\delta\|\nabla V_\ell\|_{0,I_\delta}, \qquad \|\nabla(w^\varepsilon_\ell - \hat{w}^\varepsilon_\ell)\|_{0,I_\delta} \le C\delta\|\nabla W_\ell\|_{0,I_\delta}. \tag{5.24}$$

Lemma 5.6. *We have*

$$\|\nabla(\hat{v}_\ell^\varepsilon - \hat{w}_\ell^\varepsilon)\|_{0,I_\delta} \le C\Big(|(V-W)(x_\ell)|(\|\nabla V_\ell\|_{0,I_\delta} + \|\nabla W_\ell\|_{0,I_\delta}) \\ + \|\nabla(V_\ell - W_\ell)\|_{0,I_\delta}\Big). \tag{5.25}$$

Proof. In view of the definitions of v_ℓ^ε and w_ℓ^ε, we have

$$\lambda\|\nabla(\hat{v}_\ell^\varepsilon - \hat{w}_\ell^\varepsilon)\|_{0,I_\delta}^2 \le \int_{I_\delta} \nabla(\hat{v}_\ell^\varepsilon - \hat{w}_\ell^\varepsilon)\cdot a\Big(x_\ell, \frac{x}{\varepsilon}, V(x_\ell)\Big)\nabla(\hat{v}_\ell^\varepsilon - \hat{w}_\ell^\varepsilon)\,dx \\ =: I_1 + I_2, \tag{5.26}$$

where

$$I_1 = \int_{I_\delta} \nabla(V_\ell - W_\ell)\cdot\Big[a\Big(x_\ell, \frac{x}{\varepsilon}, V(x_\ell)\Big)\nabla\hat{v}_\ell^\varepsilon - a\Big(x, \frac{x}{\varepsilon}, W(x_\ell)\Big)\nabla\hat{w}_\ell^\varepsilon\Big]\,dx$$

$$I_2 = \int_{I_\delta} \nabla(\hat{v}_\ell^\varepsilon - \hat{w}_\ell^\varepsilon)\cdot\Big[a\Big(x_\ell, \frac{x}{\varepsilon}, W(x_\ell)\Big) - a\Big(x_\ell, \frac{x}{\varepsilon}, V(x_\ell)\Big)\Big]\nabla w_\ell^\varepsilon\,dx.$$

Obviously, I_1 can be decomposed into

$$I_1 = \int_{I_\delta} \nabla(V_\ell - W_\ell)\cdot\Big[a\Big(x_\ell, \frac{x}{\varepsilon}, V(x_\ell)\Big) - a\Big(x_\ell, \frac{x}{\varepsilon}, W(x_\ell)\Big)\Big]\nabla v_\ell^\varepsilon\,dx \\ + \int_{I_\delta} \nabla(V_\ell - W_\ell)\cdot a\Big(x_\ell, \frac{x}{\varepsilon}, W(x_\ell)\Big)\nabla(\hat{v}_\ell^\varepsilon - \hat{w}_\ell^\varepsilon)\,dx.$$

Therefore we bound I_1 and I_2 as

$$|I_1| \le C|(V-W)(x_\ell)|\,\|\nabla\hat{v}_\ell^\varepsilon\|_{0,I_\delta}\|\nabla(V_\ell - W_\ell)\|_{0,I_\delta} \\ + \Lambda\|\nabla(\hat{v}_\ell^\varepsilon - \hat{w}_\ell^\varepsilon)\|_{0,I_\delta}\|\nabla(V_\ell - W_\ell)\|_{0,I_\delta},$$

and

$$|I_2| \le C|(V-W)(x_\ell)|\,\|\nabla\hat{w}_\ell^\varepsilon\|_{0,I_\delta}\|\nabla(\hat{v}_\ell^\varepsilon - \hat{w}_\ell^\varepsilon)\|_{0,I_\delta}$$

Putting the estimates for I_1 and I_2 into (5.26), and using (5.5) we come to (5.25). □

Next we establish the estimate for $(v_\ell^\varepsilon - \hat{v}_\ell^\varepsilon) - (w_\ell^\varepsilon - \hat{w}_\ell^\varepsilon)$. Let $\psi_\ell^\varepsilon := v_\ell^\varepsilon - \hat{v}_\ell^\varepsilon$ and $\tilde{\psi}_\ell^\varepsilon := w_\ell^\varepsilon - \hat{w}_\ell^\varepsilon$. Clearly, $\psi_\ell^\varepsilon, \tilde{\psi}_\ell^\varepsilon$ vanish on $\partial I_\delta(x_\ell)$ and satisfy:

$$\begin{aligned}
&-\operatorname{div}\Big(a\Big(x, \frac{x}{\varepsilon}, V(x_\ell)\Big)\nabla\psi_\ell^\varepsilon\Big)\\
&\qquad = \operatorname{div}\Big(\Big[a\Big(x, \frac{x}{\varepsilon}, V(x_\ell)\Big) - a\Big(x_\ell, \frac{x}{\varepsilon}, V(x_\ell)\Big)\Big]\nabla\hat{v}_\ell^\varepsilon\Big),
\end{aligned}\tag{5.27}$$

$$\begin{aligned}
&-\operatorname{div}\Big(a\Big(x, \frac{x}{\varepsilon}, W(x_\ell)\Big)\nabla\tilde{\psi}_\ell^\varepsilon\Big)\\
&\qquad = \operatorname{div}\Big(\Big[a\Big(x, \frac{x}{\varepsilon}, W(x_\ell)\Big) - a\Big(x_\ell, \frac{x}{\varepsilon}, W(x_\ell)\Big)\Big]\nabla\hat{w}_\ell^\varepsilon\Big).
\end{aligned}\tag{5.28}$$

Lemma 5.7. *We have*

$$\begin{aligned}
\|\nabla(\psi_\ell^\varepsilon - \tilde{\psi}_\ell^\varepsilon)\|_{0,I_\delta} \le C\delta\Big(&|(V-W)(x_\ell)|(\|\nabla V_\ell\|_{0,I_\delta} + \|\nabla W_\ell\|_{0,I_\varepsilon})\\
&+ \|\nabla(V_\ell - W_\ell)\|_{0,I_\delta}\Big).
\end{aligned}\tag{5.29}$$

Proof. Proceeding along the same line as in Lemma 5.6, using (5.27) and (5.28), we have

$$\begin{aligned}
\lambda\|\nabla(\psi_\ell^\varepsilon - \tilde{\psi}_\ell^\varepsilon)\|_{0,I_\delta}^2 \le &\Big[\int_{I_\delta} \nabla(\psi_\ell^\varepsilon - \tilde{\psi}_\ell^\varepsilon)\cdot\Big(a\Big(x_\ell, \frac{x}{\varepsilon}, V(x_\ell)\Big) - a\Big(x, \frac{x}{\varepsilon}, V(x_\ell)\Big)\Big)\nabla\hat{v}_\ell^\varepsilon\,dx\\
&+ \int_{I_\delta} \nabla(\psi_\ell^\varepsilon - \tilde{\psi}_\ell^\varepsilon)\cdot\Big(a\Big(x, \frac{x}{\varepsilon}, W(x_\ell)\Big) - a\Big(x_\ell, \frac{x}{\varepsilon}, W(x_\ell)\Big)\Big)\nabla\hat{w}_\ell^\varepsilon\,dx\Big]\\
&+ \int_{I_\delta} \nabla(\psi_\ell^\varepsilon - \tilde{\psi}_\ell^\varepsilon)\cdot\Big(a\Big(x, \frac{x}{\varepsilon}, W(x_\ell)\Big) - a\Big(x, \frac{x}{\varepsilon}, V(x_\ell)\Big)\Big)\,dx\\
=: &I_1 + I_2.
\end{aligned}\tag{5.30}$$

We decompose I_1 into

$$\begin{aligned}
I_1 = &\int_{I_\delta} \nabla(\psi_\ell^\varepsilon - \tilde{\psi}_\ell^\varepsilon)\cdot\Big[a\Big(x_\ell, \frac{x}{\varepsilon}, V(x_\ell)\Big) - a\Big(x, \frac{x}{\varepsilon}, V(x_\ell)\Big)\Big]\nabla(\hat{v}_\ell^\varepsilon - \hat{w}_\ell^\varepsilon)\,dx\\
&+ \int_{I_\delta} \nabla(\psi_\ell^\varepsilon - \tilde{\psi}_\ell^\varepsilon)\cdot\Big[a\Big(x_\ell, \frac{x}{\varepsilon}, V(x_\ell)\Big) - a\Big(x, \frac{x}{\varepsilon}, V(x_\ell)\Big)\\
&\qquad + a\Big(x, \frac{x}{\varepsilon}, W(x_\ell)\Big) - a\Big(x_\ell, \frac{x}{\varepsilon}, W(x_\ell)\Big)\Big]\nabla\hat{w}_\ell^\varepsilon\,dx.
\end{aligned}$$

Similar to (5.25), we have

$$\|\nabla(\hat{v}_\ell^\varepsilon - \hat{w}_\ell^\varepsilon)\|_{0,I_\delta} \le C\Big(|(V-W)(x_\ell)|\|\nabla V_\ell\|_{0,I_\delta} + \|\nabla(V_\ell - W_\ell)\|_{0,I_\delta}\Big).\tag{5.31}$$

We thus bound I_1 as

$$|I_1| \le C\delta\Big(|(V-W)(x_\ell)|\|\nabla V_\ell\|_{0,I_\delta} + \|\nabla(V_\ell - W_\ell)\|_{0,I_\delta}\Big)\|\nabla(\psi_\ell^\varepsilon - \tilde{\psi}_\ell^\varepsilon)\|_{0,I_\delta}.$$

$\hat{w}_\ell^\varepsilon$ can be viewed as a perturbation of w_ℓ^ε, it follows from (5.24) that

$$\|\nabla\tilde{\psi}_\ell^\varepsilon\|_{0,I_\delta} \le C\delta\|\nabla W_\ell\|_{0,I_\delta}.$$

So I_2 can be bounded as

$$\begin{aligned}|I_2| &\le C\delta|(V-W)(x_\ell)|\,\|\nabla\tilde{\psi}_\ell^\varepsilon\|_{0,I_\delta}\|\nabla(\psi_\ell^\varepsilon-\tilde{\psi}_\ell^\varepsilon)\|_{0,I_\delta}\\ &\le C\delta|(V-W)(x_\ell)|\,\|\nabla W_\ell\|_{0,I_\delta}\|\nabla(\psi_\ell^\varepsilon-\tilde{\psi}_\ell^\varepsilon)\|_{0,I_\delta}.\end{aligned}$$

Putting the estimates for I_1 and I_2 into (5.30), we get (5.7). $\square$

Define

$$\begin{aligned}\hat{V}_\ell(x) &= V_\ell(x) + \varepsilon\chi^k\Big(x_\ell,\frac{x}{\varepsilon},V(x_\ell)\Big)\frac{\partial V_\ell}{\partial x_k}(x),\\ \hat{W}_\ell(x) &= W_\ell(x) + \varepsilon\chi^k\Big(x_\ell,\frac{x}{\varepsilon},W(x_\ell)\Big)\frac{\partial W_\ell}{\partial x_k}(x),\end{aligned}$$

where $\{\chi^k\}_{k=1}^d$ are the solutions of (3.2) with coefficients replaced by $a_{ij}(x_\ell, y, V(x_\ell))$.

Denote $\theta_\ell^\varepsilon = \hat{v}_\ell^\varepsilon - \hat{V}_\ell$ and $\tilde{\theta}_\ell^\varepsilon = \hat{w}_\ell^\varepsilon - \hat{W}_\ell$. Observe that

$$\begin{cases} -\operatorname{div}\Big(a\big(x_\ell,\frac{x}{\varepsilon},V(x_\ell)\big)\theta_\ell^\varepsilon(x)\Big) = 0 & x\in I_\delta(x_\ell),\\ \theta_\ell^\varepsilon(x) = -\varepsilon\chi^k\big(x_\ell,\frac{x}{\varepsilon},V(x_\ell)\big)\dfrac{\partial V_\ell}{\partial x_k} & x\in\partial I_\delta(x_\ell).\end{cases}$$

and

$$\begin{cases} -\operatorname{div}\Big(a\big(x_\ell,\frac{x}{\varepsilon},W(x_\ell)\big)\tilde{\theta}_\ell^\varepsilon(x)\Big) = 0 & x\in I_\delta(x_\ell),\\ \tilde{\theta}_\ell^\varepsilon(x) = -\varepsilon\chi^k\big(x_\ell,\frac{x}{\varepsilon},W(x_\ell)\big)\dfrac{\partial W_\ell}{\partial x_k} & x\in\partial I_\delta(x_\ell).\end{cases}$$

Similar to (3.4), we have

$$\begin{aligned}\|\nabla\theta_\ell^\varepsilon\|_{0,I_\delta} &\le C\Big(\frac{\varepsilon}{\delta}\Big)^{1/2}\|\nabla V_\ell\|_{0,I_\delta},\\ \|\nabla\tilde{\theta}_\ell^\varepsilon\|_{0,I_\delta} &\le C\Big(\frac{\varepsilon}{\delta}\Big)^{1/2}\|\nabla W_\ell\|_{0,I_\delta}.\end{aligned}\tag{5.32}$$

Lemma 5.8. *We have*

$$\begin{aligned}\|\nabla(\theta_\ell^\varepsilon-\tilde{\theta}_\ell^\varepsilon)\|_{0,I_\delta} \le C\Big(\frac{\varepsilon}{\delta}\Big)^{1/2}\Big(&\|\nabla(V_\ell-W_\ell)\|_{0,I_\delta}\\ &+|(V-W)(x_\ell)|(\|\nabla V_\ell\|_{0,I_\delta}+\|\nabla W_\ell\|_{0,I_\delta}\Big).\end{aligned}\tag{5.33}$$

Proof. Obviously we have

$$
\begin{aligned}
\lambda\|\nabla(\theta_\ell^\varepsilon-\tilde{\theta}_\ell^\varepsilon)\|_{0,I_\delta}^2 \le & \int\limits_{I_\delta}\nabla(\theta_\ell^\varepsilon-\tilde{\theta}_\ell^\varepsilon)\cdot\Big[a\Big(x_\ell,\frac{x}{\varepsilon},W(x_\ell)\Big)-a\Big(x_\ell,\frac{x}{\varepsilon},V(x_\ell)\Big)\Big]\nabla\tilde{\theta}_\ell^\varepsilon\,dx\\
&-\int\limits_{I_\delta}\nabla(\theta_\ell^\varepsilon-\tilde{\theta}_\ell^\varepsilon)\cdot\Big(a\Big(x_\ell,\frac{x}{\varepsilon},V(x_\ell)\Big)\nabla\theta_\ell^\varepsilon-a\Big(x_\ell,\frac{x}{\varepsilon},W(x_\ell)\Big)\nabla\tilde{\theta}_\ell^\varepsilon\Big)\,dx\\
=&:I_1+I_2.
\end{aligned}
$$

Invoking $(5.32)_2$, we bound I_1 as

$$
\begin{aligned}
|I_1| &\le C|(V-W)(x_\ell)|\,\|\nabla\tilde{\theta}_\ell^\varepsilon\|_{0,I_\delta}\|\nabla(\theta_\ell^\varepsilon-\tilde{\theta}_\ell^\varepsilon)\|_{0,I_\delta}\\
&\le C\Big(\frac{\varepsilon}{\delta}\Big)^{1/2}|(V-W)(x_\ell)|\,\|\nabla W_\ell\|_{0,I_\delta}\|\nabla(\theta_\ell^\varepsilon-\tilde{\theta}_\ell^\varepsilon)\|_{0,I_\delta}.
\end{aligned}
$$

Denote by

$$
p^\varepsilon(x)=\chi^k\Big(x_\ell,\frac{x}{\varepsilon},V(x_\ell)\Big)\frac{\partial V_\ell}{\partial x_k}-\chi^k\Big(x_\ell,\frac{x}{\varepsilon},W(x_\ell)\Big)\frac{\partial W_\ell}{\partial x_k}.
$$

In light of the continuity of $\{\chi^k\}_{k=1}^d$, we have

$$
|p^\varepsilon|,\ \varepsilon|\nabla p^\varepsilon|\le C|V(x_\ell)-W(x_\ell)||\nabla V_\ell|+C|\nabla(V_\ell-W_\ell)|.
$$

Obviously, $\theta_\ell^\varepsilon-\tilde{\theta}_\ell^\varepsilon+p^\varepsilon(1-\rho^\varepsilon)\in H_0^1\big(I_\delta\big)$, where ρ^ε is defined as in (3.12). Integrating by parts, we rewrite I_2 as

$$
\begin{aligned}
I_2=&-\int\limits_{I_\delta}\nabla\Big(\varepsilon p^\varepsilon(1-\rho^\varepsilon)\Big)\cdot\Big(a\Big(x_\ell,\frac{x}{\varepsilon},V(x_\ell)\Big)-a\Big(x_\ell,\frac{x}{\varepsilon},W(x_\ell)\Big)\Big)\nabla\theta_\ell^\varepsilon\,dx\\
&+\int\limits_{I_\delta}\nabla\Big(\varepsilon p^\varepsilon(1-\rho^\varepsilon)\Big)\cdot a\Big(x_\ell,\frac{x}{\varepsilon},W(x_\ell)\Big)\nabla(\tilde{\theta}_\ell^\varepsilon-\theta_\ell^\varepsilon)\,dx.
\end{aligned}
$$

Invoking $(5.32)_1$ and proceeding as in the estimate of I_3 in Lemma 3.2, we get

$$
\begin{aligned}
|I_2|\le & C\Big(\frac{\varepsilon}{\delta}\Big)^{1/2}(|(V-W)(x_\ell)|\|\nabla V_\ell\|_{0,I_\delta}+\|\nabla(V_\ell-W_\ell)\|_{0,I_\delta})|(V-W)(x_\ell)|\|\nabla\theta_\ell^\varepsilon\|_{0,I_\delta}\\
&+C\Big(\frac{\varepsilon}{\delta}\Big)^{1/2}\Big(|(V-W)(x_\ell)|\|\nabla V_\ell\|_{0,I_\delta}+\|\nabla(V_\ell-W_\ell)\|_{0,I_\delta}\Big)\|\nabla(\theta_\ell^\varepsilon-\tilde{\theta}_\ell^\varepsilon)\|_{0,I_\delta}\\
\le & C\frac{\varepsilon}{\delta}(|(V-W)(x_\ell)|^2\|\nabla V_\ell\|_{0,I_\delta}^2+\|\nabla(V_\ell-W_\ell)\|_{0,I_\delta}^2)\\
&+C\Big(\frac{\varepsilon}{\delta}\Big)^{1/2}(|(V-W)(x_\ell)|\|\nabla V_\ell\|_{0,I_\delta}+\|\nabla(V_\ell-W_\ell)\|_{0,I_\delta})\|\nabla(\theta_\ell^\varepsilon-\tilde{\theta}_\ell^\varepsilon)\|_{0,I_\delta}.
\end{aligned}
$$

A combination of the estimates for I_1 and I_2 gives (5.33). □

In the next lemma, we shall prove that $E(V,W)$ has certain continuity with respect to V.

Lemma 5.9. *For any $V, W, Z \in X_H$ satisfying $\|V\|_{1,\infty} + \|W\|_{1,\infty} + \|Z\|_{1,\infty} \le M$, there exists a constant $C(M)$ such that*

$$
\begin{aligned}
&|K| \sum_{x_\ell \in K} \omega_\ell |E(V,Z) - E(W,Z)| \\
&\le C(M)\Big(\Big(\frac{\varepsilon}{\delta}\Big)^{1/2} + \delta\Big)\|V - W\|_{1,K}\|\nabla Z\|_{0,K}. \qquad (5.34)
\end{aligned}
$$

Proof. We decompose $E(V,Z) - E(W,Z)$ into the following terms:

$$E(V,Z) - E(W,Z) = : = I_1 + \cdots + I_6,$$

where

$$I_1 = \fint_{I_\delta} \nabla z_\ell^\varepsilon \cdot \Big[a\Big(x, \frac{x}{\varepsilon}, V(x_\ell)\Big) - a\Big(x, \frac{x}{\varepsilon}, W(x_\ell)\Big)\Big]\nabla(v_\ell^\varepsilon - \hat{v}_\ell^\varepsilon)\,dx,$$

$$I_2 = \fint_{I_\delta} \nabla z_\ell^\varepsilon \cdot a\Big(x, \frac{x}{\varepsilon}, W(x_\ell)\Big)\nabla(v_\ell^\varepsilon - \hat{v}_\ell^\varepsilon - w_\ell^\varepsilon + \hat{w}_\ell^\varepsilon)\,dx.$$

$$I_3 = -\Big(\frac{1}{|\kappa\varepsilon|^d} - \frac{1}{\delta^d}\Big)\int_{I_\delta} \nabla Z_\ell \cdot \Big(a\Big(x_\ell, \frac{x}{\varepsilon}, V(x_\ell)\Big)\nabla\hat{V}_\ell - a\Big(x_\ell, \frac{x}{\varepsilon}, W(x_\ell)\Big)\nabla\hat{W}_\ell\Big)\,dx.$$

$$
\begin{aligned}
I_4 &= \frac{1}{|\kappa\varepsilon|^d}\int_{I_\delta\setminus I_{\kappa\varepsilon}} \nabla Z_\ell \cdot \Big(a\Big(x_\ell, \frac{x}{\varepsilon}, V(x_\ell)\Big)\nabla\hat{V}_\ell - a\Big(x_\ell, \frac{x}{\varepsilon}, W(x_\ell)\Big)\nabla\hat{W}_\ell\Big)\,dx \\
I_5 &= \fint_{I_\delta} \nabla Z_\ell \cdot \Big(a\Big(x_\ell, \frac{x}{\varepsilon}, V(x_\ell)\Big)\nabla\theta_\ell^\varepsilon - a\Big(x_\ell, \frac{x}{\varepsilon}, W(x_\ell)\Big)\nabla\tilde{\theta}_\ell^\varepsilon\Big)\,dx \\
I_6 &= -\Big[\fint_{I_\delta} \nabla Z_\ell \cdot \Big(a(x_\ell, \frac{x}{\varepsilon}, V(x_\ell))\nabla v_\ell^\varepsilon - a(x_\ell, \frac{x}{\varepsilon}, W(x_\ell))\nabla w_\ell^\varepsilon\Big)\,dx \\
&\quad - \fint_{I_\delta} \nabla Z_\ell \cdot \Big(a\Big(x, \frac{x}{\varepsilon}, V(x_\ell)\Big)\nabla v_\ell^\varepsilon - a\Big(x, \frac{x}{\varepsilon}, W(x_\ell)\Big)\nabla w_\ell^\varepsilon\Big)\,dx\Big].
\end{aligned}
$$

Using (5.24) to bound I_1, and (5.29) to bound I_2, we get

$$|I_1| + |I_2| \le C\delta\Big(|\nabla(V_\ell - W_\ell)| + |(V - W)(x_\ell)|(|\nabla V_\ell| + |\nabla W_\ell|\Big)|\nabla Z_\ell|.$$

I_3 and I_4 can be bounded as

$$|I_3|, |I_4| \le C\frac{\varepsilon}{\delta}\Big(|\nabla(V_\ell - W_\ell)| + |(V - W)(x_\ell)|(|\nabla V_\ell| + |\nabla W_\ell|)\Big)|\nabla Z_\ell|.$$

Using Lemma 5.8 we bound I_5 as

$$|I_5| \le C\Big(\frac{\varepsilon}{\delta}\Big)^{1/2}\Big(|\nabla(V_\ell - W_\ell)| + |(V - W)(x_\ell)|(|\nabla V_\ell| + |\nabla W_\ell|)\Big)|\nabla Z_\ell|.$$

Proceeding as I_2 in Lemma 5.7, we bound I_6 as

$$|I_6| \le C\delta\Big(|\nabla(V_\ell - W_\ell)| + |(V - W)(x_\ell)|(|\nabla V_\ell| + |\nabla W_\ell|)\Big)|\nabla Z_\ell|.$$

Finally, we get

$$|E(V,Z)-E(W,Z)|\le C\Big(\frac{\varepsilon}{\delta}+\delta+\Big(\frac{\varepsilon}{\delta}\Big)^{1/2}\Big)\Big(|\nabla(V_\ell-W_\ell)| + |(V-W)(x_\ell)|(|\nabla V_\ell|+|\nabla W_\ell|)\Big)|\nabla Z_\ell|.$$

Summing up, we get

$$|K|\sum_{x_\ell\in K}\omega_\ell|E(V,Z)-E(W,Z)|\le C\delta\|\nabla(V-W)\|_{0,K}\|\nabla Z\|_{0,K} + C\delta\|V-W\|_{L^\infty(K)}(\|\nabla V\|_{0,K}+\|\nabla W\|_{0,K})\|\nabla Z\|_{0,K}.$$

Using the *inverse inequality* on each element, we obtain

$$\begin{aligned}\|V-W\|_{L^\infty(K)}\|\nabla V\|_{0,K}&\le CH_K^{-2/d}\|V-W\|_{0,K}H_K^{2/d}\|\nabla V\|_{L^\infty(K)}\\&=C\|V-W\|_{0,K}\|\nabla V\|_{L^\infty(K)}.\end{aligned}$$

Similarly

$$\|V-W\|_{L^\infty(K)}\|\nabla W\|_{0,K}\le C\|V-W\|_{0,K}\|\nabla W\|_{L^\infty(K)}.$$

A combination of the above estimates leads to (5.34). □

Proof for the second case in Theorem 5.5. Let $W=0$ in the above lemma, we obtain (5.22). Notice that for sufficiently small ε and δ, we have

$$e(\mathrm{HMM})\le C\Big(\Big(\frac{\varepsilon}{\delta}\Big)^{1/2}+\delta\Big)<C_0.$$

Next let $\eta=C\Big((\varepsilon/\delta)^{1/2}+\delta\Big)$, then for sufficiently small ε and δ, we have $\eta<1$, this verifies (5.13). Furthermore, let $\eta=\delta+(\varepsilon/\delta)^{1/2}+H^k$, then for sufficiently small ε,δ and H, η is smaller than η_0, and (5.16) is verified. □

Remark 5.10. *Compared to the linear case, the upper bound for $e(\mathrm{HMM})$ for the case when $\delta/\varepsilon\notin\mathbb{Z}$ degrades to $\sqrt{\varepsilon/\delta}$. This is due to the fact that A_H is non-symmetric.*

In case of $\delta=\varepsilon$, note that $\hat v_\ell^\varepsilon=\hat V_\ell$ and $\hat w_\ell^\varepsilon=\hat W_\ell$, so a direct calculation gives Lemma 5.6 for this case. Lemma 5.7 is also valid with δ replaced by ε. We also have $\theta_\ell^\varepsilon=0$ and $\tilde\theta_\ell^\varepsilon=0$. Observing that for any $V,Z\in X_H$,

$$\fint_{I_\varepsilon(x_\ell)}\nabla z_\ell^\varepsilon\cdot a\Big(x_\ell,\frac{x}{\varepsilon},V(x_\ell)\Big)\nabla\hat v_\ell^\varepsilon\,dx=\fint_K\nabla Z_\ell\cdot\mathcal{A}(x_\ell,V(x_\ell))\nabla V_\ell\,dx,$$

we may rewrite $E(V,Z)$ as

$$\begin{aligned} E(V,Z) = &\fint_{I_\varepsilon} \nabla z_\ell^\varepsilon \cdot a\Big(x, \frac{x}{\varepsilon}, V(x_\ell)\Big) \nabla (v_\ell^\varepsilon - \hat{v}_\ell^\varepsilon)\, dx \\ &+ \fint_{I_\varepsilon} \nabla z_\ell^\varepsilon \cdot \Big[a\Big(x, \frac{x}{\varepsilon}, V(x_\ell)\Big) - a\Big(x_\ell, \frac{x}{\varepsilon}, V(x_\ell)\Big)\Big] \nabla \hat{v}_\ell^\varepsilon\, dx. \end{aligned}$$

Consequently, we have for any $V, W, Z \in X_H$,

$$|K| \sum_{x_\ell \in K} \omega_\ell |E(V,Z) - E(W,Z)| \le C(M)\varepsilon \|V - W\|_{1,K} \|\nabla Z\|_{0,K}.$$

Proof of Theorem 5.5. Let $W = 0$ in the above lemma, we obtain (5.23). Notice that for sufficiently small ε, we have

$$e(\mathrm{HMM}) \le C\varepsilon < C_0.$$

Next let $\eta = C\varepsilon$, then for sufficiently small ε, we have $C\varepsilon < 1$, this proves (5.13). Furthermore, we let $\eta = \varepsilon^{1/2} + H^k$, so for sufficiently small ε and H, η is smaller than η_0. This proves (5.16). □

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