Digit Distribution in the Continued Fraction of $\zeta(n)$

Felice Kuan*

February 13, 2003

Mathematics Department
Princeton University
Princeton, NJ 08544

Abstract

Kuzmin's Theorem predicts a logarithmic relationship between a digit and its frequency in the continued fraction expansion of a number. This theorem holds for all numbers excepting a set of measure zero. We examine the values of $\zeta(n)$, where n is an integer, to produce evidence that this set follows or does not follow the theorem.

1 Gauss-Kuzmin Distribution

Let $\alpha \in \mathbb{R}$. In examining the continued fraction expansion of α, we may assume without loss of generality that $\alpha \in (0, 1)$, for it is only the first term that is affected.

We have $\alpha = [0, a_1, a_2, a_3, a_4, \ldots]$. Let k be a positive integer. Then Levy's improvement on Kuzmin's Theorem predicts

$$Prob(a_n = k) \;=\; \log_2\left(1 + \frac{1}{k(k+2)}\right) + O(e^{-Bn}) \tag{1.1}$$

for some constants A and B [M]. Since the error term shrinks as n increases, for n sufficiently large, the probability that $a_n = k$ is nearly independant of n.

Kuzmin's Theorem holds for all $\alpha \in (0, 1)$, save a set of measure zero. Rational α fail, of course, for their continued fraction expansions are finite. Quadratic irrationals also fail, for their continued fraction expansions are periodic. Some individual curiosities form their own failing set, such as e.

*E-mail: `fkuan@math.princeton.edu`

We wish to determine which other sets fail and which succeed. This paper examines the set of values of $\zeta(n)$, where n is a positive integer. Restricting n in this way makes our set countable, and therefore of measure zero. Were we to consider $\zeta(x)$ for any real x, our set would take on positive measure, and we would therefore already know that it satisfies the theorem.

2 The Riemann-Zeta Function

Since rationals fail Kuzmin's Theorem, we would like to avoid redundancy by first checking that $\zeta(n)$ is irrational $\forall n \in \mathbb{Z}^+$. Unfortunately, we do not even have the luxury of knowing this is true for all n. We look first at $\zeta(2n)$.

2.1 The Zeta Function at Positive Even Integers

It is commonly known that $\zeta(2) = \frac{pi^2}{6}$, and thus since π^2 is irrational, so is $\zeta(2)$. Though we also know that π is transcendental [ST], this implies nothing about the transcendence of $\zeta(2)$.

For fun, here are three of the most elegant proofs of $\zeta(2) = \frac{pi^2}{6}$.

2.1.1 Calabi's Method

We take the geometric series

$$(1 - x^2y^2)^{-1} = \sum_{n=0}^{\infty}(x^2y^2)^n \tag{2.1}$$

and integrate both sides over the square $S : [0,1] \times [0,1]$.

$$
\begin{aligned}
\int\int_S (1-x^2y^2)^{-1}\, dxdy &= \int\int_S (1+(x^2y^2)+(x^2y^2)^2+\cdots)\, dxdy \\
&= \int_0^1 \left[(x+\frac{1}{3}x^3y^2+\frac{1}{5}x^5y^4+\cdots\right]_0^1 dy \\
&= \int_0^1 (1+\frac{1}{3}y^2+\frac{1}{5}y^4+\cdots)\, dy \\
&= \left[y+\frac{y^3}{3^2}+\frac{y^5}{5^2}+\cdots\right]_0^1 \\
&= 1+\frac{1}{3^2}+\frac{1}{5^2}+\cdots \\
&= (1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots)-(\frac{1}{2^2}+\frac{1}{4^2}+\cdots) \\
&= (1-\frac{1}{4})\,\zeta(2) \qquad (2.2)
\end{aligned}
$$

We then make the substitution $(x,y)=(\frac{\sin u}{\cos v},\frac{\sin v}{\cos u})$. The cooperative Jacobian conveniently turns out to be $1-x^2y^2$. Since the substitution bijectively maps the triangle $T=\{u,v>0,\ u+v<\frac{\pi}{2}\}$ into the interior of S, we obtain

$$
\begin{aligned}
\int\int_S (1-x^2y^2)^{-1}\, dxdy &= \int\int_T dudv \\
&= \frac{1}{2}\cdot\frac{\pi}{2}\cdot\frac{\pi}{2} \\
&= \frac{\pi^2}{8} \qquad (2.3)
\end{aligned}
$$

Applying Eq. 2.2 to the left side produces

$$
\begin{aligned}
(1-\frac{1}{4})\,\zeta(2) &= \frac{\pi^2}{8} \\
\zeta(2) &= \frac{\pi^2}{6}. \qquad (2.4)
\end{aligned}
$$

2.1.2 Product Expansion of Sine

We have

$$
\begin{aligned}
\sin(z) &= z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right) \\
\frac{\sin(z)}{z} &= \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right).
\end{aligned}
\tag{2.5}
$$

Letting $w = z^2$ produces

$$
\frac{\sin(\sqrt{w})}{\sqrt{w}} = \prod_{n=1}^{\infty} \left(1 - \frac{w}{n^2\pi^2}\right). \tag{2.6}
$$

Expanding $\sin(\sqrt{w})$, we take the logarithm of both sides. Recall that the logarithm of a product is the sum of the logarithms.

$$
\begin{aligned}
\log\left(\frac{\sqrt{w} - \frac{(\sqrt{w})^3}{3!} + \frac{(\sqrt{w})^5}{5!} - \cdots}{\sqrt{w}}\right) &= \sum_{n=1}^{\infty} \log\left(1 - \frac{w}{n^2\pi^2}\right) \\
\log\left(1 - \left[\frac{w}{6} - \frac{w^2}{120} + O(w^3)\right]\right) &= \sum_{n=1}^{\infty} \log\left(1 - \frac{w}{n^2\pi^2}\right)
\end{aligned}
\tag{2.7}
$$

Now the Taylor expansion of $\log(1-x)$ is $-x - \frac{x^2}{2!} + \cdots$. Thus,

$$
-\frac{w}{6} + \frac{w^2}{120} - \frac{w^2}{72} + O(w^3) = -\sum_{n=1}^{\infty} \frac{w}{n^2\pi^2} - \sum_{n=1}^{\infty} \frac{w^2}{2n^4\pi^4} + \cdots \tag{2.8}
$$

Equating coefficients of w, we have

$$
\begin{aligned}
\frac{1}{6} &= \sum_{n=1}^{\infty} \frac{1}{n^2\pi^2} \\
\frac{\pi^2}{6} &= \sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2).
\end{aligned}
\tag{2.9}
$$

Also, equating coefficients of w^2, we have

$$\begin{aligned} \frac{1}{180} &= \sum_{n=1}^{\infty} \frac{1}{2n^4\pi^4} \\ \frac{\pi^4}{90} &= \sum_{n=1}^{\infty} \frac{1}{n^4} \;=\; \zeta(4). \end{aligned} \tag{2.10}$$

2.1.3 Taylor Expansion of $\ln(1-x)$

We begin with

$$-\sum_{n=1}^{\infty} \frac{x^n}{n} = \ln(1-x) \tag{2.11}$$

We take the principal branch of $\ln x$, with the cut along $(\infty, 0]$ such that $\text{Ln } x = \ln x \;\; \forall x > 0$. Then e^{ix} remains within the radius of convergence of 1, so we may take our x to be e^{ix}.

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{e^{inx}}{n} &= -\ln(1-e^{ix}) \\ \sum_{n=1}^{\infty} \frac{\cos(nx) + i\sin(nx)}{n} &= \Re\left[-\ln(1-e^{ix})\right] + i\Im\left[-\ln(1-e^{ix})\right] \end{aligned} \tag{2.12}$$

Now since $\Im\left[\log(z)\right] = \arg(z)$, taking imaginary parts of Eq. 2.12 produces

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\sin(nx)}{n} &= \Im\left[-\ln(1-e^{ix})\right] \\ &= -\Im\left[\ln(1-(\cos x + i\sin x))\right] \\ &= -\arctan\left(\frac{-\sin x}{1-\cos x}\right) \\ &= \arctan\left(\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}\right) \\ &= \arctan\left(\cot\frac{x}{2}\right) \end{aligned} \tag{2.13}$$

Integrating both sides with respect to x produces

$$
\begin{aligned}
-\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2} &= \frac{-x^2 + 2\pi x}{4} + C \\
\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2} &= \frac{x^2 - 2\pi x}{4} + C
\end{aligned} \tag{2.14}
$$

Let $f(x) = \sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2}$. In which case

$$
\begin{aligned}
f(0) &= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \zeta(2) \\
f(\pi) &= -\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \cdots
\end{aligned} \tag{2.15}
$$

Then

$$
\begin{aligned}
\frac{f(0)}{4} &= \frac{1}{2^2} + \frac{1}{4^2} + \cdots \\
&= \frac{f(0) + f(\pi)}{2} \\
-\frac{1}{2} f(0) &= f(\pi)
\end{aligned} \tag{2.16}
$$

By Eq. 2.14, we have

$$
\begin{aligned}
-\frac{1}{2} C &= \frac{\pi^2 - 2\pi^2}{4} + C \\
C &= \frac{\pi^2}{6}
\end{aligned} \tag{2.17}
$$

But of course $C = f(0) = \zeta(2)$, and thus $\zeta(2) = \frac{\pi^2}{6}$ [R].

2.2 Euler's Product Formula

We make an aside to discuss the significance of the irrationality of $\zeta(s)$. One consequence is the infinitude of primes, and it is remarkable that this prime abundance can be so plainly shown by the irrationality of a single number. We prove it here [S].

Let p be a prime. Then for $\Re(s) > 1$, $|\frac{1}{p^s}| < 1$, and we may form the geometric series

$$\frac{1}{\left(1-\frac{1}{p^s}\right)} = 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \cdots + \frac{1}{p^{Ms}} + \cdots . \tag{2.18}$$

Let M and N be positive integers such that $M > N$. Now the fundamental theorem of arithmetic states that every positive integer $n \leq N$ can be uniquely written as a product of primes. It furthermore states that each prime in this product is less than or equal to N and occurs less than M times.

Thus

$$\begin{aligned}\sum_{n=1}^{N} \frac{1}{n^s} &\leq \prod_{p \leq N} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots + \frac{1}{p^{Ms}}\right) \\ &\leq \prod_{p \leq N} \left(\frac{1}{1-p^{-s}}\right) \\ &\leq \prod_{p} \left(\frac{1}{1-p^{-s}}\right). \end{aligned} \tag{2.19}$$

As N tends to infinity, we have

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \leq \prod_{p} \left(\frac{1}{1-p^{-s}}\right). \tag{2.20}$$

Also from the fundamental theorem of arithmetic, we have

$$\prod_{p \leq N} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots + \frac{1}{p^{Ms}}\right) \leq \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{2.21}$$

As M tends to infinity, we have

$$\prod_{p \leq N} \left(\frac{1}{1-p^{-s}}\right). \tag{2.22}$$

Thus

$$\prod_{p \leq N} \left(\frac{1}{1 - p^{-s}} \right) \leq \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{2.23}$$

producing what is known as the Euler Product Formula:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p} \frac{1}{\left(1 - \frac{1}{p^s}\right)}. \tag{2.24}$$

The infinitude of primes follows from this Formula. The standard method lets s tend to 1. Then the right-hand side of Eq. 2.24 is finite if and only if there are finitely many primes. But $\zeta(1)$ is not a convergent series, i.e., the left-hand side is not finite. Thus there cannot be finitely many primes.

Another option is to let $s = 2$. Then the right-hand side of Eq. 2.24 is rational if and only if there are finitely many primes. Now as we spent the past section proving, $\zeta(2) = \frac{\pi^2}{6}$, which is irrational because π^2 is irrational. Therefore, the supply of primes is infinite. This 'special value' proof is interesting indeed, for it connects the infinitude of primes with a simple property of a single number.

2.3 Extension to $\zeta(2n)$

We return to the task of verifying that $\zeta(2n)$ is irrational. The zeta function possesses the friendly property that for all positive even integers k,

$$\zeta(k) = \frac{p}{q} \cdot \pi^k \tag{2.25}$$

where $p, q \in \mathbb{Q}$. This assures the irrationality of $\zeta(2n)$.

For proof of Eq. 2.25, the three methods discussed earlier can all be extended to apply to $\zeta(2n)$ for all $n \in \mathbb{Z}$, but they are comparatively unpleasant. Here instead is a method by Don Zagier [DZ].

Define the function

$$f(m, n) = \frac{1}{mn^3} + \frac{1}{2m^2n^2} + \frac{1}{m^3n}. \tag{2.26}$$

Then

$$f(m,n) - f(m+n,n) - f(m,m+n) = \frac{1}{m^2n^2} \tag{2.27}$$

can be shown with straightforward arithmetic.

If we sum over all $m,n > 0$, we acquire $\zeta(2)^2$ from the right-hand side of Eq. 2.27. Thus

$$\begin{aligned}
\zeta(2)^2 &= \left(\sum_{m,n>0} - \sum_{m>n>0} - \sum_{n>m>0}\right) f(m,n) \\
&= \sum_{n>0} f(n,n) \\
&= \sum_{n=1}^{\infty} \frac{1}{nn^3} + \frac{1}{2n^2n^2} + \frac{1}{n^3n} \\
&= \frac{5}{2}\,\zeta(4).
\end{aligned} \tag{2.28}$$

Then, delightfully, $\zeta(4) = \frac{\pi^4}{90}$. Now let us determine $\zeta(k)$ for even values of k larger than 4. Define

$$f(m,n) = \frac{1}{mn^{k-1}} + \frac{1}{2}\sum_{r=2}^{k-2} \frac{1}{m^r n^{k-r}} + \frac{1}{m^{k-1}n}. \tag{2.29}$$

Arithmetic again yields

$$f(m,n) - f(m+n,n) - f(m,m+n) = \sum_{0<j<k,\ j\ even} \frac{1}{m^j n^{k-j}}, \tag{2.30}$$

so in similar fashion, we find

$$\sum_{0<j<k,\ j\ even} \zeta(j)\,\zeta(k-j) = \frac{k+1}{2}\,\zeta(k). \tag{2.31}$$

Thus Eq. 2.25 is true by induction.

3 Irrationality of $\zeta(2n+1)$

The problem of whether $\zeta(n)$ is irrational for odd n is dramatically more difficult than that of even n. It was not until 1978 that R. Apéry presented the first-ever proof of the irrationality of $\zeta(3)$. Essentially, he showed that $\zeta(3)$ can be approximated by rationals too well to be itself rationally. Said rigorously, he proved that the approximation exponent of $\zeta(3)$ is greater than 1.

A number ξ has approximation exponent e if e is the smallest number such that for all $r > e$, there are only finitely many solutions to

$$\left|\xi - \frac{p}{q}\right| < \frac{C}{q^r}. \tag{3.1}$$

e is thus an indication of sorts of the closest to ξ an approximation by rational numbers can get. (e need not be an integer.)

Suppose ξ is rational, i.e., $\xi = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$. Let $\frac{p}{q} \neq \frac{a}{b}$. Then $pb - aq \neq 0$ and is an integer, in which case $|pb - aq| \geq 1$. Thus

$$\begin{aligned} \left|\xi - \frac{p}{q}\right| &= \left|\frac{a}{b} - \frac{p}{q}\right| \\ &= \frac{|pb - aq|}{bq} \\ &\geq \frac{1}{bq}. \end{aligned} \tag{3.2}$$

Now there's bound to be some e such that

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{q^e}, \tag{3.3}$$

for we can just pick something or other for e and keep decreasing it until the relation is true. Suppose $e > 1$. Then

$$\frac{1}{q^e} > \left|\xi - \frac{p}{q}\right| \geq \frac{1}{bq} \tag{3.4}$$

and therefore $q^{e-1} < b$. But b is fixed, and q is integral, thus there are only finitely many q for which this relation holds. Since p is also integral, there are only finitely many $\frac{p}{q}$ such that Eq. 3.3 is true. In other words, try to take rational numbers any closer to ξ than $\frac{1}{q}$ and you won't have an approximation by rationals anymore. Thus, the approximation exponent of ξ (that is, any rational number) must be 1 or less.

One can therefore prove a number is irrational by showing it has an approximation exponent greater than 1. This is precisely what Apéry did. Take the recursion

$$n^3 u_n + (n-1)^3 u_{n-2} = (34n^3 - 51n^2 + 27n - 5)u_{n-1} \tag{3.5}$$

for $n \geq 2$. Let $\{b_n\}$ be defined by $b_0 = 1, b_1 = 5$, and $b_n = u_n$. Let $\{a_n\}$ be defined by $a_0 = 0, a_1 = 6$, and $a_n = u_n$.

Now let d_n denote the least common multiple of $1, \ldots, n$. If we take

$$\begin{aligned} p_n &= 2d_n^3 a_n \\ q_n &= 2d_n^3 b_n, \end{aligned} \tag{3.6}$$

then Apéry showed that $p_n, q_n \in \mathbb{Z}$ and that

$$\zeta(3) - \frac{p_n}{q_n} = O(\frac{1}{q_n^{1+\delta}}), \tag{3.7}$$

where $\delta = 0.080529\ldots > 0$. In other words, Apéry found an infinite supply of $\frac{p}{q}$ such that

$$\left|\zeta(3) - \frac{p}{q}\right| < \frac{1}{q^e} \tag{3.8}$$

where $e > 1$. Thus, $\zeta(3)$ must have approximation exponent greater than 1, and therefore is irrational. For more details, see [VDP].

More than twenty years after Apéry's proof, definitive statements were finally made concerning the irrationality of $\zeta(2n+1)$. In 2000, T. Rivoal produced linear forms with rational coefficients in the values of $\zeta(s)$ at odd points. These constructions allowed him to prove that there are infinitely many irrationals among $\zeta(2n+1)$.

The generalization of Rivoal's method leads to linear forms in

$$1, \zeta(s), \zeta(s+2), \ldots, \zeta(s+2m), \tag{3.9}$$

where $s \geq 3$ is odd. A wise choice of m lets one prove the irrationality of at least one element in a set of values of $\zeta(s)$ at consecutive odd points.

Since then, Rivoal and V. Zudilin have pared down the scope of these linear forms; at the time of this writing, one can prove at least one of the numbers $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is irrational [Z].

4 Computations

The purpose of the experimental portion of this paper, which was performed on Mathematica, was to provide large-scale numerical evidence that $\zeta(n),\ n \in \mathbb{Z}^+$ satisfies Kuzmin's Theorem. The use of Mathematica enabled the scanning of up to 1,000,000 digits per continued fraction.

Let's start with the results for $\zeta(2n+1), 1 \leq n \leq 100$. The probabilities beyond $P(a_n = 5)$ were calculated using $P(a_n \in [x, y]) = 1 - P(a_n \notin [x, y])$.

	$P(a_n = 1)$	$P(a_n = 2)$	$P(a_n = 3)$	$P(a_n = 4)$	$P(a_n = 5)$
Kuzmin	0.415037	0.169925	0.093109	0.058894	0.040642
$\zeta(2n+1)$	0.414851	0.169458	0.093149	0.058749	0.043578
σ	0.005040	0.003792	0.002943	0.006773	0.002815

	$P(a_n \in [6, 10])$	$P(a_n \in [11, 100])$	$P(a_n > 100)$
Kuzmin	0.096862	0.111317	0.014214
$\zeta(2n+1)$	0.096665	0.111784	0.014413
σ	0.002815	0.003148	0.001199

That's enough evidence for me that $\zeta(2n+1)$ follows Kuzmin's Theorem. These results are the average of the results from digits 1 to 10,000 in the continued fractions of the 200 data points, $\zeta(2n+1), 1 \leq n \leq 100$. To compile these results, we counted the occurrences of the digits $1, 2, \ldots$, then used the fact that since the expected value of a sum is the sum of the expected values,

$$\text{Total digits examined} \cdot Prob(a_n = k) \;=\; \text{Number of occurrences.} \qquad (4.1)$$

The error term must also be taken into account. Being of order e^{-Bn} for some constant B, as proved by Levy, it decays as n grows large. This explains the desire to calculate as many digits of a continued fraction as possible, for the later the digit, the lesser the error.

The effects of the error term can be seen in the following chart, which lists the results of $\zeta(2n)$ using 100,000 digits (set A), or a mere 1,000 (set B). Though both sets show the adherence of $\zeta(2n)$ to Gauss-Kuzmin distribution, accuracy from the 100,000-digit set A is much sharper. Results were tabulated from the last 50,000 digits of set A and all 1,000 digits of set B. In both cases, we considered $\zeta(2n)$ for $1 \leq n \leq 1000$.

	$P(a_n = 1)$	$P(a_n = 2)$	$P(a_n = 3)$	$P(a_n = 4)$	$P(a_n = 5)$
Kuzmin	0.415037	0.169925	0.093109	0.058894	0.040642
$\zeta(2n)_A$	0.415131	0.169978	0.093062	0.058867	0.040641
$\zeta(2n)_B$	0.415385	0.169518	0.092633	0.058757	0.040696
σ_A	0.002059	0.001633	0.001296	0.001033	0.000862
σ_B	0.014614	0.011554	0.009187	0.007447	0.006264

	$P(a_n \in [6, 10])$	$P(a_n \in [11, 100])$	$P(a_n > 100)$
Kuzmin	0.096862	0.111317	0.014214
$\zeta(2n)_A$	0.096839	0.111268	0.014214
$\zeta(2n)_B$	0.096347	0.111511	0.015153
σ_A	0.001312	0.001393	0.000530
σ_B	0.009243	0.009752	0.003786

Note that the probabilities for intervals are less accurate than the probabilities for digits 1 to 5. This is perfectly understandable, since the error is the sum of all the errors for all the digits in the interval.

Note also that the standard deviations of set B suffer from the scarcity of digits and are thus unsurprisingly larger than those of set A.

How can we account for the rather mediocre performance of $\zeta(2n + 1)$, especially as compared to the mere 1,000 digits taken in set B of $\zeta(2n)$? Well, our data pool had only 200 numbers. Laziness on my part? No – Mathematica has significantly more trouble producing values for $\zeta(2n + 1)$ than for $\zeta(2n)$. Most likely, this is because good identities exist for values of $\zeta(2n)$, e.g. quick formulae using Bernoulli numbers, and Mathematica is probably using these.

Regardless, these results demonstrate that Kuzmin's Theorem certainly holds for $\zeta(2n), n \in [1, 1000]$ and $\zeta(2n + 1), n \in [1, 100]$. This is of course not proof that it should hold for all n, though I personally have little doubt that it does.

5 For Those Who Come After

The first extension question that comes to mind is what of $\zeta(s), s \in \mathbb{Q}$? (Or, if we assume Kuzmin's Theorem holds for $\zeta(n), n \in \mathbb{Z}$, what of $\zeta(s)$, where s rational but not integral?) There exists theory and some formulae about values of $\zeta(s)$ for s rational, though it's much thicker and less definate than for $\zeta(n)$. The computational portion would of course be more difficult as well.

One could also seek to improve the statistical accuracy of this paper, which might lead to discovering that Mathematica's excellent built-in programs are not the best way, especially in the case of $\zeta(2n+1)$. It would be a splendid project to investigate algorithms for calculating zeta values. Articles by J. Borwein are interesting; for odd n there is Plouffe's work with the LLL Algorithm [SQ].

The program used for this paper should be useful in any of these cases.

6 Bibliography

References

[A] R. Apéry, *Irrationalité de $\zeta(2)$ et $\zeta(3)$*, Astérisque **61**, 11-13, 1979.

[B] F. Beukers, *A Note on the Irrationality of $\zeta(3)$*, Bull. London Math. Soc. **11**, 268-272 (1979).

[D] K. Devlin, *How Euler discovered the zeta function*, http://www.maths.ex.ac.uk/˜mwatkins/zeta/devlin.pdf

[E] H. M. Edwards, *Riemann's Zeta Function*, Academic Press, New York, 1974.

[M] S.J. Miller, R. Takloo-Bighash, F. Spinu, H. Helfgott, *Notes for Princeton Junior Seminar, Fall 2002: Diophantine Analysis and Roth's Theorem.* Princeton University, 2002.

[SQ] R. Qian and D. Steinhauer, *Rational Relation Conjectures.* Princeton University, 2003.

[R] E Rowland, *Riemann Zeta Function*, http://people.ucsc.edu/˜erowland/setz.html

[S] E. Stein, *Complex Analysis* (to appear)

[ST] I. Stewart, *Galois Theory*, Chapman and Hall, London, 1973.

[VDP] A. van der Poorten, *A proof that Euler missed...Apéry's proof of the irrationality of $\zeta(3)$*, New Mathematical Intelligencer **1**, 195-203, 1979.

[MW] E. Weisstein, *Mathworld* (Riemann Zeta Function, Riemann Zeta Function Zeta(2), Apéry's Constant, Gauss-Kuzmin Distribution). http://www.mathworld.com

[DZ] D. Zagier, *Values of Zeta Functions and Their Applications*, First European Congress of Mathematics, Vol. II, 497-512, Basel, Birkhauser Verlag, 1992.

[Z] W. Zudilin, *Irrationality of values of zeta-function at odd integers*, Uspekhi Mat. Nak **56**:2, 215-216, 2001.