Possible Problems in Number Theory and Analysis for Numerical Experimentation

Carlos García, Jennifer Johnson, Steven Miller, and Peter Sarnak

Note that in the cases of the first four problems below, some experimentation has been carried out by seniors at Princeton University in their senior theses (Problem 1 Jean Steiner, Problem 2 Susan Berube, Problem 3 Martin Weisman, Problem 4 Ron Fertig). These include elementary descriptions and setting a context for the problems. There remains, even for these, much room for further investigation.

1. Let α be an irrational of "diophantine type", for example $\alpha = \sqrt{2}$. For N large, consider the N numbers in $[0, 1)$, $n^2\alpha \pmod 1$, $n = 1, 2, \ldots, N$. Order these as $0 \le x_1 \le x_2 \le \cdots \le x_N < 1$. The question is: Do the normalized consecutive spacings

   $$\Delta_j = N(x_{j+1} - x_j), \quad j = 1, \ldots, N-1$$

   behave like spacings between random numbers? So for example, does the distribution of the consecutive spacings tend to the density $e^{-x}\,dx$? This problem arises in connection with the spacing distribution for the eigenvalues of quantizations of completely integrable Hamiltonians, and was raised by physicists.

2. Let X_N be a random cubic graph on N vertices, i.e., 3 edges emanate from each vertex. Let A be the adjacency matrix for X_N, i.e., $a_{vw} = 1$ if v is joined to w and 0 otherwise. A is symmetric and its eigenvalues all lie in $[-3, 3]$. The largest eigenvalue $\lambda_0(X)$ is equal to 3. Denote by $\lambda_1(X)$ the next to largest eigenvalue of A. In the construction of highly connected sparse graphs ("expanders"), it is desirable to have $\lambda_1(X)$ as small as possible. It is known that for any sequence of X_N's

   $$\liminf_{N\to\infty} \lambda_1(X_N) \ge 2\sqrt{2}$$

   The question is whether, with probability 1, $\lambda_1(X) \to 2\sqrt{2}$ as $N \to \infty$. If the answer to the above question is yes, then the random graph would be as good (for this purpose) as the arithmetically constructed Ramanujan graph.

3. Let ρ be a nontrivial irreducible representation of the Galois group G of an extension K of $\mathbb{Q}$. Artin introduced an "L-function" $L(s,\rho)$ which he conjectured is entire. When ρ is two dimensional and "odd", a lot is known. However, for the even case, very little is known. A method to check this conjecture numerically was developed in a recent undergraduate thesis at Princeton University, and some checking was carried out. The problem is ripe for much further exploration.

4. It seems to be an empirical fact that the ranks of Mordell-Weil groups of families of elliptic curves over $\mathbb{Q}$ are generally higher than one might naively expect. There are numerous papers documenting this, but in all of these the inferences drawn are from data which is small for the type of problem (so we suggest that often the wrong inference is drawn). It is desirable to investigate much larger data sets which can be done with some specific families. Such an investigation would indicate whether "it is an artifact of small numbers (and 'low height' points)". A step in this direction has been taken in a senior thesis here last year, but much more needs to be done if what the "truth" on the question of excess rank is to be determined.

5. The local scaled spacing distribution between the eigenvalues of a random large symmetric matrix whose entries are chosen to be I.I.D. Gaussian is a well known celebrated result from random matrix theory. It is believed (and is a central open problem) that the same universal laws hold for the eigenvalue spacings if one chooses the matrix entries as I.I.D., but not necessarily Gaussian. There is not that much evidence for such a universality conjecture and numerical experiments would be illuminating (there are some done for matrices of size 20×20 going back to the 50's and 60's, but one can go a lot further with today's eigenvalue algorithms and computers).

6. It is a folklore belief that the local spacings between consecutive primes follow the laws of random numbers (as in Problem 1). That is, if P_n is the n^{th} prime, then the distribution of $\Delta_n = \frac{P_{n+1}-P_n}{\log P_n}$, $n = 1, 2, \dots$ will converge to $e^{-x}\,dx$. There is, as far as we know, no serious experimentation confirming this conjecture (in fact there are some misguided claims and computations in the physics literature).

7. Let V be a variety (for example a hypersurface) defined by integral polynomial equations in $(x_1, \ldots, x_n)$. Let $V(\mathbb{Z})$ be the integral points in V and let

$$N_V(T) = \left|\{x \in \mathbb{Z}^n | x \in V(\mathbb{Z}), ||x|| \leq T\}\right|$$

We say V is a "Hardy-Littlewood Variety" if the asymptotic behavior of $N_V(T)$ as $T \to \infty$ is given by a product of local densities which count the number of solutions in V over the real numbers and the p-adic numbers (the exact form of this product of local densities comes from the Hardy-Littlewood method). The known Hardy-Littlewood varieties are ones shown to be so by their method, which requires that n be very large compared to the degrees of the defining equations. However, it appears (from a number of points of view) that to be a Hardy-Littlewood variety is not so restrictive. A test example would be the case of nonsingular cubic forms F in six variables:

$$V : \{x \mid F(x_1, \ldots, x_6) = 0\}$$

(The case of nine is probably Hardy-Littlewood, while four variables is not!) One could test this by determining the expected Hardy-Littlewood asymptotic and comparing it with the experimental counting of $N_V(T)$.

8. The local distribution of the spacings of the high zeros of the Riemann zeta function has been throughly investigated numerically. For many applications, other issues, besides the local spacings, are central. For example, the truth of the following is of great interest:

Write the zeros of zeta as $\frac{1}{2} + i\gamma_j$, $0 \leq \gamma_1 \leq \gamma_2 \leq \gamma_3 \leq \ldots$. Fix $A > 0$; then the numbers

$$A\frac{\gamma_j}{2\pi} \log\left(\frac{\gamma_j}{2\pi e}\right)$$

are well distributed modulo 1 if and only if $A \neq \frac{1}{\nu}$, ν an integer.

9. **Cycle double cover conjecture**. It is conjectured that for every graph with no cutedge, there is a list of cycles covering every edge exactly twice. Suppose that this is false - what can we say about the smallest counterexample G? Certainly G is cubic and 3-edge-connected. Huck showed (on a computer) that all cycles of G have at least 10 edges (about - I don't remember exactly). To do this, let

C be a short cycle (length ¡ 10); delete one edge e of C; look at all the possible ways that this new graph $G \setminus e$ might have a cycle double covering (more precisely, enumerate all the ways that the remaining edges of C might be covered) and for each such way, show how to convert it to a cycle covering of G. What about variations on this? Other things than cycles? Other methods than deletion?

10. **The 5-flow conjecture**. Let G be a directed graph, and let N be an abelian group. A nowhere-zero N-flow in G means a map $f : E(G) \longrightarrow N$ so that:

    (i) the flow is conserved at every vertex v - that is, the sum of $f(e)$ over all edges e with head v equals the sum of $f(e)$ over all edges with tail v.

    (ii) $f(e)$ is nonzero for every edge e.

    Tutte conjectured that every 2-edge-connected graph has a nowhere-zero Z_5-flow (Z_5 is the additive group of integers mod 5). (It is known that they all have Z_6-flows.) If Tutte's conjecture is false, what can be said about the length of the shortest cycle in the smallest counterexample? It is known only that this shortest cycle has length at least 7, if I remember correctly. Can anything work similar to Huck's method in (1) above?

11. The chromatic polynomial of a graph is the number of ways to color the vertex set of the graph with only k available colors, expressed as a function of k (it is a polynomial in k). It is a famous old conjecture that for every graph, the coefficients of this polynomial make a unimodal sequence (except for their signs - the signs alternate); i.e., in absolute value the coefficients first increase and then decrease. This could be tested out for some particular graphs on a computer - e.g. what about complete multipartite graphs? (These are the graphs with a vertex-colouring so that every pair of vertices with different colors are adjacent.)

12. **Roots of the chromatic polynomial**. The four-color theorem says that for a planar loopless graph, 4 is not a root of the chromatic polynomial. What about roots bigger than 4? It is known that there are no roots above 5, and it is conjectured that there are no roots between 4 and 5. Perhaps this could be tested for particular graphs. e.g. what about the graph with k pentagons, each inside the next,

so that every vertex of every pentagon (except the first and last) is adjacent to two vertices of the next pentagon and to two vertices of the previous one, and we add a vertex of degree 5 inside the first and another outside the last pentagon. Also, can there be real roots arbitrarily close to 4? It is conjectured not.

13. **The points-lines-planes conjecture**. Take p points in 3-space; let q be the number of lines in 3-space that pass through at least two of them, and let r be the number of planes in 3-space that contain at least three of them, not collinear. It is conjectured that

$$\frac{q}{p} > \frac{r}{q}$$

but it is not known whether there is ANY constant $c > 0$ so that

$$\frac{q}{p} > c\frac{r}{q}$$

for all choices of the points. Maybe this could be tested on some examples?